Nominal Asynchronous Games Semantics and Linear Type Variables

Thomas Cuvillier

Queen Mary University of London, UK

Abstract

The aim of this paper is to give a nominal account of asynchronous game semantics, with an emphasis on names as resources. Based on them, we present fully complete models of multiplicative additive tensorial, and then linear logics. This result extends the one obtained obtained by Melliès by adding atomic variables, but also provides a nominal structure that allows for stronger relationship between the Bohm tree of the linear λ-term and the plays of the strategy. Additionally, the nominal framework enables a direct encoding of pointers into moves, development which nicely reflects on earlier work done on encoding pointers as names. The approach chosen follows the steps of the research done by Melliès on asynchronous game semantics while enriching it with names.

Keywords: Nominal Game Semantics, Tensorial Logic, Linear Logic, Asynchronous Game Semantics

1 Introduction

The paradigm of names as resources has been used extensively in games for providing semantics of imperative languages, being a key ingredient to provide a clean presentation of effects such as references, exceptions, or polymorphism [13]. On the other hand, names also appear as a key ingredient in modelling proofs via terms through the Curry-Howard isomorphism. For instance, the identity is mapped to the α-equivalence class of terms λx.x, the formula on the left of the sequent being interpreted as a name x that is bound under λ to be passed as a variable to the right hand side. However, the modelling of computation flow has, to the best of the author’s knowledge, not been modelled nominally when non-syntactical models have been considered.

The purpose of the paper is double: to emphasize the ability of names to model linear type systems, to introduce tensorial lambda calculus, and to present nominal asynchronous games semantics. This achieves the following contributions. First, it shortens the gap between syntactical and mathematical models, by making the strategies and the terms live within the same universe, allowing one to establish a stronger correspondence between the plays and the syntax. Second, it advocates the vision of type variables as typed resources, and emphasizes the ability of names to model them. Third, it extends the previous fully complete game model of linear logic [6] by adding type variables to it.

The cornerstone of our approach is the similarity between Mellies’s dialogue games [9], and former works on encoding pointers as names, notably [1] which describes game semantics in the nominal framework. In these two papers, the moves are separated into different entities, namely the value of the move itself, and the data that will relate it to other moves in the sequence. This data has often been encoded in the form of pointers, or indices [5]. In this paper, it appears under the structure of named cells (or cells). The cells introduced by opponent correspond to those names under a λ abstraction in the term. This reinforces our intuition of cells as being resources: each bound name is an asset that the term can use afterwards. We furthermore take care of the type variables by sorting the names according to what resources they represent. The non-atomic types are modelled by untyped names, just as the additive units. We introduce nominal conditions on the strategies to enforce a correspondence between terms and strategies. Notably, Player might be able to play a name of sort X only if opponent has introduced it before, mimicking a copy-cat link. This condition is enforced through the sequentiality structure associated with the strategy [9], that reflects on the links coming from the proof net associated with the strategy.

1 Email: t.d.cuvillier@qmul.ac.uk

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We also hope that this approach via names could be helpful in understanding more clearly the work achieved on resources in game semantics via payoff and their relational counter-part [11][10].

2 Nominal sets and renamings

We briefly recall some key notions of sorted nominal set theory, for a more complete introduction we advise looking [14], and [2] for renamings. Let us denote TVar = X, Y, ... a countable infinite set of atomic types and let us fix a family $A_{\text{cell}} \cup (A_X)_{X \in \text{TVar}}$ of pairwise disjoint, countable infinite sets of names. We denote $A_T$ the set of typed names $\{x : A_X\}$, and the names of $A_{\text{cell}}$ are called untyped. We write $\text{Fin}(A_X)$ (resp $\text{Fin}(A_{\text{cell}})$), for the set of renamings of $A_X$ (resp $A_{\text{cell}}$), that are, injective functions that change only a finite number of elements. We will denote names by $a, b, c, ...$, and let $f, g, ...$ range over renamings. Also, $(a, b)$ is the permutation swapping $a$ and $b$, and $[a \rightarrow b]$ is the renaming that maps $a$ to $b$.

We will call $A$ the set of all names : $A = A_{\text{cell}} \cup (\bigcup_X A_X)$, and consider $\text{Fin}(A) = \text{Fin}(A_{\text{cell}}) \oplus (\bigoplus_X \text{Fin}(A_X))$, where $\bigoplus$ is the disjoint union, and $\bigoplus$ the direct product. We call name permutations the bijective elements of $\text{Fin}(A)$, and their set $\text{Perm}(A)$.

**Definition 2.1**

Given $M$ a monoid, a $M$-set $S$ is a set $S$ together with a monoid action $M \times |S| \rightarrow |S|$. We briefly recall some key notions of sorted nominal set theory, for a more complete introduction we advise looking [14], and [2] for renamings.

**Definition 2.2**

A nominal renaming set (resp nominal set) $S$ is a $\text{Fin}(A)$-set (resp $\text{Perm}(A)$) such that each element of $|S|$ has a finite support set.

As supporting sets intersect, we write $\nu\gamma(x)$ for the minimal supporting set of $x$, named support of $x$. We also write $\nu\gamma(x) = y$ for $\nu\gamma(x) \cap A_T = y$ and $\nu\gamma(x) = A_{\text{cell}}$ for $\nu\gamma(x) \cap A_{\text{cell}}$. Given an element $x$ in a nominal set, we denote $\{x\}$ its orbit, defined by $\{x\} = \{\pi \cdot x \mid \pi \in \text{Perm}(A)\}$. Given two elements $x, y$ of $S$, we write $x \neq y$ if they have disjoint support ($\nu\gamma(x) \cap \nu\gamma(y) = \emptyset$). We say that two elements $x, y$ are equivalent, written $x \equiv y$ if there is a permutation $\pi$ such that $\pi \cdot x = y$. Similarly, we will sometimes use notations $\equiv_T$, $\equiv_{\text{cell}}$ meaning that there are permutations of $A_T$ (respectively $A_{\text{cell}}$) equalising the two elements. They are congruent, written $x = y$ if there are two renamings $f, g$ such that $f \cdot x = g \cdot y$. A function between two nominal sets is equivariant if $\forall x, \pi, \nu\gamma(\pi \cdot x) = \pi \cdot f(x)$. In this case, one can establish that $\nu\gamma(f(x)) \subseteq \nu\gamma(x)$.

Given a nominal set without additional structure, we define the relation $\equiv$ of compatibility for equality as follows : $a \equiv b$ if $a \equiv b$. We say that a nominal set $\Sigma$ is orbit finite if there is a finite (non-nominal) subset $T \subseteq |\Sigma|$, such that $T$ is finite, and $|T| = |\Sigma|$. In this paper, we will restrict our constructions to orbit-finite sets, that will be enough for our purposes. In most cases elements will be list of names, and the support of the element will be the set of names it contains. Furthermore, the actions from $\text{Fin}(A)$ will be evident element-wise actions.

