Bounded Functional Interpretation and Feasible Analysis

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Abstract

In this article we study applications of the bounded functional interpretation to theories of feasible arithmetic and analysis. The main results show that the novel interpretation is sound for considerable generalizations of weak König's lemma, even in the presence of very weak induction. Moreover, when combined with Cook and Urquhart's variant of the functional interpretation, one obtains effective versions of conservation results regarding weak König's lemma which have been so far only obtained non-constructively.

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1 Introduction

A new form of functional interpretation has been developed in [15], focusing on bounds rather than on precise witnesses. In that paper, the new interpretation is defined and studied, and some applications are made to systems where the bounded search operator is present. In these systems bounded first-order formulas are equivalent to quantifier-free formulas, and the analysis of the former

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is reducible to the latter. However, in feasible settings – where bounded search is unavailable – this reduction is blocked. Gödel's original functional interpretation (cf. [16]) treats bounded quantifications as ordinary quantifications, not being attuned for their specific analysis. On the other hand, the new interpretation was conceived so that it would leave (intensional) bounded formulas unaffected and, in particular, would leave unaffected first-order bounded formulas, even in feasible settings. Therefore, the new interpretation is tailored for getting conservation results over *feasible* theories of arithmetic and analysis. This issue is the focus of the present paper.

Feasible systems of arithmetic are formal theories with very restricted kinds of induction, so much so that their provably total functions (in an appropriate sense) are the polynomial time computable functions. In the context of *first-order* arithmetic, these systems were introduced by Samuel Buss in his doctoral dissertation [5] two decades ago. Three years later in [11], Fernando Ferreira introduced *second-order* feasible systems (see, also, [12]). More recently, he and António Fernandes laid down the grounds for the formalization of analysis in feasible systems – cf. [13], but also [10,14,25]. In the sequel, we work within the framework of *finite type* arithmetic and use the new form of functional interpretation to study metamathematical properties of *feasible* systems related to so-called boundedness principles in analysis.

There are four main differences between feasible systems and systems based on primitive recursion:

(1) There is no minimization functional μ_b of type $(0 \to 1) \to 1$ satisfying axioms stating that $\mu_b f^{0\to 1} n^0 =_0 \min_0 k \leq_0 n(fnk =_0 0)$ if such a $k \leq_0 n$ exists, and $=_0$ 0 otherwise. In other words, bounded searches are not permitted in general.
(2) Bounded first-order formulas are not necessarily equivalent to quantifier-free formulas and are not necessarily decidable (i.e., *tertium non datur* need not hold for them). This characteristic is, of course, related to the previous one.
(3) There is no maximization functional M of type $1 \to 1$ satisfying the equations $Mf0 =_0 f0$ and $Mf(n+1) =_0 \max_0(Mfn, f(n+1))$.
(4) The exponential function is not provably total and, as a consequence, finite initial segments of type 1 functionals are not (in general) encodable by type 0 objects (natural numbers).

To make the paper reasonably self-contained, we briefly describe the new *bounded functional interpretation* in the next section. We direct the reader to [15] for the full treatment, with proofs, of the interpretation.

2 Background

The *finite types* are defined inductively as follows: 0 is a finite type, and if ρ and σ are finite types then $\rho \to \sigma$ is also a finite type. We write t^ρ to say that term t has type ρ. In this paper we assume familiarity with the finite type arithmetical theories PV^ω, IPV^ω and CPV^ω as defined in [8] (see also [23]). PV^ω is a quantifier-free calculus, whereas IPV^ω and CPV^ω are extensions of PV^ω based on intuitionistic, respectively classical, many-sorted predicate calculus. We denote the language over which these systems are defined by $\mathcal{L}^\omega$. We point that the amount of induction present in IPV^ω and CPV^ω is

$$A(0) \wedge \forall x^0 (A(\lfloor \frac{1}{2} x \rfloor) \to A(x)) \to \forall x^0 A(x),$$

where A is a Σ^b_1-formula, i.e. a formula of the form $\exists y \leq_0 t\, (s^0 = u^0)$ in which y does not occur in the term t^0 and this term has only parameters of type 0: a so-called *zero-open term*, in the terminology of [8] (however, s and u may have parameters of arbitrary type). Note that in IPV^ω, for each quantifier-free formulas $A(\underline{x})$, one can build a term s such that $\mathsf{IPV}^\omega \vdash A(\underline{x}) \leftrightarrow s\underline{x} = 0$. Underlined variables stand for tuple of variables.

In the language of IPV^ω we can define Bezem's strong majorizability relation [2] (a modification of Howard's hereditary majorizability relation [17] that, unlike Howard's, is *provably* transitive) and prove its main properties. We write $\leq^*_\rho$ for Bezem's strong majorizability relation for type ρ. This relation is defined by induction on the types:

$$\begin{aligned} x \leq^*_0 y \quad &:\equiv x \leq_0 y \\ x \leq^*_{\rho \to \sigma} y &:\equiv \forall v^\rho \forall u \leq^*_\rho v (xu \leq^*_\sigma yv \wedge yu \leq^*_\sigma yv). \end{aligned}$$

The following is a consequence of a result in [2]:

Lemma 1 IPV^ω *proves*

(i) $x \leq^* y \to y \leq^* y$.
(ii) $x \leq^* y \wedge y \leq^* z \to x \leq^* z$.

In order to give a bounded functional interpretation of IPV^ω we introduce an extension $\mathcal{L}^\omega_\trianglelefteq$ of the language $\mathcal{L}^\omega$, obtained from the latter by the adjunction of new primitive binary relation symbols $\trianglelefteq_\rho$, one for each type ρ (we use infix notation for these symbols). The relation $\trianglelefteq_\rho$ is the *intensional* counterpart of the extensional relation $\leq^*_\rho$. The terms of $\mathcal{L}^\omega_\trianglelefteq$ are the same as the terms of the original language $\mathcal{L}^\omega$. Formulas of the form $s \trianglelefteq_\rho t$, where s and t are terms of type ρ, are the new atomic formulas of the language. We also add, as a new syntactic device, *bounded quantifiers*, i.e. quantifications of the form

$\forall x \trianglelefteq t A(x)$ and $\exists x \trianglelefteq t A(x)$, for terms t not containing x. *Bounded formulas* are those formulas in which every quantifier is bounded.

Definition 1 *The theory* $\mathsf{IPV}^\omega_\trianglelefteq$ *is an extension of* IPV^ω *with the schemata:*

$$\begin{array}{lcl} \mathsf{B}_\forall & : & \forall x \trianglelefteq t A(x) \leftrightarrow \forall x (x \trianglelefteq t \to A(x)) \\ \mathsf{B}_\exists & : & \exists x \trianglelefteq t A(x) \leftrightarrow \exists x (x \trianglelefteq t \wedge A(x)), \end{array}$$

provided that x does not occur in t. There are also two further axioms

$$\begin{array}{lcl} \mathsf{M}_1 & : & x \trianglelefteq_0 y \leftrightarrow x \leq_0 y \\ \mathsf{M}_2 & : & x \trianglelefteq_{\rho \to \sigma} y \to \forall u \trianglelefteq_\rho v (xu \trianglelefteq_\sigma yv \wedge yu \trianglelefteq_\sigma yv) \end{array}$$

and a rule $\mathsf{RL}_\trianglelefteq$

$$\frac{A_\mathrm{b} \wedge u \trianglelefteq v \to su \trianglelefteq tv \wedge tu \trianglelefteq tv}{A_\mathrm{b} \to s \trianglelefteq t}$$

where s and t are terms of $\mathsf{IPV}^\omega_\trianglelefteq$, *$A_\mathrm{b}$ is a bounded formula and u and v are variables that do not occur free in the conclusion.*

Warning 1 *The induction available in the extended theory* $\mathsf{IPV}^\omega_\trianglelefteq$ *is* exactly the same *as that of the original theory* IPV^ω*, i.e., it does* not *include induction for formulas of the form $\exists y \leq t\, A$, where A is a quantifier-free formula in which the new predicate symbols $\trianglelefteq$ occur.*

We called the new binary relations *intensional* because they are regulated by a *rule*, instead of axioms only. Note that the presence of this rule $\mathsf{RL}_\trianglelefteq$ entails the failure of the deduction theorem for the theory $\mathsf{IPV}^\omega_\trianglelefteq$ (cf. the argument of Proposition 8 in [15]).

Let the relation $\leq_\sigma$ be the usual pointwise "less than or equal to" relation, i.e. $\leq_0$ for type 0, and $x \leq_{\rho\to\sigma} y$ defined recursively by $\forall u^\rho (xu \leq_\sigma yu)$. Let also the relation $\min_1(x^1, y^1)$ be defined as $\lambda n^0. \min_0(xn, yn)$, where $\min_0$ is the usual minimum function between two numbers.

Lemma 2 $\mathsf{IPV}^\omega_\trianglelefteq$ *proves*

(i) $x \trianglelefteq y \to y \trianglelefteq y$.
(ii) $x \trianglelefteq y \wedge y \trianglelefteq z \to x \trianglelefteq z$.
(iii) $x \trianglelefteq_1 y \to x \leq^*_1 y$. *Hence,* $x \trianglelefteq_1 y \to x \leq_1 y$.
(iv) $x \trianglelefteq_1 z \to \min_1(y, x) \trianglelefteq_1 z$.

The following result is an adaptation of a result due to Howard in [17]:

Proposition 1 $\mathsf{IPV}^\omega_\trianglelefteq$ *is a* majorizability theory*, i.e., for every closed term t^ρ*

there is a closed term $\tilde{t}^\rho$ of the same type such that $\mathsf{IPV}^\omega_\trianglelefteq \vdash t \trianglelefteq_\rho \tilde{t}$.

