

# Proving Constraints on Nichols Plots

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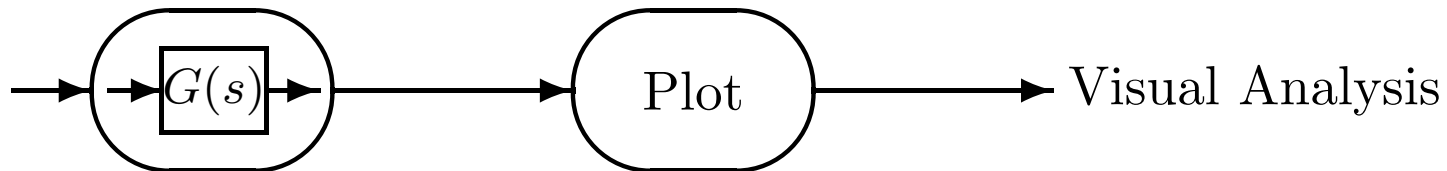
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## Outline

- Background, Objectives and Approach
- Frequency Domain
- Nichols Charts
- Exclusion Region on Nichols Charts
- Our Approach to Analysis
- Example using Maple and PVS
- Progress so far
- Future Work

## Background and Objectives

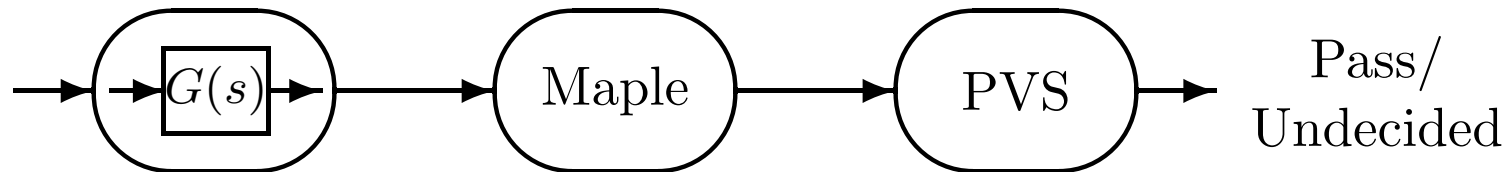
- Use theorem provers to extend the analysis of control laws
- Currently, analysis done numerically



- Problems:
  1. human errors
  2. plotting inaccuracy
  3. limited scope

## Approach

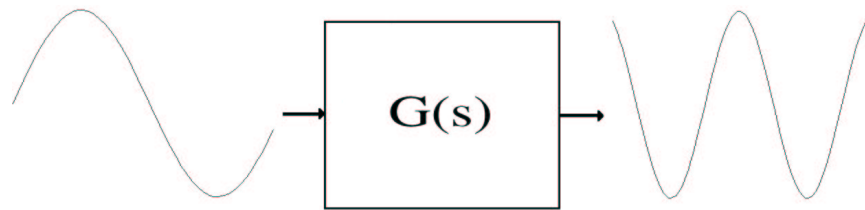
- Focussed on analysis in frequency domain
  - Nichols Charts
- Our method use symbolic techniques:



- Get assured pass or undecided
- Potential for more general application than plots e.g. parameterised input