3 The logic of Games

Tensorial logic has been introduced by Mellies [11] following his discovery of a fully complete game model of linear logic [6]. Tensorial logic was designed to be, somehow, the logic of games. If linear logic is often seen as static, then tensorial logic is its dynamic counterpart. We here introduce a variant of focalized tensorial logic where we take advantage of the the isomorphisms coming from associativity to define global connectives $\oplus$ and $\otimes$, and the isomorphisms coming from distributivity and monoidal units ($A \otimes I = A$) to restrict the types. That is, in our system 1 never appears next to a $\otimes$, neither is a $\oplus$ directly below a $\otimes$. Our sequent calculus $\text{Term}_{\text{sc-glob}}$ relies on similar ideas as those that served as foundations for ludics, except that we keep separate the additive connectives from the multiplicative ones. We introduced it as a typing system for the tensorial $\Lambda$-calculus, which can be seen as a fragment of the linear $\Lambda$-calculus [15] tailored for tensorial logic, where the binder $\Lambda$ is replaced by $\sim$. The types / propositions of our system are those coming from the following grammar:

$$N = \sim P \quad P = \bigoplus_i R_i \quad R = \bigotimes_i (N + X)_i$$

where $X \in \text{TVar}$. Following standard conventions, we will write $0$ for $\bigoplus_0$ and $I$ for $\bigotimes_0$. The formulas of $N$ are called negative whereas those of $P$ are deemed positive. We see a formula $\Gamma$ of $R$ as $\bigoplus_i F_i$ and hence deems it positive, just as we write $X$ for $\bigoplus_i \bigotimes_i X$. Similarly, a negative formula might be of the form $\sim \bigoplus_i \bigotimes_i \sim F_i$, which corresponds to a double-negation. We equip our terms with a nominal structure, that is, each variable is a name, and the action of permutations on them are defined by straightforward induction. The grammar of terms is presented below:

$$\begin{align*}
\text{TE} \ni & t, u := x \mid fu \mid \sim, t \mid \bigotimes_i t_i \mid \text{let } z \text{ be } \bigoplus_i x_i \text{ in } t \mid i n_i t_i \mid \text{ case } t \text{ of } \bigoplus_{i=1,n} (i n_i t_i) \mid \text{ abort }
\end{align*}$$

Typing context $\Gamma := \emptyset \mid x : T, \Gamma$ where $x \notin \Gamma$ where $x \in A_T$, and $f \in A_{\text{cell}}$. Given $F = F_1, ..., F_n$ a list of formulas, and $\Gamma$ a context, we write $\Gamma : F$ for $\Gamma = x_1 : F_1, ..., x_n : F_n$. In the following, $Nl$ denotes a list of $N$-formulas, $\Sigma$ a list of positive literals, $Nl, \Sigma$ the concatenation of
both lists, and $N, R, P$ are formulas as above, and $F$ represents a formula.

\[
\begin{align*}
\Gamma : \Pi, \Xi, x : F & \vdash t : F & \text{Left} & \rightarrow, f \in \mathcal{A}_{\text{cell}}, f \# \Gamma & \Gamma : \Pi, \Xi, x : F & \vdash t : \perp & \text{Right} - \\
\Gamma : \Pi, \Xi, x : F & \vdash \neg t : F & \text{Right} & \Gamma : \Pi, \Xi, x : F & \vdash \neg x, t : \neg F; &
\end{align*}
\]

For linear logic where we write $\neg$ for linear connectives.

\[
\begin{align*}
\Gamma : \Pi, \Xi, x_1 : F_1, ..., x_m : F_m & \vdash t : F & \text{Left} & \left(\bigotimes_i x_i \right) & \Gamma : \Pi, \Xi, x_1 : F_1, ..., x_m : F_m & \vdash t : F & \text{Right} & \bigoplus_i \left(\bigotimes_i x_i \right) &
\end{align*}
\]

Similarly, one can design a laxed version of the tensorial lambda-calculus presented here, and define those that are well-typed in $\text{Tens}_{\text{loc}} \otimes \text{glob}$ to be in normal form. Following a similar reasoning, any term of this laxed version of the tensorial calculus would be equivalent (up to isomorphism) to one in normal-form, and the associated rewriting system would be confluent. One can define $\alpha$-equivalence between terms the usual way, and we write $\mathcal{L}$ for the equivalence class of $t$. Note that each term of type $\neg F$ of our calculus has the following form:

\[
\neg x, \text{case } x \vdash \mathcal{L}_{\{i \mid i \neq 0\}}(t_0)
\]

where we write abort for case $t$ of $\mathcal{L}_{\emptyset}$. Focalized tensorial logic with global connectives prevents commutation of rules. Therefore its equational theory becomes straightforward.

**Proposition 3.1** • Any proof of tensorial logic can be canonically mapped to an equivalent (up to isomorphism between types) proof of $\text{Tens}_{\text{loc}} \otimes \text{glob}$.

- Two proofs of $\text{Tens}_{\text{loc}} \otimes \text{glob}$ are equivalent if and only if they are equal.

Similarly, one can design a laxed version of the tensorial lambda-calculus presented here, and define those that are well-typed in $\text{Tens}_{\text{loc}} \otimes \text{glob}$ to be in normal form. Following a similar reasoning, any term of this laxed version of the tensorial calculus would be equivalent (up to isomorphism) to one in normal-form, and the associated rewriting system would be confluent. One can define $\alpha$-equivalence between terms the usual way, and we write $\mathcal{L}_{\{i \mid i \neq 0\}}$ for the equivalence class of $t$. Note that each term of type $\neg F$ of our calculus has the following form:

\[
\neg x, \text{case } x \vdash \mathcal{L}_{\{i \mid i \neq 0\}}(t_0)
\]

where $t_0$ is as follows:

\[
t_0 = \text{let } x = (x_1, \ldots, x_m) \vdash f \in \mathcal{A}_{\text{cell}}, f \# \Gamma
\]

where $f \in \mathcal{A}_{\text{cell}}$, and each $t_i$ is also a term of negation type, or $t_i$ is a name for $A_T$. This can be, from a game semantics point of view, examined as follows: opponent brings a term corresponding to $t$, this one consists of a choice of a branch (case $x$ of), and, for this branch, of a product of terms of appropriate type (let $x = (x_1, \ldots, x_m)$ in). Proponent answers by bringing a positive formula $P$, denoted by $f$ in the term, and, for this formula, makes a choice of a branch $(\mathcal{L}_{\emptyset}, i_k, \mathcal{L}_{\{i \mid i \neq 0\}})$, and brings appropriate terms for the tensor product.