In the sequel, we will often quantify over *monotone functionals*, i.e., functionals f such that $f \trianglelefteq f$. We abbreviate the quantifications $\forall f(f \trianglelefteq f \to A(f))$ and $\exists f(f \trianglelefteq f \wedge A(f))$ by $\tilde{\forall} f A(f)$ and $\tilde{\exists} f A(f)$, respectively.

2.1 Logical Extensions

The principles that have a bounded functional interpretation were characterized in [15]. They are:

1. *Bounded Choice Principle*

$$\mathsf{bAC}^{\rho,\tau}[\trianglelefteq] \quad : \quad \forall x^\rho \exists y^\tau A(x,y) \to \tilde{\exists} f \tilde{\forall} b \forall x \trianglelefteq b \exists y \trianglelefteq fb\, A(x,y),$$

where A is an arbitrary formula of the language $\mathcal{L}^\omega_\trianglelefteq$.

2. *Bounded Independence of Premises Principle*

$$\mathsf{bIP}^\rho_{\forall \mathrm{bd}}[\trianglelefteq] \quad : \quad (\forall \underline{x} A_\mathrm{b}(\underline{x}) \to \exists y^\rho B(y)) \to \tilde{\exists} b (\forall \underline{x} A_\mathrm{b}(\underline{x}) \to \exists y \trianglelefteq b B(y)),$$

where A_b is a bounded formula and B is an arbitrary formula.

3. *Bounded Markov's Principle*

$$\mathsf{bMP}^\rho_{\mathrm{bd}}[\trianglelefteq] \quad : \quad (\forall y^\rho \forall \underline{x} A_\mathrm{b}(\underline{x}, y) \to B_\mathrm{b}) \to \tilde{\exists} b (\forall y \trianglelefteq b \forall \underline{x} A_\mathrm{b}(\underline{x}, y) \to B_\mathrm{b}),$$

where A_b and B_b are bounded formulas. If B_b is $\perp$, we get a useful version of the above principle: $\neg\neg\exists y^\rho A_\mathrm{b}(y) \to \tilde{\exists} b \neg\neg\exists y \trianglelefteq b A_\mathrm{b}(y)$, where A_b is a bounded formula. For y of type 0, we have: $\neg\neg\exists n^0 A_\mathrm{b}(n) \to \exists m^0 \neg\neg\exists n \leq m A_\mathrm{b}(n)$. In the feasible setting, we cannot (in general) replace the consequent by $\exists n A_\mathrm{b}(n)$, even when A_b is quantifier-free. This is due to the fact that bounded first-order formulas are not (in general) decidable. Cf. (2) of §1.

4. *Bounded Contra Collection Principle*

$$\mathsf{bBCC}^{\rho,\underline{\tau}}_{\mathrm{bd}}[\trianglelefteq] \quad : \quad \tilde{\forall} c (\tilde{\forall} \underline{b}^{\underline{\tau}} \exists z \trianglelefteq c^\rho \forall \underline{y} \trianglelefteq \underline{b} A_\mathrm{b}(\underline{y}, z) \to \exists z \trianglelefteq c \forall \underline{y} A_\mathrm{b}(\underline{y}, z)),$$

where A_b is a bounded formula. This principle allows the conclusion of certain existentially bounded statements from the assumption of weakenings thereof (so-called *ϵ-versions* or *ϵ-weakenings*, in a terminology that Kohlenbach introduced in [18] for more concrete situations).

5. *Majorizability Axioms*

$$\mathsf{MAJ}^\rho[\trianglelefteq] \quad : \quad \forall x^\rho \exists y (x \trianglelefteq y).$$

We use $\mathsf{bAC}^\omega[\trianglelefteq]$, $\mathsf{bIP}^\omega_{\forall\mathrm{bd}}[\trianglelefteq]$, $\mathsf{bMP}^\omega_{\mathrm{bd}}[\trianglelefteq]$, $\mathsf{bBCC}^\omega_{\mathrm{bd}}[\trianglelefteq]$ and $\mathsf{MAJ}^\omega[\trianglelefteq]$, respectively, for the aggregate of each of the above principles over all types. We denote by $\mathsf{P}[\trianglelefteq]$ the sum total of all these principles.

Proposition 2 *The theory* $\mathsf{IPV}^\omega_\trianglelefteq + \mathsf{P}[\trianglelefteq]$ *proves:*

1. The Bounded Universal Disjunction Principle

$$\mathsf{bUD}^{\rho,\tau}_{\forall\mathrm{bd}}[\trianglelefteq] \quad : \quad \tilde{\forall} b^\rho \tilde{\forall} c^\tau (\forall x \trianglelefteq b A_\mathrm{b}(x) \vee \forall y \trianglelefteq c B_\mathrm{b}(y)) \to \forall x A_\mathrm{b}(x) \vee \forall y B_\mathrm{b}(y),$$

where A_b *and* B_b *are bounded formulas.*

2. The Bounded Collection Principle

$$\mathsf{bBC}^{\rho,\tau}[\trianglelefteq] \quad : \quad \tilde{\forall} c (\forall z \trianglelefteq c^\rho \exists y^\tau A(y,z) \to \tilde{\exists} b \forall z \trianglelefteq c \exists y \trianglelefteq b A(y,z)),$$

where A *is an arbitrary formula.*

Proof. We show that the principle $\mathsf{bUD}^{\rho,\tau}_{\forall\mathrm{bd}}[\trianglelefteq]$ is a simple consequence of $\mathsf{bBCC}^{0,\rho,\tau}_{\mathrm{bd}}[\trianglelefteq]$. The antecedent of $\mathsf{bUD}^{\rho,\tau}_{\forall\mathrm{bd}}[\trianglelefteq]$ implies

$$\tilde{\forall} b \tilde{\forall} c \exists z \trianglelefteq_0 1 \forall x \trianglelefteq b \forall y \trianglelefteq c ((z = 0 \to A_\mathrm{b}(x)) \wedge (z = 1 \to B_\mathrm{b}(y))).$$

By the Contra Collection Principle it follows that

$$\exists z \trianglelefteq_0 1 \forall x \forall y ((z = 0 \to A_\mathrm{b}(x)) \wedge (z = 1 \to B_\mathrm{b}(y))),$$

and this entails the disjunction we want. Part 2 was shown in Proposition 3 of [15]. □

The Bounded Universal Disjunction Principle entails the following version of the *lesser limited principle of omniscience* LLPO, so-called by Errett Bishop in [3]:

$$\forall n^0, m^0 (\forall k \leq n A_\mathrm{b}(k) \vee \forall k \leq m B_\mathrm{b}(k)) \to \forall n A_\mathrm{b}(n) \vee \forall m B_\mathrm{b}(m),$$

where A_b and B_b are bounded first-order formulas. We cannot obtain the usual version of LLPO, the one in which the antecedent is $\forall n, m (A_\mathrm{b}(n) \vee B_\mathrm{b}(m))$, because bounded formulas are not (in general) decidable. On the other hand, the *limited principle of omniscience* LPO is *refutable* in $\mathsf{IPV}^\omega_\trianglelefteq + \mathsf{P}[\trianglelefteq]$. This is shown in [15] for stronger theories, but the proof also goes through in $\mathsf{IPV}^\omega_\trianglelefteq + \mathsf{P}[\trianglelefteq]$. Actually, the so-called *weak limited principle of omniscience* (cf. [4]) WLPO is already refuted in $\mathsf{IPV}^\omega_\trianglelefteq + \mathsf{P}[\trianglelefteq]$. Concerning the second principle of Proposition 2, observe that the Bounded Collection Principle is related to the FAN principle of Brouwer (the case $\rho = 1$ and $\tau = 0$).

Let $\mathsf{bAC}^\omega_{\mathrm{bd}}[\trianglelefteq]$ be the version of the Bounded Choice Principle in which the matrix A is bounded. The acronym $\mathsf{P}_{\mathrm{bd}}[\trianglelefteq]$ denotes the modification of $\mathsf{P}[\trianglelefteq]$

in which $\mathsf{bAC}^\omega[\trianglelefteq]$ is substituted by $\mathsf{bAC}^\omega_{\mathrm{bd}}[\trianglelefteq]$. Under $\mathsf{P}_{\mathrm{bd}}[\trianglelefteq]$ it is only possible to derive the version of the Bounded Collection Principle in which the matrix is a bounded formula (we denote this version by $\mathsf{bBC}^\omega_{\mathrm{bd}}[\trianglelefteq]$). The schema $\mathsf{P}_{\mathrm{bd}}[\trianglelefteq]$ plays an important role in the *negative translation* of formulas A into formulas A'. For definiteness, we use Kuroda's translation [21] adapted to our setting (cf. [15]), where A' is $\neg\neg A^\dagger$, with $A^\dagger$ obtained from A by maintaining unchanged atomic formulas, conjunctions, disjunctions, implications and existential quantifications (bounded or not) and inserting a double negation *after* each universal quantification (bounded or not). We denote by $\mathsf{CPV}^\omega_\trianglelefteq$ the theory $\mathsf{IPV}^\omega_\trianglelefteq$ together with all instances of the law of excluded middle $A \vee \neg A$.

We call a bounded formula of the form $\exists x_1 \leq_0 t_1 \ldots \exists x_n \leq_0 t_n\, (s^0 = t^0)$ a $\hat{\Sigma}^b_1$-formula. Note that no restriction is made on the parameters appearing in the bounding terms $t_1, \ldots, t_n$, thereby obtaining a proper extension of the class of Σ^b_1-formulas (whose formulas have bounding terms which are zero-open). It should also be observed that no restriction is made by having defined Σ^b_1-formulas with only *one* bounded existential quantifier, as shown by Cook and Urquhart in [8] (caution: Cook and Urquhart's argument relies essentially on the fact that the bounding terms are zero-open). For more information on the issue concerning parameters of bounding terms we direct the reader to Observation 1 at the end of Subsection 3.2. We let $\mathsf{MP}_{\hat{\Sigma}^b_1}$ be the following bounded version of Markov's principle: $\neg\neg A \to A$, where A is a $\hat{\Sigma}^b_1$-formula. For technical reasons which will be apparent later, we need $\mathsf{MP}_{\hat{\Sigma}^b_1}$ as it is formulated, instead of a bounded Markov's principle for mere Σ^b_1-formulas.