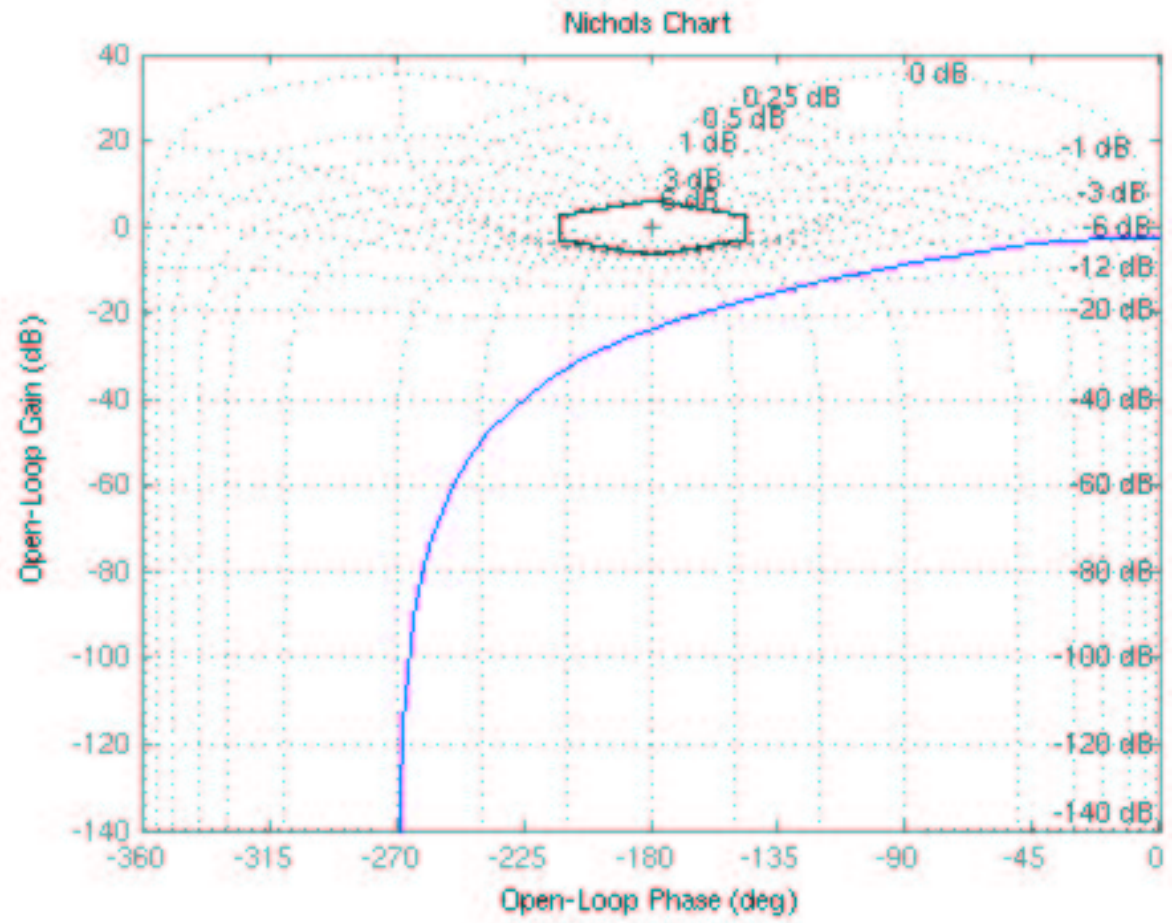
## Frequency Domain

- Expose system to sinusoidal inputs with same amplitude but varying frequencies
- Analysis done by examining the gain and phase-shift produced



- Most common method Nichols charts
- Used to determine instability and response of the system
- Shape and position of plots used in the analysis
- Desired shapes not specified explicitly
- Aim to keep a balance between stability and performance

