An Overview of Control System Engineering as Background for Research into Symbolic Reasoning

Ruth Hardy

School of Computer Science, University of St Andrews

This document discusses some of the basic concepts and methods used by control engineers in the development of control laws. It is not meant to act as an introduction as to how to use these techniques, but rather as a guide to some of the more important aspects and issues associated with this area.

In Section 1 some of the different representations used to model systems are discussed. Section 2 looks at one of these representations in more depth. Sections 3 and 4 discuss different methods of analysing various properties of systems, while Section 5 discusses methods to alter a system's response to fit criteria. Methods for discretizing systems and analysing their responses are discussed in Section 6. An extended example looking at stability analysis, using some of the methods discussed in the previous sections, is given in Section 7. Section 8 examines in more depth the use of frequency-domain analysis and discusses the possible use of formal methods in this area.

1 Modelling Dynamic Systems

One can think of a dynamic system in several ways and at several different levels of abstraction. In this section different representations are given. A simple spring/mass/damper system is used to illustrate these representations.

1.1 Physical Representation

Perhaps the most basic view of a dynamic system is the representation of the physical components that make up the system. The simple spring/mass/damper system can be represented in such a way (see Figure 1). In this system k is the coefficient of the spring, b is the coefficient of friction, m is the mass of the body on the end of the spring, $u(t)$ is the external force on the system and $y(t)$ is the resulting displacement. This representation can show the various forces that act upon the system but it tells us very little about the *behaviour* of the system under these forces.

1.2 Block Diagram

At a slightly lower level of abstraction is the block diagram. This diagram models the forces acting on the system and allows the resulting behaviour to be determined. The same spring/mass/damper system from Figure 1 would appear as in

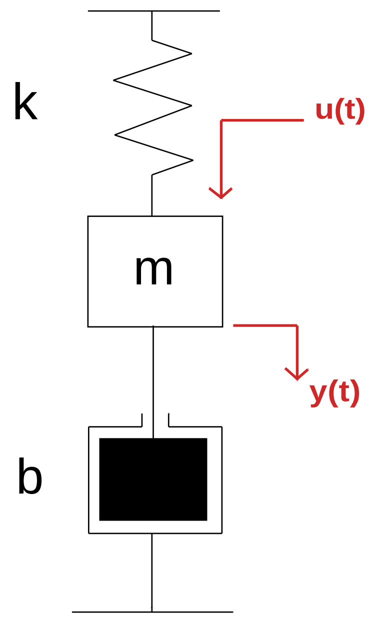


Fig. 1. Physical representation.

Figure 2 in block diagram form. The boxes containing the expressions $\frac{1}{m}$, $\frac{b}{m}$ and $\frac{k}{m}$ are gains, the other boxes are integrators. A gain simply takes the product of the input and the given expression. The terms $\dot{y}$ and $\ddot{y}$ refer to the derivative and the double derivative of y respectively.

1.3 State-space Representation

Dynamic systems can also be represented purely mathematically, and in several different ways. The state-space representation uses vector-matrix notation to represent the system of first-order differential equations that model a system. The state-space representation of the given dynamic system is as follows:

$$\text{Let } x_1(t) = y(t),\ x_2(t) = \dot{y}(t),\ \dot{x_1}(t) = x_2,\ \dot{x_2}(t) = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{u}{m}$$

$$\text{Then } \underline{\dot{x}} = \begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u = A\underline{x} + B\underline{u} \quad (1)$$

An underlined variable denotes that it is a vector.

The complexity of the equations in this representation does not increase with the numbers of inputs, outputs and state variables. This means systems with many inputs and outputs, which may interrelate in a complex way, can be represented in a relatively simple manner.

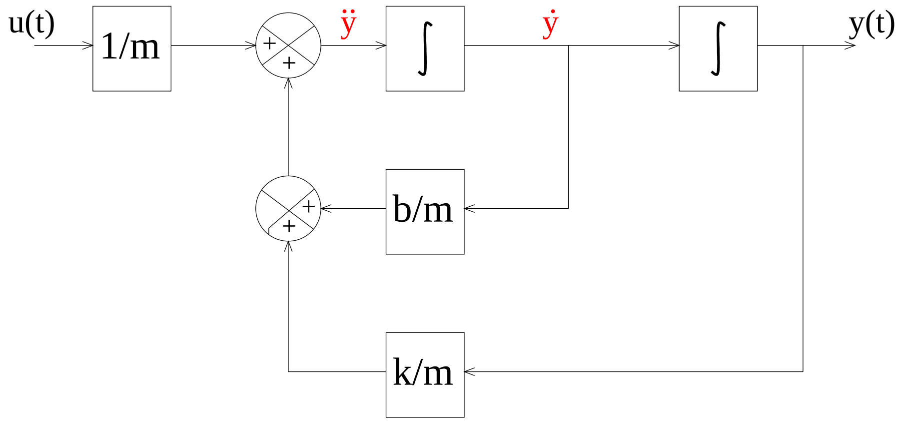


Fig. 2. Block diagram.

1.4 Differential Equation

Another possible representation of dynamic systems are differential equations. For the given spring/mass/damper system the differential equation would be

$$m\ddot{y} + b\dot{y} + ky = u \tag{2}$$

1.5 Laplace Transform

From the difference equation we can obtain another representation of the system: the Laplace transform (see Section 2). The Laplace transform models the behaviour of the system in terms of the relation between the output ($Y(s)$) and the input ($U(s)$) of the system. For our example the Laplace transform is:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{ms^2 + bs + k} = \frac{1}{|sI - A|} \tag{3}$$

where $|sI - A|$ is the determinant of the matrix $sI - A$.