Proposition 3 *If* $\mathsf{CPV}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash A$ *then* $\mathsf{IPV}^\omega_\trianglelefteq + \mathsf{MP}_{\hat{\Sigma}^b_1} + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash A'$.

Proof. The part concerning the principles $\mathsf{P}_{\mathrm{bd}}[\trianglelefteq]$ were discussed in [15]. Concerning the axioms of $\mathsf{CPV}^\omega_\trianglelefteq$, they are all universal (posing no problems regarding their negative translations because quantifier-free formulas are decidable) except for the induction axioms. It is clear that the negative translations of the induction axioms follow from $\mathsf{IPV}^\omega_\trianglelefteq + \mathsf{MP}_{\hat{\Sigma}^b_1}$. $\Box$

Definition 2 *The* 0-bounded *formulas of the language $\mathcal{L}^\omega$ form the smallest class of formulas which includes the quantifier-free formulas and is closed under propositional connectives and quantifications of the form $\forall x \leq_0 t(...)$ and $\exists x \leq_0 t(...)$, where t is a term of type 0 in which the variable x does not occur.*

Note that the 0-bounded formulas with parameters of type 0 correspond to the bounded formulas of first-order bounded arithmetic (cf. [5]).

Proposition 4 *Let*

$$\mathsf{bAC}^{0,1}_0 \;:\; \forall x^0 \exists y^1 A_0(x,y) \to \exists \Phi^{0\to 1} \forall x^0 \exists y \leq_1 \Phi x\, A_0(x,y),$$

with $A_0(x,y)$ a 0-bounded formula of the original language $\mathcal{L}^\omega$, possibly with parameters. $\mathsf{IPV}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq]$ *proves* $\mathsf{bAC}_0^{0,1}$.

Proof. Assume $\forall x^0 \exists y^1 A_0(x,y)$. By $\mathsf{bAC}^\omega_{\mathrm{bd}}[\trianglelefteq]$, $\tilde{\exists}\Phi^{0\to 1}\tilde{\forall} a^0 \forall x \trianglelefteq a \exists y \trianglelefteq_1 \Phi a \, A_0(x,y)$. Given x^0, put a^0 as x^0 and use part (iii) of lemma 2 to replace the intensional sign $\trianglelefteq_1$ by $\leq_1$. $\square$

Notice that $\mathsf{bAC}_0^{0,1}$ entails $\mathsf{bAC}_0^{0,0}$, where y has type 0. The usual choice principle,

$$\mathsf{AC}_0^{0,0} \quad : \quad \forall x^0 \exists y^0 A_0(x,y) \to \exists \Phi^{0\to 0} \forall x^0 \, A_0(x, \Phi x),$$

does not follow from $\mathsf{bAC}_0^{0,0}$ because there is no minimization functional in the feasible setting. Cf. (1) of §1.

In the stronger theories studied in [15] the above choice principle also holds for x of type 1. The fact that there is no maximization functional M in the feasible setting (cf. (3) of §1) prevents the extension of the above proof to that type.

2.2 The Bounded Functional Interpretation

We now describe the bounded functional interpretation, state its main (soundness) theorem and, finally, present a result that relates the extended language to the original language without the intensional relations $\trianglelefteq$.

Definition 3 *To each formula A of the language $\mathcal{L}^\omega_\trianglelefteq$ we associate formulas $(A)^{\mathrm{B}}$ and A_{B} of the same language so that $(A)^{\mathrm{B}}$ is of the form $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c} A_{\mathrm{B}}(\underline{b},\underline{c})$, with $A_{\mathrm{B}}(\underline{b},\underline{c})$ a bounded formula.*

1. *$(A_{\mathrm{b}})^{\mathrm{B}}$ and $(A_{\mathrm{b}})_{\mathrm{B}}$ are simply A_{b}, for bounded formulas A_{b}.*

If we have already interpretations for A and B given by $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c} A_{\mathrm{B}}(\underline{b},\underline{c})$ and $\tilde{\exists}\underline{d}\tilde{\forall}\underline{e} B_{\mathrm{B}}(\underline{d},\underline{e})$ (respectively), then we define

2. *$(A \wedge B)^{\mathrm{B}}$ is $\tilde{\exists}\underline{b},\underline{d}\tilde{\forall}\underline{c},\underline{e}(A_{\mathrm{B}}(\underline{b},\underline{c}) \wedge B_{\mathrm{B}}(\underline{d},\underline{e}))$,*
3. *$(A \vee B)^{\mathrm{B}}$ is $\tilde{\exists}\underline{b},\underline{d}\tilde{\forall}\underline{c},\underline{e}(\tilde{\forall}\underline{c}' \trianglelefteq \underline{c} A_{\mathrm{B}}(\underline{b},\underline{c}') \vee \tilde{\forall}\underline{e}' \trianglelefteq \underline{e} B_{\mathrm{B}}(\underline{d},\underline{e}'))$,*
4. *$(A \to B)^{\mathrm{B}}$ is $\tilde{\exists}\underline{f},\underline{g}\tilde{\forall}\underline{b},\underline{e}(\tilde{\forall}\underline{c} \trianglelefteq \underline{g}\underline{b}\underline{e} A_{\mathrm{B}}(\underline{b},\underline{c}) \to B_{\mathrm{B}}(\underline{f}\underline{b},\underline{e}))$.*

For bounded quantifiers we have:

5. *$(\forall x \trianglelefteq t A(x))^{\mathrm{B}}$ is $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}\forall x \trianglelefteq t \, A_{\mathrm{B}}(\underline{b},\underline{c},x)$,*
6. *$(\exists x \trianglelefteq t A(x))^{\mathrm{B}}$ is $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}\exists x \trianglelefteq t \tilde{\forall}\underline{c}' \trianglelefteq \underline{c} \, A_{\mathrm{B}}(\underline{b},\underline{c}',x)$.*

And for unbounded quantifiers we define

7. $(\forall x A(x))^{\mathrm{B}}$ *is* $\tilde{\exists}\underline{f}\tilde{\forall}a,\underline{c}\forall x \trianglelefteq a A_{\mathrm{B}}(\underline{f}a,\underline{c},x)$,
8. $(\exists x A(x))^{\mathrm{B}}$ *is* $\tilde{\exists}a,\underline{b}\tilde{\forall}\underline{c}\exists x \trianglelefteq a\tilde{\forall}\underline{c}' \trianglelefteq \underline{c} A_{\mathrm{B}}(\underline{b},\underline{c}',x)$.

In the above, it is understood that $(\exists x A)_{\mathrm{B}}$ is $\exists x \trianglelefteq a\tilde{\forall}\underline{c}' \trianglelefteq \underline{c} A_{\mathrm{B}}(\underline{b},\underline{c}',x)$. Similarly for the other clauses.

Theorem 1 (Soundness) *Assume that* $(A(\underline{z}))^{\mathrm{B}}$ *is* $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}A_{\mathrm{B}}(\underline{b},\underline{c},\underline{z})$, *where* $A(\underline{z})$ *is an arbitrary formula of* $\mathcal{L}^{\omega}_{\trianglelefteq}$ *with its free variables as displayed. If*

$$\mathsf{IPV}^{\omega}_{\trianglelefteq} + \mathsf{P}[\trianglelefteq] \vdash A(\underline{z}),$$

then there are closed monotone terms $\underline{t}$ *of appropriate types such that*

$$\mathsf{IPV}^{\omega}_{\trianglelefteq} \vdash \tilde{\forall}\underline{a}\forall\underline{z} \trianglelefteq \underline{a}\,\tilde{\forall}\underline{c}A_{\mathrm{B}}(\underline{t}\underline{a},\underline{c},\underline{z}).$$

Moreover, in the above, we can simultaneously replace $\mathsf{IPV}^{\omega}_{\trianglelefteq}$ *by* $\mathsf{IPV}^{\omega}_{\trianglelefteq} + \mathsf{MP}_{\hat{\Sigma}^b_1}$.

Proof. A Soundness Theorem like the above is the main result of [15]. There are only two differences worth discussing between the theories addressed in [15] and the theories $\mathsf{IPV}^{\omega}_{\trianglelefteq}$ and $\mathsf{IPV}^{\omega}_{\trianglelefteq} + \mathsf{MP}_{\hat{\Sigma}^b_1}$. One is the principle $\mathsf{MP}_{\hat{\Sigma}^b_1}$. Well, this principle is a universal closure of a bounded formula and, therefore, its interpretation follows from itself. The other is the induction scheme. The $^{\mathrm{B}}$-translation of an instance of the induction scheme is equivalent to

$$\tilde{\exists}g\forall b^0\Big(\forall x' \leq_0 gb(A(0) \wedge (A(\lfloor \tfrac{1}{2}x' \rfloor) \rightarrow A(x'))) \rightarrow \forall x \leq bA(x)\Big)$$

where A is a given Σ^b_1-formula. Taking $g := \lambda x.x$, it easy to derive

$$\forall b^0\Big(\forall x' \leq_0 b(A(0) \wedge (A(\lfloor \tfrac{1}{2}x' \rfloor) \rightarrow A(x'))) \rightarrow \forall x \leq bA(x)\Big)$$

in $\mathsf{IPV}^{\omega}_{\trianglelefteq}$. □

It would be more in the spirit of the bounded functional interpretation to state that the theory $\mathsf{IPV}^{\omega}_{\trianglelefteq}$ (or $\mathsf{IPV}^{\omega}_{\trianglelefteq} + \mathsf{MP}_{\hat{\Sigma}^b_1}$) proves $\tilde{\forall}\underline{a}\forall\underline{z} \trianglelefteq \underline{a}\exists\underline{b} \trianglelefteq \underline{t}\underline{a}\tilde{\forall}\underline{c}A_{\mathrm{B}}(\underline{b},\underline{c},\underline{z})$. This is in fact equivalent to what is stated in the Soundness Theorem because formulas of the form $A_{\mathrm{B}}(\underline{b},\underline{c})$ are monotone with respect to the variable(s) $\underline{b}$, i.e., whenever $\underline{b} \trianglelefteq \underline{b}'$ then $A_{\mathrm{B}}(\underline{b},\underline{c}) \rightarrow A_{\mathrm{B}}(\underline{b}',\underline{c})$.