# Example Nichols Chart



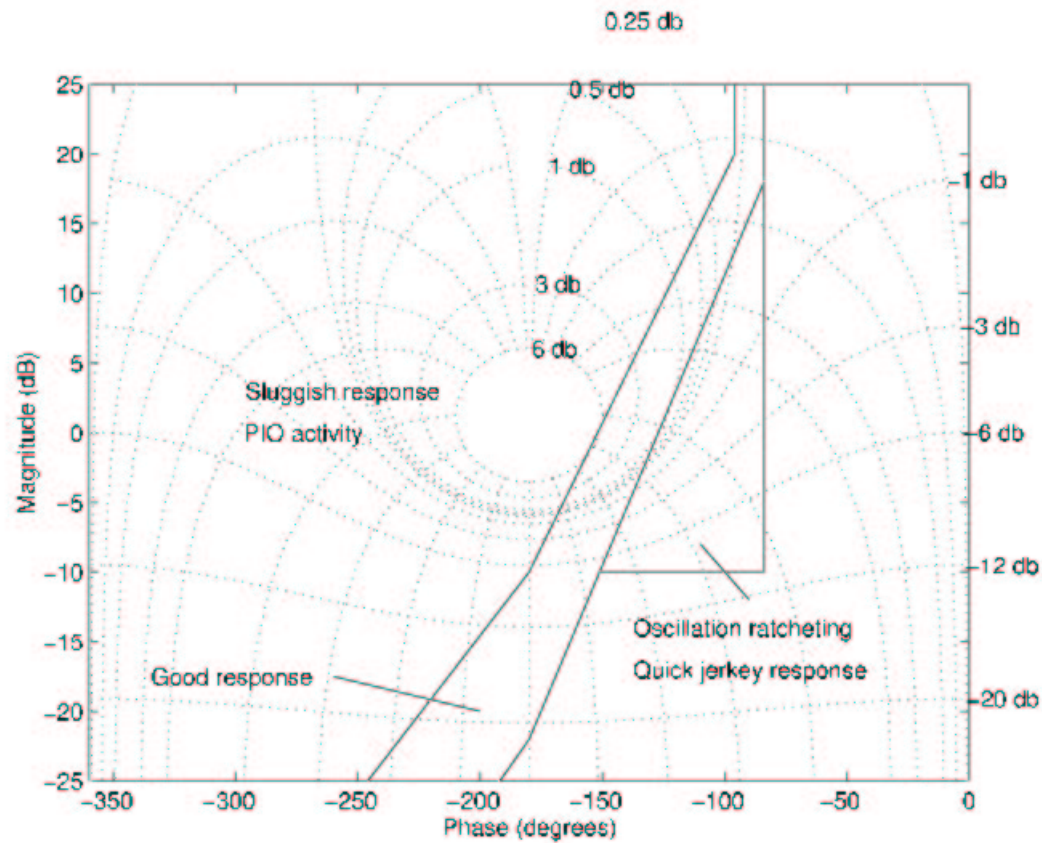
## Nichols Chart

- Substitute  $s = j\omega$  in  $G(s)$  to get  $G(j\omega)$
- Plot a single graph of gain (in decibels) against phase-shift for the varying value of frequency
- Construct by plotting the the magnitude in decibels of  $G(j\omega)$  against argument (angle) of  $G(j\omega)$
- Nichols plots have region to be avoided for stability and performance
- Generally the aim is to get the plot close to the region without actually entering it.