The Laplace transform representation of the system can be used to determine various properties of the system, such as stability. The stability of a system can be ascertained by determining the locations of the poles of $G(s)$. The poles are the roots of $q(s) = 0$ (where $q(s)$ is the denominator of $G(s)$ once all common factors of the numerator and denominator have been cancelled). The roots can be determined by solving the equation or by finding the eigenvalues of A.

2 The Laplace Transform

The Laplace transform is the most common representation used by control engineers in the development of control laws. The Laplace transform transforms functions from the real numbers to the complex plane, and allows analysis in terms of the system's complex frequency as well as its real-time response. It takes a function $f(t)$ and is used to produce the function $F(s)$ (where s is a complex variable). To take the Laplace transform of a function $f(t)$ certain properties must hold:

1. f must be a function from the reals to the reals

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

2. $f(t)$ must be zero for all t less than zero

$$f(t) = 0 \text{ for } t < 0$$

Given these criteria, the method for finding the Laplace transform is as shown in Equation 4.

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st}\, dt\, [f(t)] = \int_0^\infty f(t)e^{-st}\, dt \tag{4}$$

Just as the original function was a function of the reals ($f : \mathbb{R} \longrightarrow \mathbb{R}$), the Laplace transform of that function is a function of the complex numbers

$$\mathcal{L} : f \longrightarrow (F \in \mathbb{C} \longrightarrow \mathbb{C})$$

There are some common functions that often appear in difference equations to be transformed and so there are often tables produced in control engineering books containing their Laplace transforms (see Table 1).

$f\vert_{\mathbb{R}>0}(t)$	$F(s)$
t	$\frac{1}{s^2}$
e^{-at}	$\frac{1}{s+a}$
$const K$	$\frac{K}{s}$
$\frac{df}{dt}$	$sF(s) - f(0)$
$\vdots$	$\vdots$

Table 1. Laplace transforms.

2.1 Using the Laplace Transform

To solve a differential equation of the form

$$a_0x^n + a_1x^{n-1} + \cdots = c,\, x^i|_0 = b_i \tag{5}$$

one should apply the Laplace transform to get a solution of the form shown below.

$$X(s) = \frac{polynomial\, in\, s, a_i, b_i, c}{polynomial\, in\, s, a_i, b_i, c} \tag{6}$$

Then one should either form partial fractions and apply the inverse Laplace or alternatively, one could analyse $X(s)$ as required via poles and zeros in the complex plane. The zeros of a transfer function are found by finding the roots of the numerator.

For the following example

$$\ddot{x} + 2\dot{x} + 5x = 3,\, x(0) = \dot{x} = 0 \tag{7}$$

one would apply the Laplace transform to get:

$$s^2X(s) + 2sX(s) + 5X(s) = \frac{3}{s} \tag{8}$$

Rearranging the equation to isolate $X(s)$ gives the following:

$$X(s) = \frac{3}{s(s^2 + 2s + 5)} \tag{9}$$

Then form the partial fractions of the equation:

$$\frac{3}{5s} - \frac{3}{10}\frac{2}{(s+1)^2 + 2^2} - \frac{3}{5}\frac{s+1}{(s+1)^2 + 2^2} \tag{10}$$

Then one should apply the inverse Laplace transform (see Equation 11) and by examining the components of this equation one could see that the behaviour is decaying and oscillating.

$$x(t) = \frac{3}{5} - \frac{3}{10}e^{-t}\sin 2t - \frac{3}{5}e^{-t}\cos 2t \tag{11}$$

Alternatively, one could determine the behaviour of the differential equation by examining the position of the poles and zeros of its Laplace transform. The poles for this example are $0, -1 \pm 2i$, and can be obtained from Equation 12.

$$X(s) = \frac{3}{s(s^2 + 2s + 5)} = \frac{3}{s(s + 1 + 2i)(s + 1 - 2i)} \tag{12}$$

2.2 Laplace Transform and System Configuration

Systems do not necessarily comprise only one component and can be composed in different ways (see Figure 3). The components $G(s)$ and $H(s)$ can themselves be made up of several components composed in different ways, allowing more complex configurations of systems, such as nested loops.

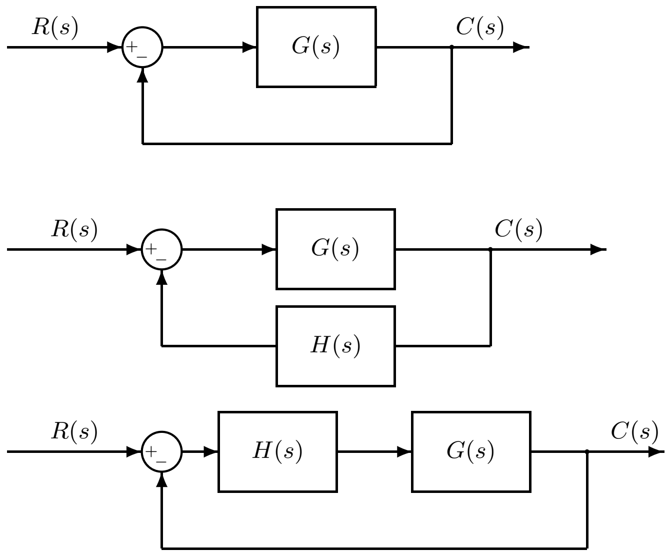


Fig. 3. Closed-loop configurations.

These different ways of composing systems entail different ways of composing the system's Laplace transforms. The simplest system contains a single plant ($G(s)$) contained in a loop. The closed-loop transfer function (a Laplace transform) of this complete system is obtained by evaluating and simplifying the following expression

$$\frac{G(s)}{1+G(s)} \tag{13}$$

The second system would have a Laplace transform obtained by evaluating and simplifying:

$$\frac{G(s)}{1+G(s)H(s)} \tag{14}$$

The third system would have a Laplace transform obtained by evaluating and simplifying:

$$\frac{G(s)H(s)}{1+G(s)H(s)} \tag{15}$$

3 Stability Analysis

As stated in Section 1.5 the stability of a system can be determined by analysing the poles of its transfer function. A system is stable if all its poles are in the left half of the complex plane, and it has a good transient response when the poles and zeros are away from the y-axis. The stability of a system can also be analysed in the frequency domain.