Definition 4 *For any given formula* A *in the language* $\mathcal{L}^{\omega}_{\trianglelefteq}$, *we define the formula* A^* *of the language of* $\mathcal{L}^{\omega}$ *by induction on* A:

(a) *If* A *is an atomic formula in which* $\trianglelefteq$ *does not occur,* A^* *is* A.
(b) *For any given type* σ, $(t \trianglelefteq_{\sigma} q)^*$ *is* $t \leq^*_{\sigma} q$.
(c) $(A \Box B)^*$ *is* $A^* \Box B^*$, *for* $\Box \in \{\wedge, \vee, \rightarrow\}$.
(d) $(QxA)^*$ *is* QxA^*, *for* $Q \in \{\forall, \exists\}$.
(e) *For any given type* σ, $(\forall x \trianglelefteq_{\sigma} tA)^*$ *is* $\forall x(x \leq^*_{\sigma} t \rightarrow A^*)$ *and* $(\exists x \trianglelefteq_{\sigma} tA)^*$ *is*

$\exists x(x \leq^*_\sigma t \wedge A^*)$.

The following is clear:

Proposition 5 (Flattening) *Let $A(\underline{z})$ be an arbitrary formula of $\mathcal{L}^\omega_\trianglelefteq$, with its free variables as displayed. We have:*

$$\mathsf{CPV}^\omega_\trianglelefteq \vdash A(\underline{z}) \Rightarrow \mathsf{CPV}^\omega \vdash A^*(\underline{z}).$$

3 Applications to Feasible Analysis

Weak König's Lemma, WKL for short, is the well-known principle saying that every infinite tree of finite sequences of 0's and 1's has an infinite path. We say that $A(s)$ defines an infinite binary tree, and write $\mathsf{Tree}_\infty(A_s)$, if

$$\forall s^0 \forall q^0 (A(s) \wedge q \preceq s \to A(q)) \wedge \forall k^0 \exists s^0 (|s| = |k| \wedge A(s)),$$

where $q \preceq s$ means that the binary expansion of q is an initial sequence of the binary expansion of s, and $|n|$ indicates the length of the binary expansion of the natural number n. Note that the quantification $\exists s^0(|s| = |k| \wedge ...)$ is bounded. If f is of type 1, $\mathsf{Tree}_\infty(f)$ stands for $\mathsf{Tree}_\infty([f(s) =_0 0]_s)$.

We formalize weak König's Lemma as follows (we write $x \leq_1 1$ instead of $x \leq_1 \lambda n^0.1^0$)

$$\forall f^1 \Big(\mathsf{Tree}_\infty(f) \to \exists x \leq_1 1 \forall k^0 \left(\overline{x}(k) \in f \right) \Big),$$

$\mathsf{Tree}_\infty(f)$ as above, and $s \in f$ abbreviates $f(s) =_0 0$. Finally, given k^0 and $x \leq_1 1$, $\overline{x}(k)$ is the (code of the) binary sequence $\langle x(0), x(1), x(11), \ldots, x(1^{|k|-1}) \rangle$. Here $11 \cdots 1$ stands for the natural number whose binary expansion is $11 \cdots 1$.

A strengthening of WKL in the feasible setting consists in admitting binary trees defined by a 0-bounded formula, instead of mere *set* trees. Let $A(s)$ be a 0-bounded formula with a distinguished variable s of type 0 (parameters are allowed). The *schema* $\Sigma^{\mathsf{b}}_\infty$-$\mathsf{WKL}$ is the following collection of formulas, one for each 0-bounded formula $A(s)$:

$$\Sigma^{\mathsf{b}}_\infty\text{-}\mathsf{WKL} \quad : \quad \mathsf{Tree}_\infty(A_s) \to \exists x \leq_1 1 \forall k^0 A(\overline{x}(k)).$$

Proposition 6 *The theory* $\mathsf{IPV}^\omega_\trianglelefteq + \mathsf{P}[\trianglelefteq]$ *proves* $\Sigma^{\mathsf{b}}_\infty$-$\mathsf{WKL}$.

Proof. Let $A(s)$ be given such that $\mathsf{Tree}_\infty(A_s)$. Given k^0, take s such that $|s| = |k|$ and $A(s)$. Let $\hat{s}$ be the type 1 function defined so that $\hat{s}(n)$ is the $|n|$-th bit of the binary expansion of s for n less than or equal to k, and is 0 otherwise. Using rule $\mathsf{RL}_{\trianglelefteq}$, it is easy to show that $\hat{s} \trianglelefteq_1 1$. We have just argued that $\forall k \exists x \trianglelefteq_1 1 \forall n \leq k\, A(\overline{x}(n))$. By $\mathsf{bBCC}^{\omega}_{\mathrm{bd}}[\trianglelefteq]$, we get $\exists x \trianglelefteq_1 1 \forall k A(\overline{x}(k))$. The result now follows from (iii) of Lemma 2. $\square$

In the subsequent subsections we will, in fact, state a rather general principle from which $\Sigma^{\mathsf{b}}_{\infty}$-$\mathsf{WKL}$ follows, and prove a pertinent conservation result. We will also relate our discussion with theories of feasible arithmetic and analysis.

3.1 A Uniform Boundedness Principle

In [7], Andrea Cantini studied the principle of *strict-*Π^1_1 *reflection*, abbreviated by $\mathsf{s}\Pi^1_1$-reflection, in the context of second-order feasible theories. For our purposes, this principle is:

$$\forall x \leq_1 1\, \exists y^0\, A_0(x, y) \to \exists z^0 \forall x \leq_1 1\, \exists y \leq_0 z\, A_0(x, y),$$

where $A_0(x, y)$ is a 0-bounded formula. Strict-Π^1_1 reflection is, in fact, a form of the FAN theorem within the second-order setting. Cantini shows that the above principle (classically) entails $\Sigma^{\mathsf{b}}_{\infty}$-$\mathsf{WKL}$ (say, over CPV^{ω}). He also proves a conservation result concerning this principle and, *a fortiori*, concerning $\Sigma^{\mathsf{b}}_{\infty}$-$\mathsf{WKL}$. We generalize this conservation result in the sequel.

The following principle was introduced by Kohlenbach in [20]. We present a slight variant thereof:

Definition 5 *The* Uniform Σ^0_1-Boundedness Principle*, abbreviated by* Σ^0_1-UB*, is the following scheme,*

$$\forall h^{0\to 1}\Big(\forall k^0 \forall f \leq_1 hk \exists e^1 A_0(f, h, k, e) \to \exists g^{0\to 1} \forall k^0 \forall f \leq_1 hk \exists e \leq_1 gk A_0(f, h, k, e)\Big),$$

where A_0 *is a 0-bounded formula (which may contain parameters of arbitrary type).*

Note that $\mathsf{s}\Pi^1_1$-reflection follows from Σ^0_1-UB over IPV^{ω}. The same is true for the *bounded collection scheme*:

$$\forall r^0\Big(\forall n \leq_0 r \exists y^0 A_0(n, y) \to \exists z^0 \forall n \leq_0 r \exists y \leq_0 z A_0(n, y)\Big).$$

(As a consequence of Theorem 2 below, we shall see that the bounded collection scheme has no effect on the Π^0_2-consequences of our starting theory, a result originally due to Buss in [6].) Since we are allowing parameters of arbitrary type in A_0, Σ^0_1-UB is *false* in the set-theoretic model of the functionals of finite type. E.g., the following patently false principle, a version of Kohlenbach's so-called *principle* F introduced in [20], is a consequence of Σ^0_1-UB:

$$\forall \Phi^2 \exists n^0 \forall f \leq_1 1(\Phi(f) \leq_0 n).$$

When only parameters of type 0 or 1 are permitted, Σ_1^0-UB is nonetheless true in the set-theoretic model. The reason for this is well-known. Fix h. Since A_0 is 0-bounded and has parameters restricted to type 0 and 1, only an initial segment of the type 1 functional e^1 is needed to fulfill A_0. This ensures the *continuity* of the functional that associates to each $f^1 \in \{f : f \leq_1 hk\}$ the (say) lexicographic least e^1 such that $A_0(f, h, k, e)$. Due to the compactness of the previous set, we can bound the e's. The uniformity in terms of k^0 is a consequence of the axiom of choice. An upshot of this argument is that the principle reduces to the *original* Uniform Σ_1^0-Boundedness Principle of Kohlenbach

$$\forall h^{0\to 1}\Big(\forall k^0 \forall f \leq_1 hk \exists s^0 A_0(f, h, k, s) \to \exists g^1 \forall k^0 \forall f \leq_1 hk \exists s \leq_0 gk A_0(f, h, k, s)\Big)$$

in the set-theoretic model. As a matter of fact, Σ_1^0-UB is a consequence of the above principle already in the (classical) theory E-$\mathsf{G}_3\mathsf{A}^\omega$ of [19]. This theory is related with the third level of Grzegorczyk's hierarchy of primitive recursive functions, and the reduction follows from results in the first part of section 9 of [15]. In the feasible setting, however, the above principle is seemingly weaker than Σ_1^0-UB because initial segments of type 1 functionals cannot (in general) be coded by numbers. Cf. (4) of §1.