Linear logic can be defined as a special kind of tensorial logic where the negation is involutive. Consequently, the negation of a negative formula becomes a positive formula, as opposed to tensorial logic. We build a sequent calculus for linear logic $\text{Lin}_{\text{loc}} \otimes \text{glob}$ following a similar recipe as for $\text{Tens}_{\text{loc}} \otimes \text{glob}$. The grammar for formulas is as follows:

\[
\begin{align*}
N & = \emptyset, O | X^{\bot} & O & = \emptyset, P | \bigoplus_i R_i & \bigotimes_i \emptyset & = X & P & = \bigoplus_i R_i & X \otimes \emptyset & = \bigotimes_i N_i
\end{align*}
\]

where $X \in \mathcal{V}ar$. $P$-formulas are positive, and $N$ are negative. Note that $T$ corresponds to $\emptyset$ and $\bot$ as $\emptyset_0$. In the following, $\mathcal{P}L$ denotes a list of $P$-formulas as above, $\Xi$ a sequence of negative literals, and $N, O, R, P$ formulas as above. In the following, $\mathcal{P}L$ denotes a list of $P$-formulas as above, $\Xi$ a sequence of negative literals, and $N, O, R, P$ formulas as above.

\[
\begin{align*}
\vdash X^{\bot}, X & \quad \text{Ax} & \vdash \mathcal{P}L, \Xi, P & \quad \text{foc} & \vdash \mathcal{P}L, N & \quad \text{unfoc} & \\
\vdash \mathcal{P}L, P_1, \ldots, P_m, \emptyset & \quad \emptyset & \vdash \mathcal{P}L, \emptyset, N & \quad \emptyset & \vdash \mathcal{P}L, \Xi, N & \quad \emptyset & \vdash \mathcal{P}L, \Xi, N
\end{align*}
\]
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The sequent calculus is similar to the one of tensorial logic, except that the left-right negation rules have been replaced with focus / unfocus rules. We recall the focalized translation, that relate proofs of linear logic with proofs of tensorial logic. Given a positive formula of linear logic \( P \) its translation into tensorial logic \( (P)^F \) is defined as follows:

\[
P = X \Rightarrow (P)^F = X = \bigoplus_i \bigotimes_j N_{ij} \Rightarrow (P)^F = \bigoplus_i \bigotimes_j (N_{ij})^F
\]

The axiom \( \cdot \) is denoted by \( \alpha \overline{\cdot} \), and \( \overline{\cdot} \) is the identity linear relation. A category \( C \) is a category with object nominal sets of separated lists of names and as morphisms \( A \rightarrow B \) is defined as follows:

\[
A \rightarrow B = \{L_1, L_2 \mid L_1 \in A, L_2 \in B, L_1 \# L_2 \}
\]

We choose to model atomic variables by separated lists, that seem to be suitable candidates to correspond to atomic formulas and units, and a finite number of categorical operations that are the analogues of the ideas to model linear logic. We present a concrete category where each object can be formed from basic building blocks, for instance the identity relation. Formally, our objects are nominal sets of list that are coming from the following grammar:

\[
L ::= a | \text{inl}(L) | \text{inr}(L) | L_1 \# L_2 \quad \text{if } L_1 \# L_2
\]

where \( a \in A \). We say that a list of names \( x_1, \ldots, x_n \) is separated if \( i \neq j \) and \( x_i \neq x_j \).

**Definition 4.1** NLRel\((A, B)\) is the category with object nominal sets of separated lists of names and as morphisms \( A \rightarrow B \) nominal linear relation, that are, relations \( R \) such that \( (a, b) \in R \) then \( \nu(a) = \nu(b) \).

Linear nominal relations compose as relations and the identity relation is linear nominal. The category NLRel can be equipped with a monoidal product, written \( \star \), and referred to as separated concatenation.

- \( A \star B = A \star B = \{L_1, L_2 \mid L_1 \in A, L_2 \in B, L_1 \# L_2 \}
- \( R_1 \star R_2 = (a, b, c, d) \rightarrow (a, c, d) \rightarrow (a, b, d) \rightarrow R_2, a \# b \wedge c \# d \)

The unit of the monoidal product \( I \) is the set whose single element is the empty list \( e \). The \text{inl}, \text{inr} constructors are called patterns, and will allow us to form coproducts, denoted by \( \oplus \).

- \( A \oplus B = \text{inl}(A) \cup \text{inr}(B) = \{L \mid L \in A \cup \text{inr}(L) \mid L \in B \}
- \( R_1 \oplus R_2 = (\text{inrL}(L_1, L_2) \mid L_1 \in R_1, L_2 \in R_2, L_1 \in A, L_2 \in B) \cup (\text{inrL}(L_1, L_2) \mid L_1, L_2 \in R_2, L_1 \in A, L_2 \in B) \)

where \( \cup \) expresses the union of disjoint sets. The unit of the coproduct \( 0 \) is the empty set \( \emptyset \). The category (NLRel,\( \oplus, I, \oplus, 0 \)) is a monoidal category with coproduct, with tensor product distributing over the coproduct. As a result, NLRel defines a model of the (\( \otimes, \oplus \)) fragment of MALL, where \( [1] = I, [X] = \Lambda X, [0] = 0 \) and the following linking.

Each element of a morphism of NLRel between denotations of (\( \otimes, \oplus \))-formulas defines an additive resolution (a choice for every \( \beta \) of a left or right subformula) together with a linking on it, that is, a bijective function (ie, a permutation) from the literals of the domain to the codomain. For instance, an element \((a_1, a_2, a_2, a_1) \) of \( X \otimes X \rightarrow X \otimes X \) defines the following linking.
Tracing

We would like to build a model of linear logic from NLRel. A way to do so is the Int construction that results in a compact-closed category, that is the simplest instance of star-autonomous category. In order to do so, we need to equip NLRel with a trace structure. We remind here that, given a category C, a trace on C is a natural family of functions $\text{Tr}^{C}_{A,B} : C(A \otimes U, B \otimes U) \to C(A, B)$ satisfying some additional constraints that we not present here (see [4]). Taking 𝑅 as above, defined by $(a_1,a_2)\mathcal{R}(a_2,a_1)$, then defining $\text{Tr}^{X_{1},X_{2}}_{\mathcal{R}}(\mathcal{R})$ (where $X_2$ are the right hand side $X$’s), requires us to equate $a_2$ and $a_1$, but the element $(a_1,a_1,a_1,a_1)$ is not part of the relation. In order to bypass this, we close the relation under renamings.