3.1 Stability Analysis with Poles and Zeros

There are various methods available for determining whether a system is stable without having to work out its poles.

The Routh-Hurwitz stability criterion states certain criteria for the coefficients of the polynomial, which show whether the system is stable without having to explicitly work out the poles. These criteria are necessary and sufficient for the system to be stable. It provides a general method for determining whether any of the poles of any sized polynomial lie in the right half of the plane (i.e. whether your system is unstable).

Liapunov stability can be used when the system is expressed in its state-space form. Given a system $\dot{x} = Ax$, where x is a state vector and A is a constant matrix, it is stable if given any real and symmetric matrix Q there exists a real, symmetric matrix P such that $A * P + PA = -Q$.

Nyquist stability uses a polar plot of $G(j\omega)H(j\omega)$, where $G(s)$ and $H(s)$ are the components in the closed-loop system, j is the complex variable, and ω is frequency. It requires $Z = N + P$ where Z is the number of zeroes of $1+G(s)H(s)$ in the left half of the complex plane, N is the number of clockwise circuits the plot makes of the point $(-1, 0)$, and P is the number of poles in the left half of the complex plane.

3.2 Stability Analysis in the Frequency Domain

There are two methods by which an open-loop system is analysed in the frequency domain using sinusoidal signals as inputs. The input sinusoids all have the same amplitude but varying frequencies. The analysis is done by comparing the output sinusoids to the input sinusoids.

The first of these methods is the Bode diagram. This technique plots two graphs — one plotting the phase-shift of the output sinusoids against frequency and the other plotting the gain (usually in decibels) of the output sinusoids against frequency. The system is unstable if at any frequency the gain is 1 (or converted to decibels $0dB$) *and* the phase is $-180°$.

The second of these methods is the Nichols chart. A Nichols chart plots the gain (in decibels) against the phase-shift of the output sinusoids for the varying value of the frequency. A Nichols chart can be constructed from a Bode diagram by reading off the values of the phase and gain for a particular frequency then plotting them, or by plotting the argument (angle) of the complex number $G(j\omega)$ (the phase) against the magnitude of $G(j\omega)$ having taken its log to the base 10 and multiplied by 20 to get a value in decibels. The system is unstable if the plot passes through the point (-180, 0).

4 Response Analysis

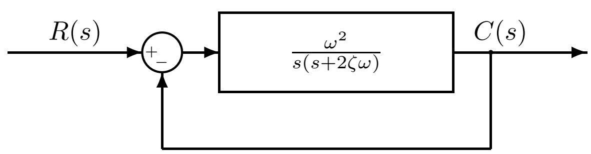
Response analysis can be done in two ways depending on one's objectives — the response of the system to inputs over time; or the response of the system in the frequency domain. This section first discusses response analysis in the real-time domain then shows various requirements that one may have for the time response of a system, then discusses frequency domain analysis.

4.1 Time Response Analysis

To analyse the real-time response of a system one can examine the shape of the curve produced by exposing the system to various inputs. There are three types of inputs for time-response analysis, as shown in Figure 4.

There are three types of responses that a system could have to these inputs and a system may have a different response depending on the chosen input. It would be desirable for the system to have the same response to all classes of inputs, since one usually does not know exactly what inputs the system will encounter in practice. The three responses that a system could have are shown in Figure 5.

Example The time response of the following system is to be analysed given a step input.



The closed-loop Laplace transform of this system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{16}$$

where ω_n is the natural frequency of the system. The analysis of a system can be done through the analysis of its poles. The poles of this system can be found by finding the roots of $s^2 + 2\zeta\omega_n s + \omega_n^2$. Given any quadratic equation there are three categories that the roots could fall into — two complex roots; two real, equal roots; two real, non-equal roots — and each of these categories correspond to a different type of response that the system could have (see Table 2). Using the table one could easily see what the type of response a particular system of this form would have by examining the value of ζ.

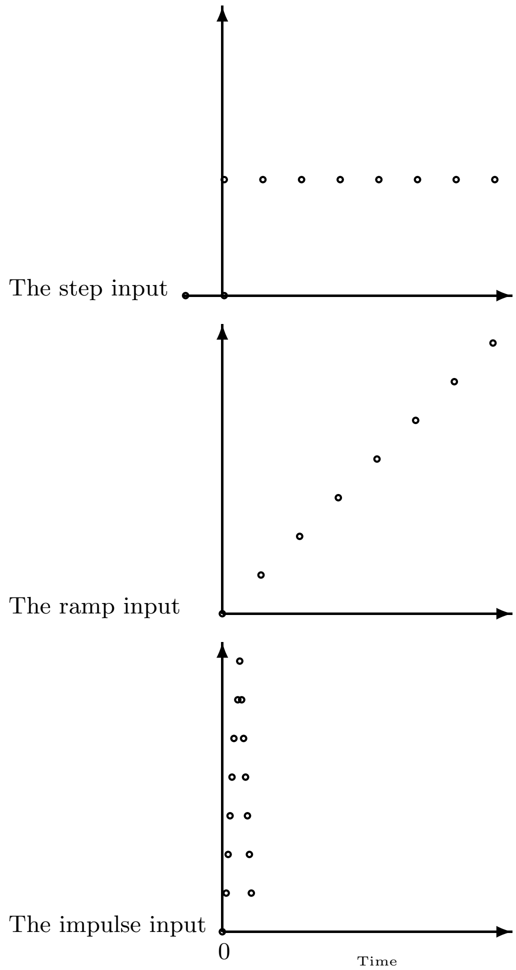
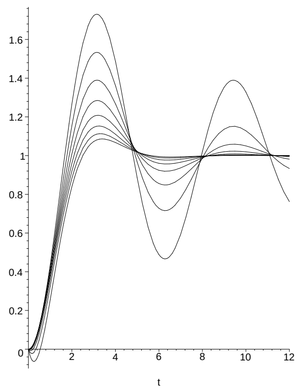