We prefix by the letter E the name of a theory to mean that we add full extensionality to it. Given s and t terms of type $\rho \equiv \rho_1 \to (\ldots \to (\rho_k \to 0)\ldots)$ we say that $s =_\rho t$ if $\forall y_1^{\rho_1} \ldots \forall y_k^{\rho_k} (s y_1 \ldots y_k =_0 t y_1 \ldots y_k)$. *Full extensionality* is the collection of axioms of the form $\forall z^{\rho\to\tau} \forall x^\rho, y^\rho (x =_\rho y \to zx =_\tau zy)$.

Here is the promised conservation result, a generalization of the uniform boundedness principle of Kohlenbach [20] to the feasible setting:

Theorem 2 *Let $\sigma \in \{0, 1\}$ and ρ be any type. Suppose that*

$$\mathsf{E\text{-}CPV}^\omega + \mathsf{bAC}_0^{0,1} + \Sigma_1^0\text{-}\mathsf{UB} \vdash \forall x^\sigma \exists y^\rho A_0(x, y),$$

where A_0 is a 0-bounded formula (its free variables as displayed) and ρ is an arbitrary type. Then, there is a closed monotone term $q^{\sigma\to\rho}$ of $\mathcal{L}^\omega$ such that

$$\mathsf{CPV}^\omega \vdash \forall a^\sigma \forall x \leq_\sigma^* a \exists y \leq_\rho^* qa\, A_0(x, y).$$

When x is of type 0, then the conclusion can be simplified to

$$\mathsf{CPV}^\omega \vdash \forall x^0 \exists y \leq_\rho^* qx\, A_0(x, y).$$

Proof. Suppose $A \equiv \forall x^\sigma \exists y^\rho A_0(x, y)$ is a theorem of the theory E-CPV^ω + $\mathsf{bAC}_0^{0,1} + \Sigma_1^0$-$\mathsf{UB}$. In the presence of full extensionality, each instance of Σ_1^0-UB is easily seen to be equivalent to

$$(*)\begin{cases} \forall h^{0\to1}\Big(\forall k^0\forall f \leq_1 hk\exists e^1\, A_0(f,h,k,e) \to \\ \qquad \exists g^{0\to1}\forall k^0\forall f^1\exists e \leq_1 gk\, A_0(\min(f,hk),h,k,e)\Big). \end{cases}$$

Therefore, $\mathsf{E\text{-}CPV}^\omega + \mathsf{bAC}_0^{0,1} + (*) \vdash A$. Since the quantifiers in each $(*)$ are (essentially) of type 1 or less, it follows by elimination of extensionality (cf. H. Luckhardt in [22]) that $\mathsf{CPV}^\omega + \mathsf{bAC}_0^{0,1} + (*) \vdash A$. Hence, by Proposition 4, $\mathsf{CPV}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] + (*) \vdash A$.

We claim that $\mathsf{CPV}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq]$ proves each $(*)$. Fix $h^{0\to1}$. Assume that

$$\forall k^0\forall f^1(\forall n^0(fn \leq_0 hkn) \to \exists e^1 A_0(f,h,k,e)).$$

By $\mathsf{bIP}^\omega_{\forall\mathrm{bd}}[\trianglelefteq]$ and $\mathsf{bMP}^\omega_{\mathrm{bd}}[\trianglelefteq]$ we get

$$\forall k^0\forall f^1\tilde{\exists} m^0\tilde{\exists} e^1(\forall n \leq_0 m(fn \leq_0 hkn) \to \exists e' \trianglelefteq_1 eA_0(f,h,k,e')).$$

And in particular

$$\forall k^0\forall f^1 \trianglelefteq_1 \tilde{h}k\tilde{\exists} m^0\tilde{\exists} e^1(\forall n \leq_0 m(fn \leq_0 hkn) \to \exists e' \trianglelefteq_1 eA_0(f,h,k,e')),$$

where $h \trianglelefteq_1 \tilde{h}$ (we are using $\mathsf{MAJ}^\omega[\trianglelefteq]$ here). By $\mathsf{bBC}^\omega_{\mathrm{bd}}[\trianglelefteq]$, the monotonicity on m and e and the transitivity of $\trianglelefteq_1$ it follows that

$$\forall k^0\tilde{\exists} m^0\tilde{\exists} e^1\forall f^1 \trianglelefteq_1 \tilde{h}k(\forall n \leq_0 m(fn \leq_0 hkn) \to \exists e' \trianglelefteq_1 eA_0(f,h,k,e')).$$

Using $\mathsf{bAC}^\omega_{\mathrm{bd}}[\trianglelefteq]$ and the transitivity of $\trianglelefteq_1$ we may conclude that there are monotone $g^{0\to1}$ and $l^{0\to0}$ such that

$$\forall k^0\forall f \trianglelefteq_1 \tilde{h}k(\forall n \leq_0 lk(fn \leq_0 hkn) \to \exists e \trianglelefteq_1 gkA_0(f,h,k,e)),$$

which implies

$$\forall k^0\forall f \trianglelefteq_1 \tilde{h}k((f \leq_1 hk) \to \exists e \trianglelefteq_1 gkA_0(f,h,k,e)).$$

We are now ready to check $\forall k^0\forall f^1\exists e \leq_1 gkA_0(\min_1(f,hk),h,k,e)$. Take k^0 and f^1. By (iv) of Lemma 2, $\min_1(f,hk) \trianglelefteq_1 \tilde{h}k$. Also $\min_1(f,hk) \leq_1 hk$. Hence, $\exists e \trianglelefteq_1 gkA_0(\min_1(f,hk),h,k,e)$. The claim follows because of (iii) of Lemma 2.

We showed that $\mathsf{CPV}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash A$. By Proposition 3, $\mathsf{IPV}^\omega_\trianglelefteq + \mathsf{MP}_{\hat{\Sigma}^b_1} + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash A'$. Using $\mathsf{bMP}^\omega_{\mathrm{bd}}[\trianglelefteq]$, $\mathsf{IPV}^\omega_\trianglelefteq + \mathsf{MP}_{\hat{\Sigma}^b_1} + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash \forall x^\sigma\tilde{\exists} b^\rho\neg\neg\exists y \trianglelefteq_\rho b(A_0(x,y))^\dagger$. Now, by the Soundness Theorem, there is a closed monotone term $q^{\sigma\to\rho}$ such that

$$\mathsf{IPV}^\omega_\trianglelefteq + \mathsf{MP}_{\hat{\Sigma}^b_1} \vdash \tilde{\forall} a^\sigma\forall x \trianglelefteq_\sigma a\neg\neg\exists y \trianglelefteq_\rho qa(A_0(x,y))^\dagger.$$

The theorem now follows from Proposition 5. $\square$

The bounded functional interpretation is efficient in dealing with principles like $\Sigma^{\mathsf{b}}_{\infty}$-$\mathsf{WKL}$ because it only cares for bounds. On the other hand, it is too coarse to get precise witnesses. It analyzes just up to a point. Beyond that point, in order to obtain precise (polynomial time) witnesses, one must turn to the more fine-grained interpretation of Cook and Urquhart [8]. However, this latter interpretation is unable to deal with bounded collection or $\Sigma^{\mathsf{b}}_{\infty}$-$\mathsf{WKL}$. A strategy emerges, though. Firstly, remove principles like $\Sigma^{\mathsf{b}}_{\infty}$-$\mathsf{WKL}$ using the bounded functional interpretation. Afterwards, obtain the polynomial time computable witnesses using Cook and Urquhart's interpretation. In this section, we put this strategy of division of labor into action.

The principle $\mathsf{AC}^{0,0}_{\mathsf{qf}}$ of the language $\mathcal{L}^{\omega}$ is $\forall x^0 \exists y^0 A_{\mathrm{qf}}(x,y) \to \exists f^1 \forall x^0\, A_{\mathrm{qf}}(x, fx)$, where A_{qf} is *quantifier-free*, possibly with parameters (compare with the choice principle $\mathsf{AC}^{0,0}_{0}$ of Subsection 2.1). We consider the following (intuitionistic) consequence of $\mathsf{AC}^{0,0}_{\mathsf{qf}}$,

$$\mathsf{AC}^{0,\mathsf{b}0}_{\hat{\Sigma}^b_1} \quad : \quad \forall x^0 \exists y \leq_0 1 A_{\Sigma}(x,y) \to \exists f \leq_1 1 \forall x^0 A_{\Sigma}(x, fx),$$

where A_{Σ} is a $\hat{\Sigma}^b_1$-formula. Next, we show that $\mathsf{AC}^{0,\mathsf{b}0}_{\hat{\Sigma}^b_1}$ is sufficient for proving (in a classical context) the following weak form of comprehension

$$\nabla^{\mathsf{b}}_1\text{-}\mathsf{CA} \quad : \quad \begin{cases} \forall h^1 \forall g^1 \Big(\forall k^0 (\exists v \leq_0 hk A_{\mathrm{qf}}(k,v) \leftrightarrow \forall w \leq_0 gk B_{\mathrm{qf}}(k,w)) \to \\ \qquad \exists f^1 \forall k^0 (fk =_0 0 \leftrightarrow \exists v \leq_0 hk A_{\mathrm{qf}}(k,v)) \Big), \end{cases}$$

where $A_{\mathrm{qf}}(k,v)$ and $B_{\mathrm{qf}}(k,w)$ are quantifier-free formulas (possibly with parameters). The parameter-free version of this schema, for h and g polytime computable functions, says that the sets in $\mathsf{NP} \cap \mathsf{co}\text{-}\mathsf{NP}$ exist.