We introduce the following notation. Given a well-typed set $A$ of annotated separated lists, we write $\hat{A}$ for the corresponding set of annotated lists such that the separation property is dropped. Given a relation $\mathcal{R} : A \to B$, then we set: $\hat{\mathcal{R}} = \{(f \cdot l_1, f \cdot l'_1) \mid f \in \text{Fin}(A), (l_1, l'_1) \in \mathcal{R} : \hat{A} \to \hat{B}\}$. For an element $x \in A$, we write $x \in \hat{A}$ if $x$ belongs to the image of the embedding $A \to \hat{A}$. We can define the trace of a linear nominal relation $\mathcal{R} : A \otimes C \to B \otimes C$ by:

$$\text{Tr}^{C}_{A,B}(\mathcal{R}) = \{u \upharpoonright A \times B \mid u \in R \cup u \upharpoonright \hat{A} \times \hat{B} \in A \times B \wedge u \upharpoonright \hat{C} \times \hat{C} \in \text{id}_C\}.$$  

The use of renamings in tracing allows us to equate names in the internal part, while keeping the separation property on the external one. We prove that this trace extends the usual one on the category of permutations.

**Proposition 4.2** $\text{Tr}^{C}_{A,B}$ defines a trace on the category NLRel.

Nominal Linear polarized relations

Thanks to the previous construction, we can build a compact closed category starting from nominal linear relations. The Int construction proceeds in adding polarities to the category. The category that we present here is isomorphic to the result of the Int construction, where we have taken advantage that it is sufficient to adjoin polarities to the building blocks of the category. Formally, the objects of NLPRel are nominal sets of elements coming from the following grammar.

$L := (a,p) \mid \text{in}(L) \mid \text{inr}(L) \mid L_1 \times L_2$

where $a \in A$, and $p \in \{-1, 1\}$, and $L_1\#_{\text{pol}}L_2$. The $p$ next to each atom is called the its polarity. Given a list $L$, we define by $\text{Pos}(L)$ the restriction of $L$ to its atoms of positive polarity. We write $L_1\#_{\text{pol}}L_2$ if $(\text{Pos}(L_1)\#\text{Pos}(L_2)) \wedge (\text{Neg}(L_1)\#\text{Neg}(L_2))$, and note $\star_{\text{pol}}$ the associated polarized separated concatenation operation:

$$A \star_{\text{pol}} B = \{L_1 \times L_2 \mid L_1 \in A, L_2 \in B, L_1\#_{\text{pol}}L_2\}.$$  

We can define the operation $(.)^+$ on lists, that consists in inverting the polarity of every atom present in the list. We extend this operation to sets: $A^+ := \{L^+ \mid L \in A\}$.

**Definition 4.3** NLPRel is the compact closed category with objects annotated polarized separated lists of atoms, and as morphisms $\mathcal{R} : A \to B$ nominal relation $\mathcal{R} \subseteq A^+ \star_{\text{pol}} B$ such that $\forall x \in \mathcal{R}, \nu(\text{Pos}(x)) = \nu(\text{Neg}(x))$.

Note that the $\star_{\text{pol}}$ extends directly to morphisms, looking at them as subsets. The coproduct $\oplus$ lifts from NLRel to NLPRel. Furthermore, since it distributes over the tensor, it forms, as proven in [3] a biproduct. Hence the category is compact-closed with products, that is, a degenerate model of MALL. The previous denotation function $\llbracket \rrbracket$ extends straightforwardly for MALL and NLPRel, by assigning $X$ to $\{(a,1) \mid a \in A_X\}$ when $X$ in TVar. The interpretation of proofs follows as well from the star-autonomous structure of NLPRel. For instance, the following proof-structure

$$\sigma : \text{X} \otimes \text{X}^\perp \rightarrow \text{X} \otimes \text{X}^\perp$$  

is modelled by $\mathcal{R} = \{(a,-1), (b,1), (a,1), (b,-1)) \mid a,b \in A_X, a \neq b\}$. We can also curry the morphisms, like $Q : X \otimes X^\perp \rightarrow 1$, the curried version of the identity $Q = \{(a,-1),(a,1)\} \mid a \in A_X$. Polarized separated relations do not compose as relations. This follows from the definition of trace using closure under renamings. For instance, the relational composition of $\mathcal{R}, \mathcal{Q}$ leads to the empty relation, as in $\mathcal{R}$ the resources $a,b$ are distinct, whereas they are equated in $\mathcal{Q}$. A simple computation following the Int definition yields:

$$\mathcal{R} \cdot Q = \{r \in \mathcal{R}_{\text{Rel}}\hat{Q} \mid r \in A \star_{\text{pol}} C\}$$  

Therefore, the composition of separated, polarized nominal relations behaves according the the following paradigm: closure under renaming + relational composition + projections on separated elements.
5 Nominal Structures

The goal of this section is to present nominal structures that extend the current ones already established for games for tensorial logic, by translating them within a nominal universe. Dialogue games were introduced in [9], as the objects supporting the game semantics of tensorial logic, as well as basic structure underlying the free dialogue category. We devise their nominal sibling, providing us with the appropriate arenas to let the strategies play with names. Afterwards, we will enforce them to capture the required linearity between negative and atomic type variables, and hence incorporate atomic variables in the already established semantics of tensorial logic.

5.1 Nominal trees and dialogue games

Nominal trees

Dialogue games form a special class of directed rooted trees. We define nominal trees by mimicking the set theoretic definitions within nominal sets.

Definition 5.1 A nominal graph \((V, E, f)\) consists of two orbit finite nominal sets \(V\) and \(E\), called vertices and edges, and a nominal function \(f : E \rightarrow V \times V\) (written \(v_1 \xrightarrow{e} v_2\) for \(f(e) = (v_1, v_2)\)).

Note that our graphs are all directed. Given an edge \(e : u \rightarrow v\) of a nominal graph, one can prove that \(v(e) = v(u) \cup v(v)\). Within the category of sets, a rooted tree (abbreviated tree in the future) is a graph in which there exists a node \(u\), called the root, such that for any vertex \(v\), there is exactly one path from \(u\) to \(v\).

Definition 5.2 A nominal-tree is a nominal graph that is a tree.

Just as their non-nominal counterpart, there is a equivalence between trees and a subclass of partial orders. A nominal partial order (nposet \((S, \preceq)\)) is a poset such that the \(\preceq\) relation is nominal.

Proposition 5.3 Let \(T\) be a nominal tree. Then \(T\) gives rise to a nposet \((S, \preceq)\) such that the down-closure of each element is well-ordered. Reciprocally, let \((S, \preceq)\) be a nposet such that the down-closure of each element is well-ordered. Then this one provides a description of an A-nominal tree.