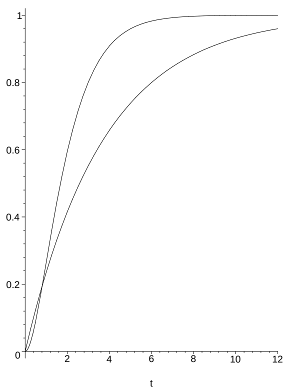
Fig. 4. Classes of inputs for time-response analysis.

Roots	Response	Parameter	Inverse Laplace
Complex	under damped	$0 < \zeta < 1$	$1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n t + \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)$
Real and Equal	critically damped	$\zeta = 1$	$1 - e^{-\omega_n t}(1 + \omega_n t)$
Real and Non-equal	over damped	$\zeta > 1$	$1 - e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$

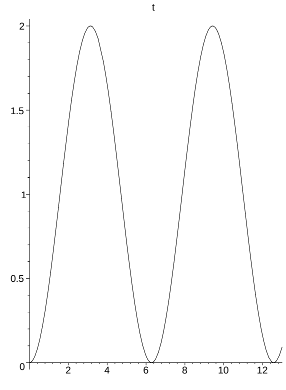
Table 2. Response analysis.



Under Damped



Over Damped



Critically Damped

Fig. 5. Response categories.

4.2 Response Requirements

When designing a system one usually has a more detailed set of requirements for its real-time response; for example, one may have a desired time in which the system should become steady. These requirements can be expressed in the form of a response specification. It is usual for response specifications to contain the following information:

1. Desired delay time t_d — the time required for the response to first reach half the desired final value.
2. Desired rise time t_r — the time required for the response to rise from either 10% to 90%, 5% to 95% or 0% to 100% of its desired final value, depending on the system.
3. Desired peak time t_p — the time required for the response to reach the first peak of the overshoot.
4. Maximum percent M_p — the percentage of the final value that the maximum peak can overshoot the desired final output by.
5. Desired settling time t_s — the time required for the response to reach and stay within a specified range of the final value (usually 2% or 5%).

If all of these values are specified then the shape of the response curve is virtually determined (see Figure 6).

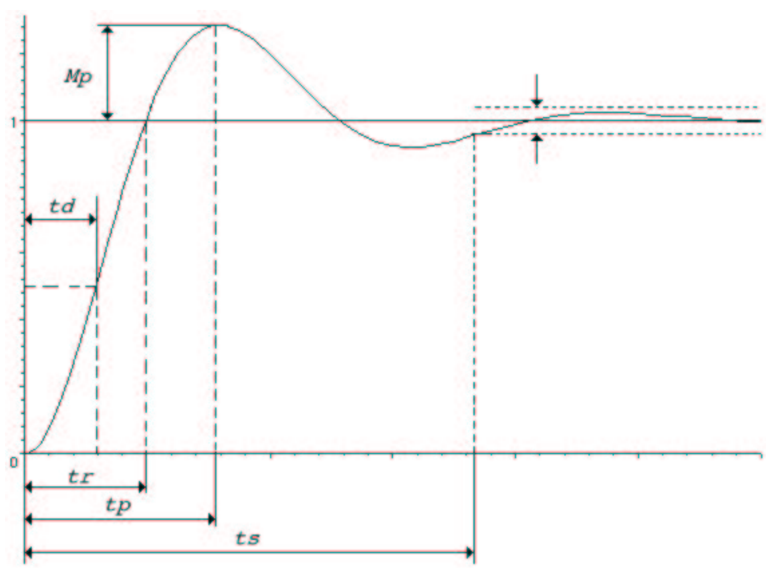


Fig. 6. Response curve.

4.3 Frequency-Domain Response Analysis

The response of a system can be determined by examining the gain and phase-shift produced by exposing the system to sinusoidal inputs. The methods most common for this type of analysis are Bode diagrams (e.g. see Figure 7) or Nichols charts (e.g. see Figure 8). As discussed in Section 3, these graphs can be used to determine instability, but they can also be used to determine aspects of the response of the system. The shape and position of these plots can be used in the analysis. The shapes are not specified explicitly — in practice it is understood that there is a general shape that the plots in Bode diagrams and Nichols charts should take. These shapes are aimed at keeping a balance between stability and performance — since the more stable a system is, the worse its performance.

In general, the desirable shape for a Bode diagram for a generic quadratic is that the plot of the gain should have a slight peak before falling off, and then should fall off at a rate of about $-20dB$ per decade (increase in frequency of a factor of 10) before flattening off. The position of the plots on the graph are generally in terms of the gain and phase margins.

The gain margin is the reciprocal of the gain at the frequency where phase is $-180°$. The phase margin is the difference between the phase at the frequency where the gain is 1 (or 0 if scale is in decibels) and $-180°$. In general there is considered to be a reasonable balance between stability and bandwidth if the gain margin is greater than 3 and the phase margin is between $30°$ and $60°$.

The Nichols plot has a region which is to be avoided, though depending on your purpose the region is not always the same. One region is a hexagon going through the points $(-180, -6)$, $(-145, -3)$, $(-145, 3)$, $(-180, 6)$, $(-215, 3)$, $(-215, -3)$ (see Figure 8). This region roughly corresponds to the regions to be avoided in the Bode diagram. The aim for a good balance between stability and performance is to get the plot as close to this area as possible without actually entering it.