Lemma 3 *$\mathsf{CPV}^{\omega} + \mathsf{AC}^{0,\mathsf{b}0}_{\hat{\Sigma}^b_1} \vdash \nabla^{\mathsf{b}}_1$-$\mathsf{CA}$.*

Proof. Let h^1 and g^1 be given functionals and assume that

$$\forall k^0 (\exists v \leq_0 hk A_{\mathrm{qf}}(k,v) \leftrightarrow \forall w \leq_0 gk B_{\mathrm{qf}}(k,w)).$$

By classical logic we have

$$\forall k^0 \exists n \leq_0 1 \Big((n = 0 \to \exists v \leq_0 hk A_{\mathrm{qf}}(k,v)) \wedge (n \neq 0 \to \neg \exists v \leq_0 hk A_{\mathrm{qf}}(k,v)) \Big),$$

which, by our assumption, is equivalent to

$$\forall k^0 \exists n \leq_0 1 \Big((n = 0 \to \exists v \leq_0 hk A_{\mathrm{qf}}(k,v)) \wedge (n \neq 0 \to \exists w \leq_0 gk \neg B_{\mathrm{qf}}(k,w)) \Big).$$

By $\mathsf{AC}^{0,\mathsf{b}0}_{\hat{\Sigma}^b_1}$ we get

$$\exists f \leq_1 1 \forall k^0 \Big((fk = 0 \to \exists v \leq_0 hk A_{\mathrm{qf}}(k,v)) \wedge (fk \neq 0 \to \exists w \leq_0 gk \neg B_{\mathrm{qf}}(k,w)) \Big).$$

By assumption again, this is equivalent to

$$\exists f \leq_1 1 \forall k^0 (fk = 0 \leftrightarrow \exists v \leq_0 hk A_{\mathrm{qf}}(k,v)).$$

$\square$

In the presence of the choice principle $\mathsf{bAC}^{0,0}_0$, $\mathsf{AC}^{0,\mathsf{b}0}_{\hat{\Sigma}^b_1}$ even implies the stronger form of "recursive" comprehension Δ^{PT}_1-CA

$$\forall k^0 (\exists v^0 A_{\mathrm{qf}}(k,v) \leftrightarrow \forall w^0 B_{\mathrm{qf}}(k,w)) \to \exists f^1 \forall k^0 (fk =_0 0 \leftrightarrow \exists v^0 A_{\mathrm{qf}}(k,v)),$$

where $A_{\mathrm{qf}}(k,v)$ and $B_{\mathrm{qf}}(k,w)$ are quantifier-free formulas (possibly with parameters).

Lemma 4 $\mathsf{CPV}^\omega + \mathsf{bAC}^{0,0}_0 + \mathsf{AC}^{0,\mathsf{b}0}_{\hat{\Sigma}^b_1} \vdash \Delta^{\mathsf{PT}}_1$-$\mathsf{CA}$.

Proof. By Lemma 3, it is sufficient to prove $\mathsf{CPV}^\omega + \mathsf{bAC}^{0,0}_0 + \nabla^{\mathsf{b}}_1$-$\mathsf{CA} \vdash \Delta^{\mathsf{PT}}_1$-$\mathsf{CA}$. Suppose that $\forall k^0 (\exists v^0 A_{\mathrm{qf}}(k,v) \leftrightarrow \forall w^0 B_{\mathrm{qf}}(k,w))$. In particular one has $\forall k^0 \exists w^0, v^0 (B_{\mathrm{qf}}(k,w) \to A_{\mathrm{qf}}(k,v))$. By $\mathsf{bAC}^{0,0}_0$, there is a function h^1 such that $\forall k^0 \exists w, v \leq_0 hk (B_{\mathrm{qf}}(k,w) \to A_{\mathrm{qf}}(k,v))$. It is now easy to conclude that,

$$\forall k^0 (\exists v \leq_0 hk A_{\mathrm{qf}}(k,v) \leftrightarrow \forall w \leq_0 hk B_{\mathrm{qf}}(k,w)).$$

By ∇^{b}_1-CA, there is a functional f^1 so that $\forall k^0 (fk =_0 0 \leftrightarrow \exists v \leq_0 hk A_{\mathrm{qf}}(k,v))$. It is easy to argue that $\forall k^0 (\exists v \leq_0 hk A_{\mathrm{qf}}(k,v) \leftrightarrow \exists v^0 A_{\mathrm{qf}}(k,v))$. We get the result. $\square$

In the following, we denote by $(\star)$ the strengthening of the scheme $\mathsf{AC}^{0,\mathsf{b}0}_{\hat{\Sigma}^b_1}$ whereby one changes the extensional bound of f by the intensional one, i.e.

$$(\star) \quad : \quad \forall x^0 \exists y \leq_0 1 A_\Sigma(x,y) \to \exists f \trianglelefteq_1 1 \forall x^0 A_\Sigma(x, fx).$$

Lemma 5 (Soundness) *Assume that* $(A(\underline{z}))^{\mathrm{B}}$ *is* $\tilde{\exists} \underline{b} \tilde{\forall} \underline{c} A_{\mathrm{B}}(\underline{b}, \underline{c}, \underline{z})$, *where* $A(\underline{z})$ *is an arbitrary formula of* $\mathcal{L}^\omega_{\trianglelefteq}$ *with its free variables as displayed. If*

$$\mathsf{IPV}^\omega_{\trianglelefteq} + \mathsf{MP}_{\hat{\Sigma}^b_1} + \mathsf{P}[\trianglelefteq] + (\star) \vdash A(\underline{z}),$$

then there are closed monotone terms $\underline{t}$ *of appropriate types such that*

$$\mathsf{IPV}^\omega_{\trianglelefteq} + \mathsf{MP}_{\hat{\Sigma}^b_1} + (\star) \vdash \tilde{\forall} \underline{a} \forall \underline{z} \trianglelefteq \underline{a} \tilde{\forall} \underline{c} A_{\mathrm{B}}(\underline{t}\underline{a}, \underline{c}, \underline{z}).$$

Proof. The lemma is proved like the Soundness Theorem. Ignoring parameters, it is sufficient to show that, for each instance P of $(\star)$,

$$\forall x^0 \exists y \trianglelefteq_0 1 A_\Sigma(x,y) \to \exists f \trianglelefteq_1 1 \forall x^0 A_\Sigma(x, fx),$$

there is a closed monotone term s^1 such that $\mathsf{IPV}^\omega_\trianglelefteq + \mathsf{MP}_{\hat{\Sigma}^b_1} + (\star) \vdash \forall k^0 P_\mathrm{B}(s,k)$, where $(P)^\mathrm{B}$ is $\tilde{\exists} u^1 \forall k^0 P_\mathrm{B}(u,k)$. We have that $(P)^\mathrm{B}$ is equivalent to

$$\tilde{\exists} u^1 \forall k^0 \Big(\forall x \trianglelefteq_0 uk \exists y \trianglelefteq_0 1 A_\Sigma(x,y) \to \exists f \trianglelefteq_1 1 \forall x \trianglelefteq_0 k A_\Sigma(x, fx) \Big),$$

where condition 6 of Definition 3 is used crucially. In order to witness $(P)^\mathrm{B}$ we must produce a monotone closed term s^1 such that,

$$\forall k^0 \Big(\forall x \trianglelefteq_0 sk \exists y \trianglelefteq_0 1 A_\Sigma(x,y) \to \exists f \trianglelefteq_1 1 \forall x \trianglelefteq_0 k A_\Sigma(x, fx) \Big),$$

Let $s^1 := \lambda k^0.k$. Fix k^0 and assume $\forall x \trianglelefteq_0 k \exists y \trianglelefteq_0 1 A_\Sigma(x,y)$. Hence

$$\forall x^0 \exists y \trianglelefteq_0 1 (x \trianglelefteq_0 k \to A_\Sigma(x,y)),$$

which by $(\star)$ gives $\exists f \trianglelefteq_1 1 \forall x^0 (x \trianglelefteq_0 k \to A_\Sigma(x, fx))$. This implies the desired conclusion. $\Box$

We can now prove the following variant of Theorem 2:

Theorem 3 *Let $\sigma \in \{0,1\}$ and ρ be any type. Suppose that*

$$\mathsf{E\text{-}CPV}^\omega + \mathsf{bAC}^{0,1}_0 + \mathsf{AC}^{0,\mathsf{b}0}_{\hat{\Sigma}^b_1} + \Sigma^0_1\text{-}\mathsf{UB} \vdash \forall x^\sigma \exists y^\rho A_0(x,y),$$

where A_0 is a 0-bounded formula (its free variables as displayed). Then, there is a closed monotone term $q^{\sigma \to \rho}$ of $\mathcal{L}^\omega$ such that

$$\mathsf{CPV}^\omega + \mathsf{AC}^{0,\mathsf{b}0}_{\hat{\Sigma}^b_1} \vdash \forall a^\sigma \forall x \leq^*_\sigma a \exists y \leq^*_\rho qa\, A_0(x,y).$$

Proof. Let $A \equiv \forall x^\sigma \exists y^\rho A_0(x,y)$. Assume the hypothesis. We can follow the proof of Theorem 2 up until the point where we can conclude that $\mathsf{CPV}^\omega_\trianglelefteq + \mathsf{P}_\mathrm{bd}[\trianglelefteq] + \mathsf{AC}^{0,\mathsf{b}0}_{\hat{\Sigma}^b_1} \vdash A$. By (*iii*) of Lemma 2, $\mathsf{CPV}^\omega_\trianglelefteq + \mathsf{P}_\mathrm{bd}[\trianglelefteq] + (\star) \vdash A$. Using the negative translation (Proposition 3) it is easy to see that:

$$\mathsf{IPV}^\omega_\trianglelefteq + \mathsf{MP}_{\hat{\Sigma}^b_1} + \mathsf{P}_\mathrm{bd}[\trianglelefteq] + (\star) \vdash \forall x^\sigma \exists z^\rho \neg\neg \exists y \trianglelefteq_\rho z\, (A_0(x,y))^\dagger.$$