Nominal trees are conservative, meaning that for any pair of vertices \(u, v\) such that there exists an edge \(e : u \rightarrow v\), then \(v(u) \subseteq v(v)\); the names are propagated from the root upwards. Therefore, we consider that it makes sense to consider trees where vertices are lists. We call such trees structured. Given \(T\) a nominal set of finite lists, we define its closure by \(\text{Clo}(T) = \{l | \exists l_2 \in T, \exists l_1 = l_2\} \) together with the prefix ordering. Then \(\text{Clo}(T)\) is a structured nominal tree.

Let us note that structured trees are closed under unions. Given two vertices \(v = v_1, v_2, ..., v_m\), \(v' = v'_1, v'_2, ..., v'_n\) in a structured trees, we refine the relation of compatibility for equality to \(v \equiv v'\) if \(\forall i \leq m, n. v_i \equiv v'_i\). Given sets \(A, B\), we write \(A.B\) for the set of lists \([a.b | a \in A, B \in B]\).

Nominal Dialogue Games

Given a list \(l\), we write \(\Gamma_l\) for the last element of the list. Given a structured nominal tree \(A\), we write \(\overline{A}\) for its set of lists that are non empty, and pattern for finite sequences of inr, inr.

Definition 5.4 A nominal dialogue game is a structured nominal tree \(A\) whose vertices, except for the root, are either labelled cells or values (that is, with a function label : \(\overline{A} \rightarrow \{\text{cells}, \text{values}\}\), and such that \(x \uplus y \Rightarrow \text{label}(x) \neq \text{label}(y)\). Each cell \(c\) is such that \(c^3\) is of the shape pattern(\(\alpha\)) (often written \(\alpha\) for short), where \(\alpha \in \lambda\). Furthermore, this graph is equipped with a function \(\Lambda : \overline{A} \rightarrow \{-1, 1\}\), called polarity, such that for each cell \(\alpha\) and value \(v\):

\[
\alpha \uplus v \Rightarrow \Lambda(\alpha) = \Lambda(v) \quad v \uplus \alpha \Rightarrow \Lambda(\alpha) \neq \Lambda(v)
\]

We request that the initial vertices are cells that consists of a unique name members of \(\mathbb{A}_{\text{cell}}\), in particular, all initial cells are equivalent. A dialogue game is said to be positive if all the initial cells are of positive polarity, and negative if all the initial cells are of negative polarity.

The values and cells of positive polarity are those belonging to proponent, often called player, whereas those of negative polarity are related to opponent. In this graph, each cell represents a question, that is, a request of data, or simply a variable in the tensorial calculus. On the other hand, the values represent the answer to the question; this translate into branching information \(\mathcal{I}_q\) in the tensorial calculus. For historical reasons, the polarity of the cell does not correspond to the protagonist asking the question, but the protagonist answering it. That is, proponent will asks questions, or bring cells, of polarity \(-1\), but will answer questions of polarity \(1\). We write that a cell is typed if the support of its last element belongs in \(\mathbb{A}_T\), and untyped if it belongs in \(\mathbb{A}_{\text{cell}}\).

Denotation of formulas

We denote each formula of TENS by a positive dialogue game. Negative dialogue games will be later used to denote the arena on which the strategies play. As the polarity function and label function is, for each vertex of the dialogue
game, computed following its distance from the root, we omit their description. We remind the graphical representation introduced in [11] for dialogue games. The values are represented as filled circles:

Whereas the cells are drawn as smaller, plain circles.

The interpretation of formulas is given below. First, we define the sum of simple dialogue games, that are those such that each initial cell justifies a unique value. The sum is defined by merging their initial cell, while keeping values distinct. This is a slight variant of the rooted sum. Note that we draw a ellipse to denote a set of cells.

Formally, given \( n \) dialogue games with unique initial values \( A_i \), we define \( \bigoplus_i A_i \) by:

\[
\bigcup_i \{ \alpha \in (v_i, l) | \alpha \cdot v_i \cdot l \in A_i \}
\]

The second step consists in defining the negation. First, given a dialogue game \( A \), we define \( A^* \) as being just like \( A \) but with a reversed polarity function, \( \lambda \cdot A^* = -\lambda \cdot A \). To recover a positive game, the negative dialogue game is lifted by a cell and value. This is the analogous as writing \( \neg \bigoplus_1 \bigotimes_1 A \) for \( \neg A \).

The tensor product of two games \( A \) and \( B \) is defined by merging their initial cells and considering a concatenation of their values. For instance, we display the product in the case of two dialogues games having two initial values.

Formally, given \( n \)-dialogue games \( A_i \), the definition of the tree \( \bigotimes_i A_i \) is:

\[
\bigcup_i \text{Clo}(\{ \alpha \in (v_i, l) | \alpha \cdot v_i \cdot l \in A_i \\})
\]

We now give the interpretation of the units. Those are designed to make the required equations \( A \oplus 0 = A \) and \( A \otimes I = A \) hold. The game \( I \) carries as single value the empty list \( \{\} \). The game for 0 consists of a set cells \( \{0\} = \text{Clo}(\{\}) \). The remaining units \( \top \) and \( \bot \) are defined through the equations \( \top = \neg 0 \) and \( \bot = \neg I \). Finally, we give the interpretation of an atomic type \( X \). We think of it as a typed resource brought by proponent. Therefore, we define \( \{X\} = \text{Clo}(\{\}, A_X) \).

Finally, we denote any formula of \( \text{Tens} \) by induction following \( I \cdot A = \bigoplus \{ A \}, I \bigotimes A = \bigotimes \{ A \}, \neg A = \neg \{ A \} \).

5.2 Nominal Event Structure

In this section, we present the definition of nominal event structures. This will enable us to associate to each dialogue game an event structure, through the intermediate description of moves. Relying on it, we will define the set of positions.

**Definition 5.5** A nominal event structure \( E = (|E|, \leq, \sim) \) consists of a nposet \((|E|, \leq)\) together with a symmetric ir-reflexive nominal conflict relation \( \sim \) such that \( m \sim n \) and \( n \leq p \Rightarrow m \sim p \). A nominal event structure is linear if
Let \( p \) be a position in an orbit finite linear event structure \( E \). Then
\[
\forall\, e \colon e \leq e' \implies e \sim e' .
\]

We restrict to orbit finite event structures, that are those whose nominal sets \( |E| \) are orbit finite. We write \( \uparrow \) for the complement of the conflict relation: \( e \uparrow e' \iff \neg(e \sim e') \). If \( e \uparrow e' \), we say that \( e, e' \) are compatible. In particular, if \( e \leq e' \) then \( e \uparrow e' \). To simplify, we write \( e \in E \) for \( e \in |E| \). For orbit finite linear event structures, the axiom of finite causes holds automatically: \( \forall e \in E. |e| \leq e \) is finite. Just as trees, linear event structures are conservative.