5 Basic Control Actions

Often when one is designing a system one finds that the system as it is cannot meet the desired response. In these cases a controller must be added to modify the system response. There are different types of controllers but the three main control actions are proportional gain, integral gain, and derivative gain (see Table 3).

The three different control actions have different effects on a system (see Table 4). The control action that should be used depends on which requirement(s) the system is failing to meet. One may add a combination of controllers if necessary. For example, if a system is underdamped (i.e has a long rise time) one may add a proportional controller to decrease the rise time. From Table 4, one can see that along with decreasing the rise time a proportional controller also increases the overshoot. One may find that the proportional controller increases the overshoot to an unacceptable level. In this case one could add a derivative

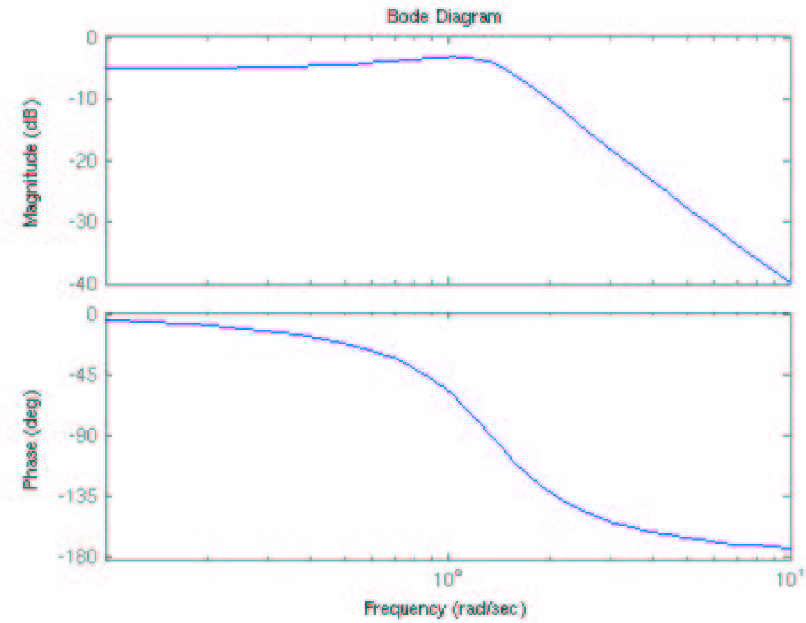


Fig. 7. Bode diagram.

Proportional	$E(s)$	K_p	$U(s)$	$u(t) = K_p e(t)$
Integral	$E(s)$	$\frac{K_i}{s}$	$U(s)$	$\frac{du(t)}{dt} = K_i e(t)$
Derivative	$E(s)$	$K_d s$	$U(s)$	$u(t) = K_d \frac{de(t)}{dt}$

Table 3. Control actions.

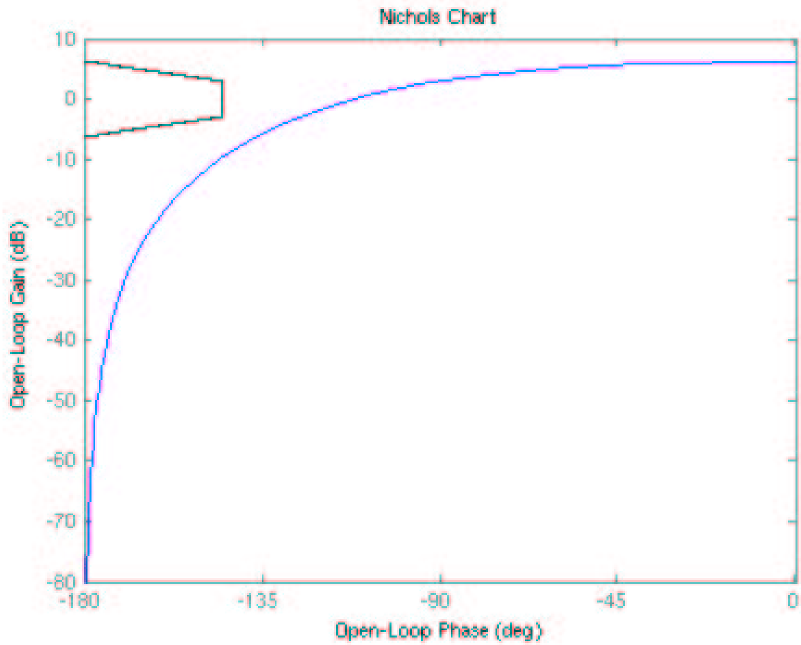


Fig. 8. Nichols chart.

controller, which decreases overshoot but makes very little difference to the rise time. By adjusting the values of K_p and K_d one can increase or decrease the effect that the controllers have on the system.

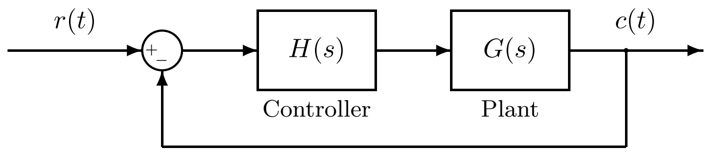
	Rise time t_r	Overshoot M_p	Settling time t_s
K_p	$\searrow$	$\nearrow$	$\approx$
K_i	$\searrow$	$\nearrow$	$\nearrow$
K_d	$\approx$	$\searrow$	$\searrow$

Table 4. PID effects on response.