Note that $\mathsf{MP}_{\hat{\Sigma}^b_1}$ is used in accounting for the negative translation of $(\star)$. (At this juncture, a restricted bounded Markov's principle for mere Σ^b_1-formulas would not suffice.) It follows from Lemma 5 that there is a closed monotone term $q^{\sigma \to \rho}$ such that

$$\mathsf{CPV}^\omega_\trianglelefteq + (\star) \vdash \forall a^\sigma \forall x \trianglelefteq_\sigma a \exists y \trianglelefteq_\rho qa\, A_0(x,y).$$

An inessential generalization of Proposition 5 now yields the conclusion (note that $f \leq_1^* 1$ iff $f \leq_1 1$). □

The following vast generalization of a Parikh-type bounding result (see [24]) is now immediate:

Corollary 1 *Suppose that*

$$\mathsf{E\text{-}CPV}^\omega + \mathsf{bAC}_0^{0,1} + \mathsf{AC}_{\hat{\Sigma}_1^b}^{0,\mathsf{b}0} + \Sigma_1^0\text{-}\mathsf{UB} \vdash \forall x^0 \exists y^\rho A_0(x, y),$$

where A_0 is a 0-bounded formula (its free variables as displayed). Then there is a closed term $q^{0 \to \rho}$ such that,

$$\mathsf{CPV}^\omega + \mathsf{AC}_{\hat{\Sigma}_1^b}^{0,\mathsf{b}0} \vdash \forall x^0 \exists y \leq_\rho^* qx\, A_0(x, y).$$

Whereas the bounded functional interpretation is powerless to analyze away $\mathsf{AC}_{\hat{\Sigma}_1^b}^{0,\mathsf{b}0}$ since this principle asks for a precise witness, the interpretation of Cook and Urquhart analyzes away $\mathsf{AC}_{\mathsf{qf}}^{0,0}$ effortlessly (this observation is due to the second author in [23]) and, *a fortiori*, $\mathsf{AC}_{\hat{\Sigma}_1^b}^{0,\mathsf{b}0}$. According to [8], the same is also the case concerning $\mathsf{MP}_{\hat{\Sigma}_1^b}$ (actually, even Markov's principle for existential formulas, without the bound).

We are now ready to prove the main theorem of this paper:

Theorem 4 (Main Theorem) *Suppose that*

$$\mathsf{E\text{-}CPV}^\omega + \mathsf{bAC}_0^{0,1} + \mathsf{AC}_{\hat{\Sigma}_1^b}^{0,\mathsf{b}0} + \Sigma_1^0\text{-}\mathsf{UB} \vdash \forall x^0 \exists y^0\, A_{\mathrm{qf}}(x, y),$$

where A_{qf} is a quantifier-free formula (its variables as displayed). Then there is a closed term $t^{0 \to 0}$ of $\mathcal{L}^\omega$ such that

$$\mathsf{PV}^\omega \vdash A_{\mathrm{qf}}(x, tx).$$

Proof. Assume the hypothesis. By Corollary 1, the theory $\mathsf{CPV}^\omega + \mathsf{AC}_{\hat{\Sigma}_1^b}^{0,\mathsf{b}0}$ proves $\forall x^0 \exists y^0 A_{\mathrm{qf}}(x, y)$. We now use Cook and Urquhart's interpretation (their Soundness Theorem 9.3 in [8], and Oliva's observation above) to finish the proof. □

Observation 1 *In the formulation of the amount of induction present in IPV^ω (see Section 2), we have demanded that the parameters of the bounding terms t be of type 0 only (i.e., t must be zero-open). The only place where this requirement is used is in Cook and Urquhart's proof of their Soundness Theorem. Except for the Main Theorem above, all the results stated so far also hold if we have induction on notation for $\hat{\Sigma}_1^b$-formulas instead of induction on notation for mere Σ_1^b-formulas.*

By Theorem 6.17 of [8], we can even have the equational theory PV instead of PV^ω in the conclusion of the theorem (the former theory bears to the polynomial-time computable functions the same relation that Skolem's primitive recursive arithmetic bears to the primitive recursive functions).

In [12], the first author showed that if $\mathsf{BTFA}+\Sigma^{\mathsf{b}}_{\infty}\text{-}\mathsf{WKL} \vdash \forall x^0 \exists y^0 A(x,y)$, where A is a Σ^b_1-formula, then there is a term t such that $\mathsf{PTCA} \vdash \forall x^0 A(x, t(x))$, where PTCA is a first-order version of Cook's PV. Ferreira's proof has an essential model-theoretic component. Disregarding notational differences, by Lemma 4 and Cantini's observations, $\mathsf{BTFA} + \Sigma^{\mathsf{b}}_{\infty}\text{-}\mathsf{WKL}$ is included in the major theory of the theorem above. Since the arguments given in this paper are proof-theoretic and based on functional interpretations, we have met the challenge of Avigad and Feferman (posed at the end of §7 of [1]) to obtain, via these means, conservation results concerning weak König's lemma in a feasible setting.

4 Finite Functions

The theorems of the previous section can be extended by replacing the notion of 0-bounded formula by a wider notion, that of FIN-*bounded* formula.

Definition 6 *The* FIN-bounded *formulas of the language $\mathcal{L}^\omega$ form the smallest class of formulas that includes the quantifier-free formulas and is closed under propositional connectives and quantifications of the form $\forall x \leq_0 t(...)$, $\exists x \leq_0 t(...)$, $\forall f^1(\mathrm{FIN}(f,q,r) \rightarrow (...))$ and $\exists f^1(\mathrm{FIN}(f,q,r) \wedge (...))$ where t is a term of type 0 in which the variable x does not occur, and q and r are terms of type 0 in which the variable f does not occur. The formula $\mathrm{FIN}(f^1, k^0, m^0)$ abbreviates $\forall n^0(n > k \rightarrow fn =_0 0) \wedge \forall n^0(fn \leq m)$.*

In primitive recursive arithmetic, the finite functions f such that $\mathrm{FIN}(f, q, r)$ can be encoded by natural numbers bounded by a (primitive recursive) function of q and r. Therefore, within primitive recursive arithmetic, the quantifications $\forall f^1(\mathrm{FIN}(f,q,r) \rightarrow (...))$ and $\exists f^1(\mathrm{FIN}(f,q,r) \wedge (...))$ are bounded. This is not the case in the context of feasibility (cf. (4) of §1).

The case $\mathrm{FIN}(f, q, 1)$ corresponds to characteristic functions of sets whose elements are all less than $q + 1$. The study of quantifications over finite sets bounded by a given number of elements in the context of feasible analysis was first addressed by A. Fernandes in his doctoral dissertation [9] using model-theoretical methods. In this last section, we study the "FIN-quantifications" in feasible analysis with the methods of this paper.

Lemma 6 *Let $\mathcal{F}$ be the functional of type $1 \to (0 \to (0 \to 1))$ defined by*

$$\mathcal{F}(f^1,k^0,m^0)(n^0) =_0 \begin{cases} \max_0(fn,m) & \textit{if } n \leq_0 k \\ 0 & \textit{otherwise} \end{cases}$$

The theory $\mathsf{IPV}^\omega_\trianglelefteq$ *proves the following:*

a) $\forall f^1 \forall k^0, m^0\, (\mathcal{F}(f,k,m) \trianglelefteq_1 \lambda n^0.m)$*;*
b) $\forall f^1 \forall k^0, m^0\, (FIN(\mathcal{F}(f,k,m),k,m))$*;*
c) $\forall f^1 \forall k^0, m^0\, (FIN(f,k,m) \to \forall x^0(f(x) = \mathcal{F}(f,k,m)(x)))$*.*

Proof. Part a) is a straightforward consequence of $\mathsf{RL}_\trianglelefteq$. Parts b) and c) are clear. □

In order to extend Theorem 2, Theorem 3 and Main Theorem 4 we perform a 'sandwich argument'.

Definition 7 *To each* FIN*-bounded formula* A *of* $\mathcal{L}^\omega$ *we associate formulas* A_{ll}*,* A_l*,* A_c*,* A_r *and* A_{rr} *according to the following recursive clauses:*