**Proposition 5.6** Let \( E \) be a linear event structure, and \( e, e' \in E \) such that \( e \leq e' \). Then \( v(e) \subseteq v(e') \).

We introduce the notion of position. We recall that in a partially ordered set \( P \), given \( D \subseteq P \), we write \( D \downarrow \) for the downward closure of \( D \), defined as \( \{ p \in P \mid \exists d \in D. p \leq d \} \).

**Definition 5.7** A position \( p \) of a nominal event structure is a finitely supported set of events that is conflict-free \( \forall e, e' \in p. e \uparrow e' \) and downward closed: \( p = p \downarrow \).

We write \( \text{Pos}(E) \) for the nominal set of positions of \( E \). We define the actions of renaming on \( \text{Pos}(E) \) as follows: \( f : p = \{ f \cdot e \mid e \in p \} \). In an orbit finite linear event structure, the positions have a useful representation.

**Proposition 5.8** Let \( p \) be a position in an orbit finite linear event structure \( E \). Then \( \exists e_1, \ldots, e_n \in E \) such that \( p = \{ e_1, \ldots, e_n \} \).

### 5.3 Nominal event structure associated with dialogue games

Following [9], we wish to associate to each dialogue game of section 5.1 an event structure, where each event corresponds to a move. The conflict relation is designed in such a way that a position can only comprise moves coming from one side of the additive connective \( \oplus \), while allowing the exploration of both sides of the multiplicative one \( \otimes \).

**Definition 5.9** Given a dialogue game \( A \), \( M_A = (M_A, \leq_A, \sim_A) \) is a nposet with a relation \( \sim_A \) on \( M_A \), called conflict.

- The set \( M_A \) consists of all triples \((\alpha, v, S)\), where \( \alpha, v \) are nodes of \( A \) and \( \alpha + v \), label(\( \alpha \)) = cell. \( S \) is a representative of \( \{ s \mid v \vdash s \} \), and therefore is a set of cells. We will call moves the elements of \( M_A \).
- We define the justifying relation between moves by \( m = (\alpha, v, S) \vdash_A (\alpha', v', S') \iff \alpha' \in S \). In this case we say that \( m \) justifies \( m' \), written \( m \vdash m' \). A move is initial if it is not justified by any move. We write \( \leq \) for the reflexive transitive closure of \( \vdash \), leading to a partially ordered set \((M_A, \leq)\).
- We define the relation of compatibility for equality, written \( \equiv \), by \( (\alpha, v, S) \equiv (\alpha', v', S') \iff (v \equiv v') \wedge (\forall c \in S. \forall c' \in S'. c \equiv c') \).
- The conflict relation between two moves is defined by \( m \sim_A m' \iff ((m \equiv m') \implies (\text{label}(v \cap v') = \text{cell})) \), where \( m = (\alpha, v, S) \) and \( m' = (\alpha', v', S') \).

A move \((\alpha, v, S)\) answers the question \( \alpha \) with the value \( v \), and brings forth \( S \) new questions. It is important to notice that \((M_A, \leq_A, \sim_A)\) is almost an event-structure, but not quite. The root of the problem lies in the fact that \( m_1 \leq m_2 \) does not necessarily entails \( \forall (m_1) \leq \forall (m_2) \). For instance, let us consider the figure 5.3. Let us pick two initial moves \( m_1 = (\alpha_1, v_1, \{1 \in C_1, 2 \in C_2\}) \) and \( m_2 = (\alpha_1, v_1, \{1 \in C_1', 2 \in C_2\}) \) such that \( C_1 \neq C_1' \). Finally, we consider a third move \( m_3 = (\{1 \in C_3, 2 \in C_2\}) \). Then we have \( m_1 \sim m_2 \) as they are not compatible for equality since \( 1 \in C_1 \neq 1 \in C_1' \). On the other hand \( m_1 \vdash m_3 \) and \( m_2 \vdash m_3 \), so \( m_1 \sim m_3 \) and \( m_2 \sim m_3 \); the conflict relation does not satisfy the axiom of event structure. To solve that, we consider events to be lists of moves.

**Definition 5.10** Given a dialogue game \( A \), and its associated set of moves \((M_A, \leq_A, \sim_A)\), we define the event structure \( \text{Event}(A) = (|E_A|, \leq_A, \sim_A) \) by overloading the notations \( \leq_A, \sim_A \) as follows:

- A nominal event \( e \in E_A \) is a list \( m_1, m_2, \ldots, m_n \) where \( m_i \in M_A \), \( m_1 \vdash m_2 \vdash \cdots \vdash m_n \), and \( m_1 \) is an initial move.
Two events are compatible for equality if their moves are: $e_1 = m_1, ..., m_n \vdash e_2 = n_1, ..., n_k$ if $\forall i \leq n, k, m_i \vdash n_i$.

Two events are in conflict $e \sim e'$ if $(e \vdash e') \Rightarrow r e \equiv r e'$.

As expected, $e \leq e'$ if $e$ is a prefix of $e'$.

The triple $\text{Event}(A)$ is a nominal event structure. Furthermore, let us note that the polarity function extends to the event structure. Given $r e \equiv m = (a, v, S)$, we set: $\lambda(e) = \lambda(m) = \lambda(v) = \lambda(a)$. The positions of $\text{Event}(A)$ form a prime algebraic domain that we denote $(\text{Pos}(A), \sqsubseteq, \bot)$. The positions of the dialogue game $A$ can be seen as (not-equivariant) subtrees of $A$, subject to some additional properties, such as, the leaves of the sub-tree can be values only if they do not justify cells in the original tree. Despite the fact that the set of moves does not form an event structure, it nevertheless seems simpler to work with moves rather than events. This proposition helps us achieve that.

**Proposition 5.11**: A move-position of $M_A$ is a set of moves $p$ such that $p$ is conflict free $\forall m, m' \in p, \neg((m \sim m'))$ and $p$ is almost downward closed $\forall m \in p, \forall m' \leq m, \exists n, n' \in p$. In that case, as $\neg(n' \sim m)$ then $n' \leq m$. Then there is a correspondence between the sets of positions and the set of move-positions.

Therefore, in the future, we will refer to move-positions as simply positions. One can notice that these games are actually fairly close to the nominal games from [1], where a move was also defined as a triple whose first and final element were names. However, this time, because of the linearity, we restrict the amount of threads that can depart from a move.