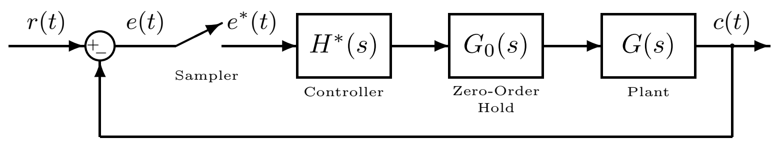
6 Discrete Models

The representations looked at in the previous sections have been continuous, but many of these systems, though modelled continuously, would need to have parts of them digitally implemented. It is therefore important to be able to represent these systems discretely and analyse the responses of these discretized systems. In general terms a continuous system is discretized by adding a sampler and some form of hold to the continuous system — usually a zero-order-hold, which sustains the sampled value throughout the sample period, until the next value is received.

Consider the following continuous system:



If this system were to have a digital controller in the implementation then the model would need to become similar to the one shown below:



The '*' notation signifies a discrete component or signal. This alteration to the model means that an alteration is also needed to its mathematical representation — the Laplace transform in the continuous model becomes the Z-transform for the discrete model.

6.1 Z-transform

To form the Z-transform of a system one can either form the Laplace transform, and from that form the Z-transform, or one can form the Z-transform from the original function $f(t)$. The general formula for obtaining the Z-transform from the original function is as follows:

$$Z\{f(t)\} = Z\{f^*(t)\} = \sum_{k=0}^{\infty} f(kT)z^{-k} \tag{17}$$

where z is a complex variable, T is the sampling period, and k ranges over the integers.

As with the Laplace transform there are many common components used in the equations, so books often contain tables containing the transformations from $x(t)$ to $X(s)$ to $X(z)$ (see Table 5).

The way in which the closed-loop Z-transform is formed depends not only on the configuration of the components in the loop but also on which components are discrete. To form the Z-transform of a system from its Laplace transform one cannot simply take the Z-transform of the components and combine them in the appropriate manner for the structure of the system. One must first take the

$x(t)$	$X(s)$	$X(z)$
$u(t)$, (UNITSTEP)	$\frac{1}{s}$	$\frac{z}{z-1}$
$\int_0^\infty u(t)\,dt$	$\frac{1}{s}$	$\frac{Tz}{z-1}$, (USING FORWARD-RECTANGULAR INTEGRATION)
$\frac{du}{dt}$	s	$\frac{z-1}{Tz}$, (USING FORWARD DIFFERENCE RULE)
t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1^2)}$
e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
⋮	⋮	⋮

Table 5. Z-transforms.

product of the Laplace transform for the zero-order hold and the Laplace transforms of the continuous components. In the system shown above, the Z-transform would be formed by first finding the product of the zero-order hold and the plant and then taking the Z-transform of the result $(Z\{G_0(s)G(s)\} = G_0G(z))$ *not* by taking the Z-transforms of the components separately then taking the product $(Z\{G_0(s)\}Z\{G(s)\} = G_0(z)G(z))$. The Z-transforms of the components could then be composed in a similar way to the way in which Laplace transforms are composed.

$$\frac{C(z)}{R(z)} = \frac{H(z)G_0G(z)}{1 + H(z)G_0G(z)} \tag{18}$$

6.2 Stability and Response Analysis of Discrete Systems

As with stability analysis of continuous systems the stability of a discrete system can be determined by analysing the poles of its Z-transform. A system is stable if all its poles lie within the unit circle in the complex plane.

The method available for determining whether a system is stable without having to work out its poles is the Jury criteria. These stability criteria state certain conditions on the coefficients of the polynomial in z. Like the Routh-Hurwitz method, the Jury conditions provide a general method for determining if any of the poles of any sized polynomial lie in the region of instability.

As with the analysis of continuous systems, Bode diagrams and Nichols charts can be used to analyse the stability and response of discrete systems. One may also have requirements for the time response of a system.

7 Extended Example — Stability of a PID Cruise Controller

To show some of the methods discussed in previous sections in practice an example was analysed for stability in both its discrete and continuous forms. The chosen example was a rather simple cruise control system with a proportional-integral-derivative controller. No consideration was given to response requirements for the system.

The stability analysis used the Routh-Hurwitz criterion and the Jury conditions, and the results of the two were compared.

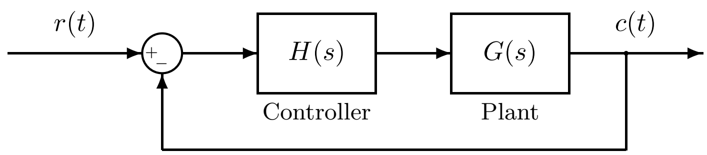
7.1 Representation

The Laplace transforms of the separate components are:

$$H(s) = \frac{K_d s^2 + K_p s + K_i}{s} \tag{19}$$

$$G(s) = \frac{1}{ms + b} \tag{20}$$

These components form a closed loop of the following type:



The closed-loop Laplace transform is:

$$\frac{H(s)G(s)}{1 + H(s)G(s)} = \frac{K_d s^2 + K_p s + K_i}{(K_d + m)s^2 + (K_p + b)s + K_i} \tag{21}$$

The Z-transforms of the components are $H(z)$ and $G_0G(z)$ as follows:

$$H(z) = K_p + \frac{K_i T z}{z - 1} + \frac{K_d(z - 1)}{Tz} \tag{22}$$

$$G_0(s)G(s) = \frac{1 - e^{-sT}}{s} \frac{1}{ms + b}$$

$$\therefore G_0G(z) = \frac{1 - e^{-\frac{b}{m}T}}{b(z - e^{-\frac{b}{m}T})} \tag{23}$$