## Nichols Chart: Exclusion Region

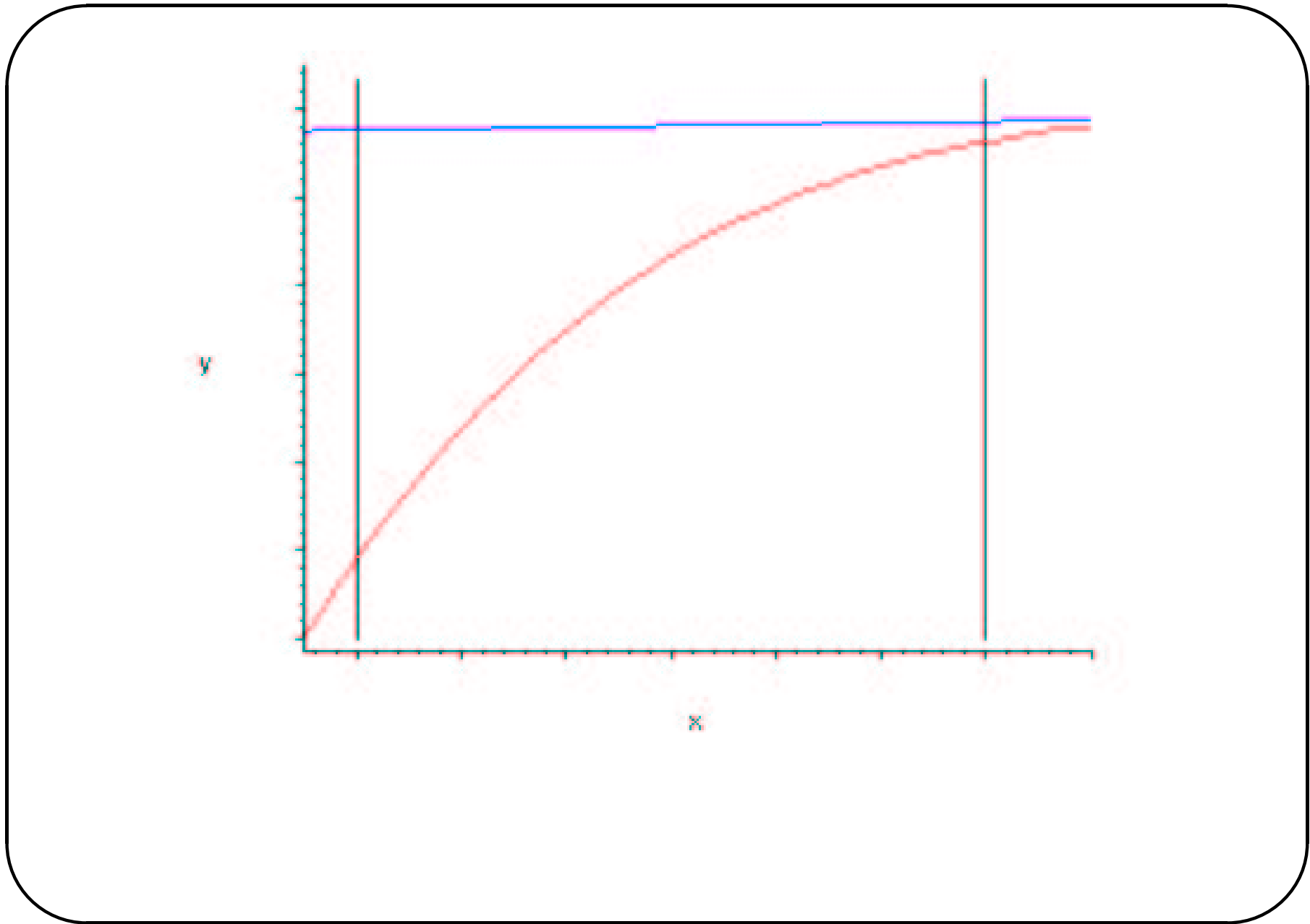
- Exclusion region varies according to purpose
- Excluded hexagon (Flight Control Systems, 2000)
- Extended excluded hexagon (Flight Control Systems, 2000)
- Region within which handling qualities are desirable (Robust Flight Control Design Challenge Problem Formulation and Manual, 1997)

# Example Nichols Chart: GARTEUR Document



## Frequency Domain Analysis

- Curve for Nichols plot is  $y = f(x)$
- Line bounding exclusion region is  $y = mx + c$
- Investigate simplest case to give idea of what is necessary / practical
- Problem: prove curve lies above or below line in region
- Prove  $f(x) < mx + c$  for  $a \leq x \leq b$
- Method:
  1. devise suite of tests for different curves
  2. program Maple and PVS to discharge them