### 5.4 Nominal Asynchronous Games

We could straight away define a graph from the event structure having as nodes the positions of the event structures, and edges the events, or moves. However, this graph is not totally well-fitted for our setting. Indeed, in it, two untyped cells present in a position could share the same name, which might be an unwanted configuration, as it would correspond to a term with a name that appears under a $\lambda$-abstraction in two different places. To prevent that, we restrict the set of positions. We write $\forall_\text{cell} w$ if $\forall (v \lor v')(w) \in _A \Rightarrow \emptyset$.

**Definition 5.12**: A set of moves $p$ is legal if $\forall m, m' \in p$, such that $m = (a, v, S), m' = (a', v', S')$, then $\forall s \in S, s' \in S'. s \not\equiv s' \Rightarrow r s \sqsubseteq \forall_\text{cell} r s'".

Legality is a too strong condition to capture precisely $\alpha$-equivalence. However, its strength is harmless for a definability-soundness result. To ensure that two positions are name compatible for union, we introduce a "post-compatible" relation $C_{\text{post}}$, that selects those legal positions whose joints are legal. That is, $p C_{\text{post}} p'$ if $p \not\equiv p'$ and $p \cup p'$ is legal.

**Definition 5.13**: Given a dialogue game $A$, and its associated event structure $\text{Event}(A)$, the nominal graph $\text{graph}(A)$ is defined as:

- having vertices the legal positions of $A$.
- having edges $x \rightarrow y$ every time there is a move $m$ such that $y = x \cup \{m\}$. In that case, we write $r e \equiv m$.

We furthermore would like to endow our graph with a notion of homotopy between paths, to emphasize when their differences are bureaucratic. In a graph coming from an event structure endowed naturally with a notion of homotopy, any two paths that are co-initial and co-final are homotopic. Therefore, we introduce this simplified definition.

**Definition 5.14**: Given a dialogue game $A$, we will speak of the arena $A$ for the graph $\text{Legal}(A)$ together with the homotopy relation $\sim$ between paths defined as $s \sim t$ if $s, t$ have same source and destination.

We say that the arena $A$ is a nominal asynchronous graph.

### 5.5 Asynchronous Bohm Graph

Just as the arena $A$ relies on the intermediate description of tree associated to a formula, one can assign to each ($\alpha$-equivalence of) term a nominal tree, called Böhm tree, and this one also gives rise to an asynchronous graph. We advise [9] and [12] as references. For simplicity, we consider only the multiplicative terms (whose types do not include $\otimes$), and we write $\neg((x_1, ..., x_n)$ for $\neg(x_1, ..., x_n)$ in. We present the Böhm graph associated with the term $\neg(u, f). f(u) \otimes \neg(w \otimes g). g(w)$ of type $\neg(X \otimes \neg X) \otimes \neg(Y \otimes \neg Y)$ in figure 2.

We remind the structure of the dialogue game associated with $\neg(X \otimes \neg X) \otimes \neg(Y \otimes \neg Y)$ in figure 3. This asynchronous Böhm graph can be seen as a sub-graph of the arena denoting its type as follows:

- each opponent move $\Omega \rightarrow \neg((x_1, ..., x_n)$, in the asynchronous Böhm graph corresponds to moves $m$ such that $\neg m \equiv (a, v, (x_1, ..., x_n))$, where $a$ is a name brought by proponent in a position corresponding to $\Omega$ in the arena.
- each player move move $U_j \rightarrow f((x_1, ..., x_n)$, in the asynchronous Böhm graph corresponds to proponent moves $m$ such that $\neg m \equiv f((a_1, ..., a_n)$, where $a_1, ..., a_n$ are names of $\forall_\text{cell}$ in the arena.

The main difference between the both structures is that the $U_j$ and the $\Omega_j$ have been replaced by names in the arena. This reflects on the symmetry of the arenas, and the asymmetry of the $\lambda$-calculus. The $\lambda$-term puts emphasis on the
names brought by opponent, and reflects how the player is going to behave with those names. In a symmetric setting, the term would also need to bring new names, that correspond to the different cells opponent could play in.

The asynchronous graph associated with the term $t$ corresponds to the subgraph of the arena where the following conditions have been added:

- $O_1 \leq P_1$ and $O_2 \leq P_2$
- Equality between the typed name of $O_1$ and $P_1$: $A = \mu$ and $\rho = \chi$.

Similarly, it can be seen as a set of sequences $\{P_0,O_2,P_3,P_2,P_0,O_2,P_3,P_1\}$. The goal of the next section 6 is to precisely characterise those sets of sequences that originate from an $\alpha$-equivalence of term, or, equivalently, from a proof.

## 6 Strategies

We will present our category of games where:

- the objects are positive dialogue games
- The morphisms $A \rightarrow B$ are strategies on the negative dialogue games $A \rightarrow B = (A \otimes B)^\ast$, where $(.)^\ast$ is the operation that inverts the polarity function.

In this paper, the strategies mostly differ to the strategies of tensorial logic previously presented in the literature by their nominal name, and by the type-coherence condition, that allows them to be “logical”, although this condition is then strengthened using sequentiality structures in the sub-section 6.1.

A play in a simple dialogue game is a path in its graph such that its starting node is the empty position, written $\star$.

**Definition 6.1** Given a dialogue game $A$, a strategy $\sigma : A$ is a non-empty set of legal, balanced plays of even lengths of the arena (that is, the asynchronous legal graph) associated to $A$ such that:

- the plays are alternating,
- the strategy is closed under prefix. If $s \cdot m \cdot n \in \sigma$ and then $s \in \sigma$,
- the strategy is nominal deterministic: $\forall s_1,m_1,n_1,s_2,m_2,n_2 \in \sigma$. $s_1 \cdot m_1 \equiv s_2 \cdot m_2 \Rightarrow s_1 \cdot m_1 \equiv s_2 \cdot m_2$.
- the strategy is equivariant: $\forall s \in \sigma$, $s \equiv t \Rightarrow t \in \sigma$.

Type-coherence prevents, as the separation property in 4, resources from being mixed up. Furthermore, it keeps the strategy from playing a resource (type name) that has not been made available by Opponent.

**Definition 6.2** A strategy is **typed coherent** if it satisfies the two following conditions:

- (Frugality): Opponent never introduces twice the same typed name.
- (semi-linearity): The strategy does not introduce typed names: $s \cdot m \cdot n \in \sigma \Rightarrow \nu_T(s \cdot m) = \nu_T(s \cdot m \cdot n)$

For the next definition, it is worth to note that given two plays $s \cdot m_1 \cdot n_1$ and $s \cdot m_2 \cdot n_2$ belonging to a strategy $\sigma$, if $m_1 \not\equiv m_2$ and $n_1 \not\equiv n_2$, then there exists a permutation of $\pi \in \text{Perm}(A_{\text{cell}})$ such that $\pi \cdot (s \cdot m_1 \cdot n_1) = s \cdot m'_1 \cdot n'_1$ $C_{\text{post}}$ $s \cdot m_2 \cdot n_2$.