The closed-loop Z-transform is given by:

$$\text{let } q = 1 - e^{-\frac{b}{m}T} \text{ , } n = 1 + e^{-\frac{b}{m}T}$$

$$\text{then } \frac{H(z)G_0G(z)}{1 + H(z)G_0G(z)} =$$

$$\frac{qK_pTz(z - 1) + qK_i(Tz)^2 + qK_d(z - 1)^2}{bTz^3 + (qK_pT + qK_iT^2 + qK_d - nbT)z^2 + (bTe^{-\frac{b}{m}T} - 2qK_d - qK_pT)z + qK_d} \tag{24}$$

7.2 Stability Analysis

When examining the poles of the closed-loop Laplace transform one is only concerned with the denominator of the equation. Therefore we are only concerned with

$$(K_d + m)s^2 + (K_p + b)s + K_i \tag{25}$$

Using the Routh-Hurwitz stability criteria for a quadratic equation we were able to determine that the system would be stable if all the coefficients are the same sign i.e. if $K_d + m > 0$ and $K_p + b > 0$ and $K_i > 0$, or $K_d + m < 0$ and $K_p + b < 0$ and $K_i < 0$.

The analysis of the closed-loop Z-transform is rather more complicated as, the equation is not only a cubic instead of a quadratic, but it also contains exponential terms:

$$bTz^3 + (K_pT(1-e^{-\frac{b}{m}T}) + K_iT^2(1-e^{-\frac{b}{m}T}) + K_d(1-e^{-\frac{b}{m}T}) - bT(1+e^{-\frac{b}{m}T}))z^2 +$$

$$(2K_d(e^{-\frac{b}{m}T} - 1) + K_pT(e^{-\frac{b}{m}T} - 1) + bTe^{-\frac{b}{m}T})z + K_d(1 - e^{-\frac{b}{m}T}))$$

The Jury conditions for a cubic of the form $az^3 + bz^2 + cz + d$, when a is positive, are $|d| < a$ and $a + b + c + d > 0$ and $-a + b - c + d < 0$ and $|d^2 - a^2| > |bd - ac|$ therefore the Jury conditions for the PID-cruise-control example are

1. $|K_d(1 - e^{-\frac{b}{m}T})| < bT$
2. $K_iT^2(1 - e^{-\frac{b}{m}T}) > 0$
3. $2K_pT(1-e^{-\frac{b}{m}T}) + K_iT^2(1-e^{-\frac{b}{m}T}) + 4K_d(1-e^{-\frac{b}{m}T}) - 2bT(1+e^{-\frac{b}{m}T}) < 0$
4. $|(K_d(1-e^{-\frac{b}{m}T}))^2 - (bT)^2| > |(K_pT(1-e^{-\frac{b}{m}T}) + K_iT^2(1-e^{-\frac{b}{m}T}) + K_d(1-e^{-\frac{b}{m}T}) - bT(1+e^{-\frac{b}{m}T}))K_d(1-e^{-\frac{b}{m}T}) - bT(2K_d(e^{-\frac{b}{m}T} - 1) + K_pT(e^{-\frac{b}{m}T} - 1) + bTe^{-\frac{b}{m}T})|$

It is difficult to see from these inequalities exactly what these conditions *mean*. By producing graphs with varying values of the parameters it is easier to visualise the relationships between the different parameters of the discrete model and the relationship between the discrete and continuous models.

7.3 Conclusions

By comparing the continuous and discretized versions of the PID cruise controller we were able to come to several conclusions. Firstly, the range of values of the parameters that give stability is much larger in the continuous system than in the discretized system. As the value of T in the discrete system increases (i.e. the sampling rate decreases), the range of values that give a stable system decreases. There are values that give stability in the discrete system that do not give stability in the continuous system.

These observations lead to the conclusion that if a system is modelled and analysed in the continuous domain, but then implemented digitally, the analysis may not be accurate.

8 Extended Example — Frequency Domain Analysis and Theorem Provers

When designing a critical system (such as flight control laws) it is important that the stability and response analysis of the system be as accurate as possible. For this reason when we looked at response analysis using Nichols charts we looked at the possibility of using theorem provers in this analysis.

To do this we began by looking at a very simple case of open-loop Laplace transform — a numeric numerator k and a quadratic denominator of the form $s^2 + bs + c$. Looking at numeric examples it became obvious that in the majority of cases the curve was monotonic increasing ($\frac{dy}{dx} > 0$) and convex ($\frac{d^2y}{dx^2} < 0$), therefore the curves did not double back on themselves. Because of this we decided that a good starting point for our theorem proving would be to devise tests to prove that a curve ($y = f(x)$) lies beneath (or over) a line ($y = mx + c$) in the interval $(a, b]$.

Several different cases of curves were identified and different tests were devised for the cases:

- monotonic decreasing (see Figure 9)
  1. prove $f(x)$ is monotonic decreasing in $[a, b]$: $\frac{df}{dx} \leq 0$
  2. prove $f(x)$ is below $mx + c$ at the end point a: $f(a) < ma + c$
- monotonic increasing and concave (see Figure 10)
  1. prove $f(x)$ is monotonic increasing in $[a, b]$: $\frac{df}{dx} > 0$
  2. prove $f(x)$ is concave in $[a, b]$: $\frac{d^2f}{dx^2} \geq 0$
  3. prove $f(x)$ is below $mx + c$ at each end point: $f(a) < ma + c$ and $f(b) < mb + c$
- monotonic increasing, convex and steeper at b than the line (see Figure 11)
  1. prove $f(x)$ is monotonic increasing in $[a, b]$: $\frac{df}{dx} > 0$
  2. prove $f(x)$ is convex in $[a, b]$: $\frac{d^2f}{dx^2} < 0$
  3. prove $f(x)$ is steeper than $mx + c$ at the end point b: $\frac{df}{dx}|_{x=b} > m$
  4. prove $f(x)$ is below $mx + c$ at the end point b: $f(b) < mb + c$
- monotonic increasing, convex and less steep at a than the line (see Figure 12)
  1. prove $f(x)$ is monotonic increasing in $[a, b]$: $\frac{df}{dx} > 0$
  2. prove $f(x)$ is convex in $[a, b]$: $\frac{d^2f}{dx^2} < 0$
  3. prove $f(x)$ is less steep than $mx + c$ at the end point a: $\frac{df}{dx}|_{x=a} < m$
  4. prove $f(x)$ is below $mx + c$ at the end point a: $f(a) < ma + c$
- monotonic increasing, convex and neither steeper at b than the line or less steep at a than the line (see Figure 13)
  1. no proof devised yet