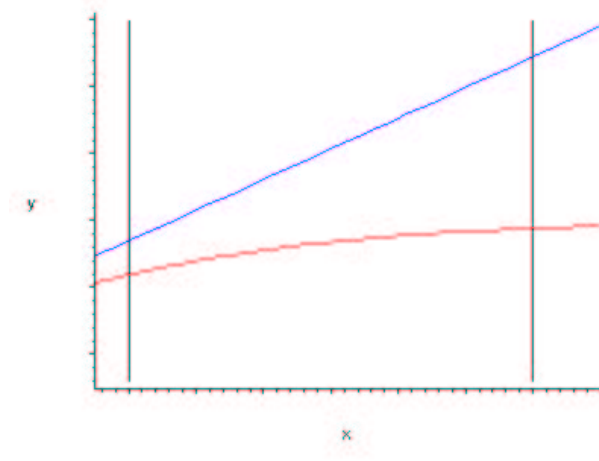
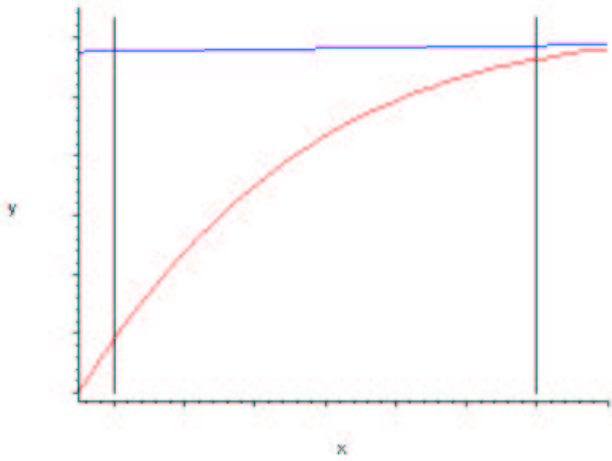
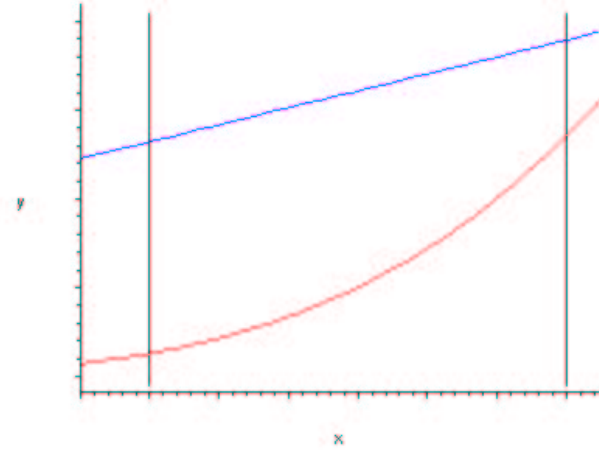
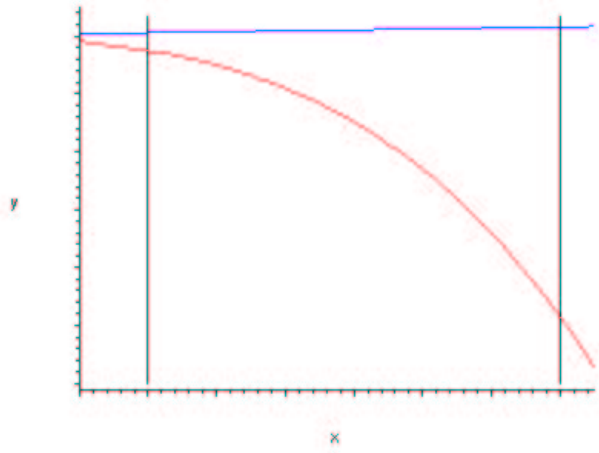
## Sample Test

- $y = f(x)$  is monotonic increasing, convex and steeper at  $b$  than the line
- Proof obligations:
  1. prove  $f(x)$  is sufficiently differentiable
  2. prove  $f(x)$  is continuous in  $[a, b]$
  3. prove  $f(x)$  is monotonic increasing in  $[a, b]$ , that is  $\frac{df}{dx} > 0$
  4. prove  $f(x)$  is convex in  $[a, b]$  that is  $\frac{d^2f}{dx^2} < 0$
  5. prove  $f(x)$  is steeper than  $mx + c$  at the end point  $b$  that is  $\frac{df}{dx}|_{x=b} > m$
  6. prove  $f(x)$  is below  $mx + c$  at the end point  $b$  that is  $f(b) < mb + c$

## Curve Cases

- Several different cases of curves were identified
  1.  $y = f(x)$  is monotonic decreasing
  2.  $y = f(x)$  is monotonic increasing and concave
  3.  $y = f(x)$  is monotonic increasing, convex and steeper at  $b$
  4.  $y = f(x)$  is monotonic increasing, convex and less steep at  $a$
- Different proof methods were devised for each case