(1) If A *is atomic,* A_{ll}*,* A_l*,* A_c*,* A_r *and* A_{rr} *are all the same and equal to* A*.*
(2) $(A \Box B)_\natural$ *is* $A_\natural \Box B_\natural$*, where* $\Box \in \{\wedge, \vee\}$ *and* $\natural \in \{ll, l, c, r, rr\}$*.*
(3)a) $(A \to B)_{ll}$ *is* $A_{rr} \to B_{ll}$*;*
b) $(A \to B)_l$ *is* $A_r \to B_l$*;*
c) $(A \to B)_c$ *is* $A_c \to B_c$*;*
d) $(A \to B)_r$ *is* $A_l \to B_r$*;*
e) $(A \to B)_{rr}$ *is* $A_{ll} \to B_{rr}$*.*
(4) $(Qz \leq_0 tA(z))_\natural$ *is* $Qz \trianglelefteq_0 t[A(z)]_\natural$*, where* $Q \in \{\forall, \exists\}$ *and* $\natural \in \{ll, l, c, r, rr\}$*.*
(5)a) $[\forall f(FIN(f,q,r) \to A(f))]_{ll}$ *is* $\forall f[A(\mathcal{F}(f,q,r))]_{ll}$*;*
b) $[\forall f(FIN(f,q,r) \to A(f))]_l$ *is* $\forall f \trianglelefteq_1 \lambda n.r[A(\mathcal{F}(f,q,r))]_l$*;*
c) $[\forall f(FIN(f,q,r) \to A(f))]_c$ *is* $\forall f[A(\mathcal{F}(\mathcal{F}(f,q,r),q,r))]_c$*;*
d) $[\forall f(FIN(f,q,r) \to A(f))]_r$ *is* $\forall f \trianglelefteq_1 \lambda n.r[A(\mathcal{F}(\mathcal{F}(f,q,r),q,r))]_r$*;*
e) $[\forall f(FIN(f,q,r) \to A(f))]_{rr}$ *is* $\forall f[A(\mathcal{F}(\mathcal{F}(\mathcal{F}(f,q,r),q,r),q,r))]_{rr}$*.*
(6)a) $[\exists f(FIN(f,q,r) \wedge A(f)]_{ll}$ *is* $\exists f[A(\mathcal{F}(\mathcal{F}(\mathcal{F}(f,q,r),q,r),q,r))]_{ll}$*;*
b) $[\exists f(FIN(f,q,r) \wedge A(f)]_l$ *is* $\exists f \trianglelefteq_1 \lambda n.r[A(\mathcal{F}(\mathcal{F}(f,q,r),q,r))]_l$*;*
c) $[\exists f(FIN(f,q,r) \wedge A(f)]_c$ *is* $\exists f[A(\mathcal{F}(\mathcal{F}(f,q,r),q,r))]_c$*;*
d) $[\exists f(FIN(f,q,r) \wedge A(f)]_r$ *is* $\exists f \trianglelefteq_1 \lambda n.r[A(\mathcal{F}(f,q,r))]_r$*;*
e) $[\exists f(FIN(f,q,r) \wedge A(f)]_{rr}$ *is* $\exists f[A(\mathcal{F}(f,q,r))]_{rr}$*.*

Observe that the original formula A as well as A_{ll}, A_c and A_{rr} are in the language $\mathcal{L}^\omega$, whereas the formulas A_l and A_r are *bounded* formulas of $\mathcal{L}^\omega_\trianglelefteq$.

Lemma 7 *Let* A *be a FIN-bounded formula of the language* $\mathcal{L}^\omega$*.*

(i) $\mathsf{E\text{-}IPV}^\omega$ *proves that* A*,* A_{ll}*,* A_c *and* A_{rr} *are all equivalent.*
(ii) $\mathsf{IPV}^\omega_\trianglelefteq$ *proves* $A_{ll} \to A_l$*,* $A_l \to A_c$*,* $A_c \to A_r$ *and* $A_r \to A_{rr}$*.*

Proof. By b) and c) of Lemma 6, (i) immediately follows in the presence of full extensionality. The proof of (ii) requires simultaneous induction, and uses a) of Lemma 6. $\square$

The principles $\mathsf{bAC}^{0,1}_{\mathrm{FIN}}$ and Σ^0_1-$\mathsf{UB}_{\mathrm{FIN}}$ are the counterparts of $\mathsf{bAC}^{0,1}_0$ and Σ^0_1-UB (respectively), where the 0-bounded matrix A_0 is replaced by a FIN-bounded matrix A_{F}. Note that the generalization of the principle of strict-Π^1_1 reflection whereby the 0-bounded matrix A_0 is replaced by a FIN-bounded matrix A_{F}, is a consequence of Σ^0_1-$\mathsf{UB}_{\mathrm{FIN}}$. Hence, by an (adaptation of an) argument of Cantini [7] (see also the beginning of §3.1), Σ^0_1-$\mathsf{UB}_{\mathrm{FIN}}$ classically entails weak König's lemma for trees defined by FIN-bounded formulas.

We can now generalize Theorem 2.

Theorem 5 *Let $\sigma \in \{0,1\}$. Suppose that*

$$\mathsf{E\text{-}CPV}^\omega + \mathsf{bAC}^{0,1}_{\mathrm{FIN}} + \Sigma^0_1\text{-}\mathsf{UB}_{\mathrm{FIN}} \vdash \forall x^\sigma \exists y^\rho A_{\mathrm{F}}(x,y),$$

where A_{F} is a FIN*-bounded formula (its free variables as displayed). Then, there is a closed monotone term $q^{\sigma\to\rho}$ of $\mathcal{L}^\omega$ such that*

$$\mathsf{E\text{-}CPV}^\omega \vdash \forall a^\sigma \forall x \leq^*_\sigma a \exists y \leq^*_\rho qa\, A_{\mathrm{F}}(x,y).$$

Proof. The argument is a combination of the proof of Theorem 2 with a 'sandwich argument'. Suppose that $\forall x \exists y A_{\mathrm{F}}(x,y)$ is a theorem of $\mathsf{E\text{-}CPV}^\omega$ + $\mathsf{bAC}^{0,1}_{\mathrm{FIN}} + \Sigma^0_1$-$\mathsf{UB}_{\mathrm{FIN}}$. By the previous lemma, then so is $\forall x \exists y [A_{\mathrm{F}}(x,y)]_c$. In virtue of the presence of full extensionality and by the previous lemma, it is clear that we may replace $\mathsf{bAC}^{0,1}_{\mathrm{FIN}}$ by the scheme

$$(\$) \quad : \quad \forall x^0 \exists y^1 [B_{\mathrm{F}}(x,y)]_c \to \exists \Phi \forall x \exists y \leq_1 \Phi x [B_{\mathrm{F}}(x,y)]_{rr},$$

for FIN-bounded formulas B_{F}. It is also clear that we may replace Σ^0_1-$\mathsf{UB}_{\mathrm{FIN}}$ by:

$$(**) \begin{cases} \forall h^{0\to 1} \Big(\forall k^0 \forall f^1 \leq hk \exists e^1 \, [B_{\mathrm{F}}(f,h,k,e)]_c \to \\ \qquad \exists g^{0\to 1} \forall k^0, f^1 \exists e \leq_1 gk \, [B_{\mathrm{F}}(\min_1(f,hk),h,k,e)]_{rr} \Big). \end{cases}$$

In sum, the theory $\mathsf{E\text{-}CPV}^\omega + (\$) + (**)$ proves $\forall x \exists y [A_{\mathrm{F}}(x,y)]_c$. Using Luckhardt's elimination technique, it follows that $\mathsf{CPV}^\omega + (\$) + (**)$ already proves $\forall x \exists y [A_{\mathrm{F}}(x,y)]_c$. Now, we claim that $\mathsf{CPV}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq]$ proves each instance of (\$) and $(**)$. Let us argue this for (\$). Assume that $\forall x \exists y [B_{\mathrm{F}}(x,y)]_c$. By the previous lemma, $\forall x \exists y [B_{\mathrm{F}}(x,y)]_r$. Since $[B_{\mathrm{F}}(x,y)]_r$ is a bounded formula, by $\mathsf{bAC}^\omega_{\mathrm{bd}}[\trianglelefteq]$ and part *(iii)* of Lemma 2 we get $\exists \Phi \forall x \exists y \leq \Phi x [B_{\mathrm{F}}(x,y)]_r$. According to the previous lemma, we may conclude $\exists \Phi \forall x \exists y \leq \Phi x [B_{\mathrm{F}}(x,y)]_{rr}$, as wanted. The case $(**)$ is similar, using $\mathsf{MAJ}^\omega[\trianglelefteq]$, $\mathsf{bMP}^\omega_{\mathrm{bd}}[\trianglelefteq]$, $\mathsf{bAC}^\omega_{\mathrm{bd}}[\trianglelefteq]$ and

$\mathsf{bBC}^\omega_{\mathrm{bd}}[\trianglelefteq]$ (following an argument in the proof of Theorem 2) and, again, the previous lemma.

We showed that $\mathsf{CPV}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq]$ proves $\forall x \exists y [A_{\mathrm{F}}(x,y)]_c$ and, hence (by the above lemma) $\forall x \exists y [A_{\mathrm{F}}(x,y)]_r$. Since the formula $[A_{\mathrm{F}}(x,y)]_r$ is bounded, using the negative translation and the Soundness Theorem, it is not difficult to see that

$$\mathsf{CPV}^\omega_\trianglelefteq \vdash \forall a^\sigma \forall x \trianglelefteq_\sigma a \exists y \trianglelefteq_\rho qa \, [A_{\mathrm{F}}(x,y)]_r$$

for a certain closed term q. By the above lemma, we may substitute the matrix $[A_{\mathrm{F}}(x,y)]_r$ by $[A_{\mathrm{F}}(x,y)]_{rr}$. A modification of Proposition 5 yields

$$\mathsf{CPV}^\omega \vdash \forall a^\sigma \forall x \leq^*_\sigma a \exists y \leq^*_\rho qa \, [A_{\mathrm{F}}(x,y)]_{rr}.$$

Therefore, by full extensionality and the above lemma:

$$\mathsf{E\text{-}CPV}^\omega \vdash \forall a^\sigma \forall x \leq^*_\sigma a \exists y \leq^*_\rho qa \, A_{\mathrm{F}}(x,y).$$

□

Theorem 3 and Main Theorem 4 can be similarly extended. We finish the paper with the improved formulation of the latter theorem.

Theorem 6 *Suppose that*

$$\mathsf{E\text{-}CPV}^\omega + \mathsf{bAC}^{0,1}_{\mathrm{FIN}} + \Sigma^0_1\text{-}\mathsf{UB}_{\mathrm{FIN}} + \mathsf{AC}^{0,\mathsf{b}0}_{\hat{\Sigma}^b_1} \vdash \forall x^0 \exists y^0 \, A_{\mathrm{qf}}(x,y),$$

where A_{qf} is a quantifier-free formula (its variables as displayed). Then there is a closed term $t^{0\to0}$ of $\mathcal{L}^\omega$ such that

$$\mathsf{PV}^\omega \vdash A_{\mathrm{qf}}(x,tx).$$

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