For all the tests it is a requirement that the curve is sufficiently differentiable and continuous. Since there was no test for the final case devised, our tests are incomplete but sound and useful. If you can prove one of the cases for your

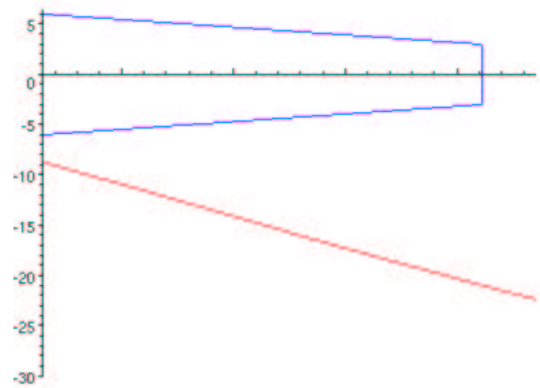


Fig. 9. Monotonic decreasing curve.

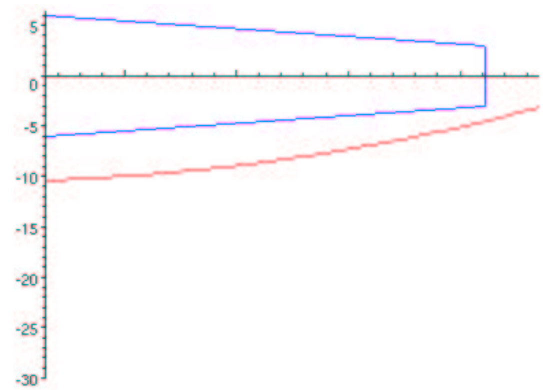


Fig. 10. Monotonic increasing concave curve.

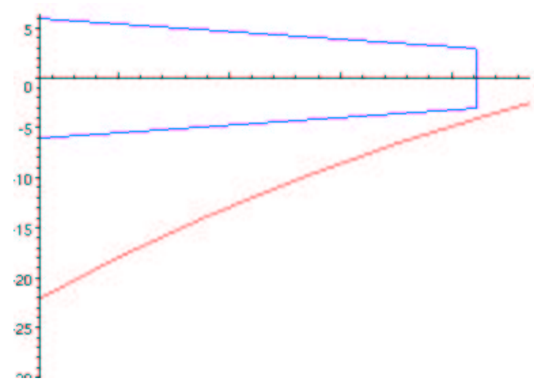


Fig. 11. Monotonic increasing convex curve (case a).

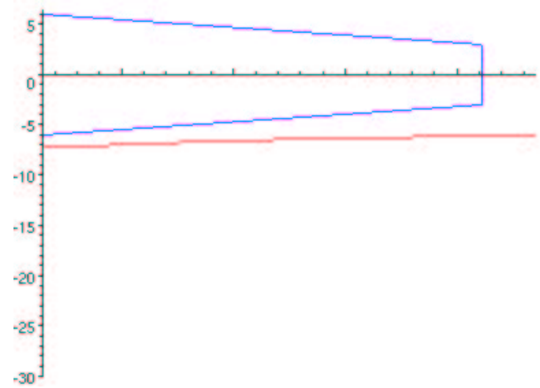


Fig. 12. Monotonic increasing convex curve (case b).

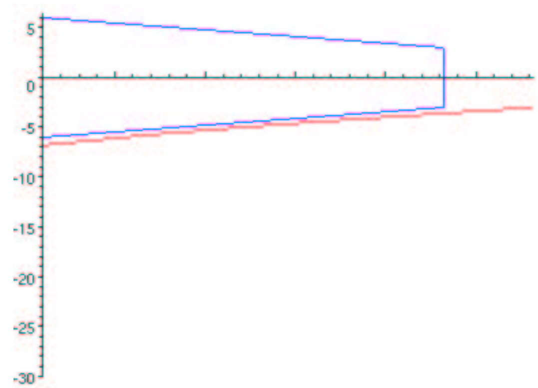


Fig. 13. Monotonic increasing convex curve (case c).

system then it is definitely below the line, if not then it may or may not be below the line.

These tests were fairly easy to formally prove for numeric examples, given that the prover had sufficient support; however, to prove the general case would have been rather more difficult.

The next step was to look at open-loop Laplace transforms with a numeric numerator k and a cubic denominator of the form s^3+bs^2+cs+d. By examining these curves it became apparent that it would be more complex to prove the properties about them — these curves, though mostly monotonic increasing, are more likely than quadratics to change from convex to concave, and this can happen within the region of interest.

Proofs for Laplace transforms with numeric parameters, which do not have points of inflection within the region of interest, are still practicable. The cases where there were points of inflection within the region $(a, b]$ were split into their separate cases and tested within the relevant region — if c is the point of inflection and $a < c < b$ then cases were split into two proofs — one proof in the region $(a, c]$ and one in the region $(c, b]$. However, for this to work the *exact* point of inflection is needed for the proof to be complete.

Finding the exact point of inflection is a problem that can not easily be overcome (if at all), so investigations began into different approaches to the proofs. Investigations are ongoing into different methods for proving curves are below (or above) a line without the need for case splits.

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