# Curve Cases



## Example

- Transfer function  $G(s) = \frac{3}{s^3+5s^2+10s+4}$ ,  
 $G(j\omega) = \frac{3}{-j\omega^3-5\omega^2+10j\omega+4}$
- $y = 20 \log_{10}(|G(j\omega)|) = 20 \log_{10}\left(\frac{3}{\sqrt{\omega^6+5\omega^4+60\omega^2+16}}\right)$
- $x = \text{argument}(G(j\omega)) = \arctan(\omega^3 - 10\omega, -5\omega^2 + 4)$
- Exclusion region used  $-\pi \leq x \leq \frac{-3\pi}{4}$  so  $1.7 \leq \omega \leq 3.2$
- Proof obligations
  - $y$  and  $x$  are continuous for  $1.7 \leq \omega \leq 3.2$
  - $\frac{df}{dx} = \frac{\omega(3\omega^4+10\omega^2+60)}{(\ln(2)+\ln(5))(5\omega^4+38\omega^2+40)} > 0$  for  $1.7 \leq \omega \leq 3.2$
  - $\frac{d^2f}{dx^2} = \frac{-20(15\omega^8+292\omega^6+80\omega^4-1080\omega^2+2400)(\omega^6+5\omega^4+60\omega^2+16)}{(\ln(2)+\ln(5))(5\omega^4+38\omega^2+40)^3} < 0$   
 for  $1.7 \leq \omega \leq 3.2$
  - $\vdots$

## PVS

- Prototype Verification System
- Specification and verification tool
- Based on classical, higher order logic
- Many predefined theories
- Supports high level of automation
- Strongly typed

## Example in PVS

```
mcubic2: THEORY
  BEGIN
  w: VAR nreal
  dfdx(w): real = 20*w...
  d2fdx2(w): real = -20*292*w^6...

  lemmaA : LEMMA
    FORALL w : 0 < w IMPLIES dfdx(w) > 0
  lemmaB : LEMMA
    FORALL w : 17/10 < w AND w < 32/10 IMPLIES dfdx(w) > 0
  lemmaC : LEMMA
    FORALL w : 0 < w IMPLIES d2fdx2(w) < 0
  lemmaD : LEMMA
    FORALL w : 17/10 < w AND w < 32/10 IMPLIES d2fdx2(w) < 0
  lemmaE : LEMMA
    dfdx(1798192213/1000000000) - 12/(31416/10000) > 0
  END mcubic2
```

## Proofs

- Make use of high level of built-in automation
  - lemmaA proved using strategies
- Use previously proved lemmas to prove others
  - lemmaB proved using lemmaA
- Case splits needed
  - lemmaC case splits needed  
(CASE "253440 > w!1\*w!1\*103040") and  
(CASE "34160\*w!1^4 > 103040")
- Lemmas used with instantiated values
  - lemmaC used (LEMMA "both\_sides\_times\_pos\_gt1") and instantiated it with  
(INST -1 "w!1\*w!1" "253440" "103040\*w!1\*w!1") to give  
"253440\*w!1\*w!1 > 103040\*w!1\*w!1\*w!1\*w!1"

## Progress so far

- Identified tests for many common cases of curves
  - Monotonic decreasing
  - Monotonic increasing and concave
  - Monotonic increasing, convex and steeper at  $b$
  - Monotonic increasing, convex and less steep at  $a$
- Curves with points of inflection split into proofs for different regions
- No test yet for monotonic increasing, convex curves not fitting above criteria
- Identified general method in Maple for test generation
- Experiments in PVS to discharge proof obligations

## Future Work

- More comprehensive suite of tests
- Improved Maple and PVS support
- Automated interface between Matlab / Maple / PVS
- Identify more general uses for this technique
  - Bode plots
  - Parametric curves i.e.

$$G(s) = \frac{1}{3s + n}$$