**Definition 6.3** A (typed coherent) strategy is **forward consistent**, if $\forall s \in \sigma$ and $m_1,m_2,n_1,n_2$ such that $s \cdot m_1 \cdot n_1$, $s \cdot m_2 \cdot n_2 \in \sigma$, $m_1 \not\equiv m_2$, $m_1 \uparrow n_1 \not\equiv n_2$. Moreover, if $m_1,n_1$ are such that $s \cdot m_1 \cdot n_1 C_{\text{post}} s \cdot m_2 \cdot n_2$ and $m_1 \cdot n_1 \not\equiv s \cdot m_2 \cdot n_2$ then we have $s \cdot m_1 \cdot n_1 \uparrow s \cdot m_2 \cdot n_2$ and $s \cdot m_1 \cdot n_1, s \cdot m_2 \cdot n_2 \in \sigma$. 

![Fig. 3. Dialogue game of the type: $\neg(\chi \otimes X) \otimes \neg(Y \otimes Y)$](image-url)
A strategy is **backward consistent** if for all cells \( m, n \in \sigma \), such that \( \neg (m \vdash \top) \) and \( \neg (n \vdash \top) \), then \( \neg (m \vdash \top) \). Let us consider the following proof: di

6.1 Sequentiality Structure

Strategies as defined now fail to distinguish between \( \neg 0 \) and \( I \): both are perceived as a single move. However, they play different roles in the proofs; \( \neg 0 \) is reflected on the additional cell that \( \neg 0 \) has compared to \( I \). In order to account for it, sequentiality structure were introduced in [9], as being set of functions associated with a strategy that link cells to the domain they capture. For instance, let us consider the following proof:

The play modelling this proof will go as follows. First, opponent will play a move introducing the cells that correspond to the context \( \Gamma, A \). Then, player will play a move bringing the cells corresponding to \( \neg 0 \) and \( A \). It will furthermore brings a function, that will map the cells of \( \Gamma \) to \( \neg 0 \), and the cell of \( A \) to \( A \). Formally, we say that a cell \( \alpha \) is **accessible** from a position \( x \in \text{Legal}(A) \) in a dialogue game if:

\[
\exists m \in x.m = (\beta, v, S), \alpha \in \Gamma^\gamma, \text{ and } \neg (\exists m' \in x.m' = (\alpha, v', S')).
\]

In other terms, the cell \( \alpha \) is maximal within the position. Given a position \( x \), we write \( \text{Acc}_x \) for the set of accessible cells, that we partition according to the polarity of the cell \( \text{Acc}_x = \text{Acc}_x^+ \cup \text{Acc}_x^- \).

**Definition 6.4** A strategy with sequentiality structure \( (\sigma, \phi) \) is an **innocent strategy** \( \sigma \) together with a family of sequentiality functions \( \phi = \{ \phi_x : \text{Acc}_x^+ \to \text{Acc}_x^- \mid x \in \sigma^* \} \) such that:

* \( \phi \) is closed under permutations: \( \phi_{\pi(x)} = \pi \circ \phi_x \).

* \( \forall s : \star \to x \overset{m, n}{\longrightarrow} y \in \sigma, \forall \alpha \in \text{Acc}_x^+ \cap \text{Acc}_y^-, \phi_x^{-1}(\alpha) = \phi_y^{-1}(\alpha) \).

And it is furthermore **well-typed** if a cell of type \( X \) can only capture a single other cell, and this one is of the same type.

\[
\forall x \in \sigma^*, \forall \alpha \in \text{Acc}_x^+, \nu(\nu(\phi(\alpha)^+) \in \mathbb{A}_X \Rightarrow \nu(\nu(\alpha^+) \in \mathbb{A}_X \land |\phi^{-1}(\phi(\alpha))| = 1).
\]
In particular, this enforces “linear” use of resources: Proponent will play not be able to play twice the same typed name. Finally, each typed name played by Opponent will either be matched to their sibling played by Proponent, or to an untyped name that will correspond to the abort program. Strategies of Game with well-typed sequentiality structure form a sub-dialogue category that we denote GameSeq.

7 Full completeness

The construction we have carried gives us an instance of the free dialogue category (whose general construction has been devised in [9]) on the category VAR, that has as objects the variables of TVar and as only morphisms the identities.

**Theorem 7.1** There is a correspondence between the equivalence classes of proofs of $\Gamma \vdash A$ and the morphisms of GameSeq($\Gamma$, $\Gamma$, $A$).

Furthermore, we can project the strategies onto nominal linear relations, and obtain full-completeness for linear logic. We remind that a model is fully-complete if it is sound and every morphism is the denotation of a proof. The projection $\text{proj}$ works in two steps:

- it sends each maximal position $x$ of the strategy to its lists of available cells $\text{Acc}_x$
- If this list contains untyped cells, then it is sent to $\emptyset$.

In [6], positions without untyped cells were called external positions.

**Theorem 7.2** Each linear polarized relation $[A] \rightarrow [B]$ that comes from the projection of a strategy GameSeq($A^\bot$, $B^\bot$) is the denotation of a proof of linear logic.

Nominal linear polarized relations are, however, not precise enough to differentiate between non-equivalent proofs of linear logic. For instance, any strategy defining a proof of $\text{MLL}_u$ (that is, the restriction of $\text{MLL}$ whose only atomic types are $\bot$, 1) will be denoted by the empty list.

References

A Proofs regarding nominal linear polarized relations

Definition A.1 The category $\text{Var}$ is the discrete category with objects $X \in \text{TVar}$, and whose only morphisms are the identities.

Proposition A.2 Let $\mathcal{R} : A \otimes U \rightarrow B \otimes U$. Let $x \in \text{Tr}_{A,B}^U(\mathcal{R})$, then $\nu(x \uparrow A) = \nu(x \uparrow B)$.

Proof. The proof is based on the above proposition ???. Let $e : x$ an element that appears as witness of the trace, and hence such that $x \in \mathcal{R}$. Then let us consider $\phi$ the morphism of $\text{NomLin}$ associated to $x$. Then $e : x$ corresponds, as explained above, to tracing $\phi$. Therefore, $e : x \uparrow A \rightarrow B$ belongs in a graph of a function $\psi \in \text{NomLin}(A \rightarrow B)$ and therefore has same support in $A$ and $B$.

We can now state with confidence that this definition of trace seems appropriate, as it is a simple extension of the one previously defined in the case without biproduct, and acts in a compatible way with linear nominal relations. Only remains to prove that this family of functions formally defines a trace.

Proposition A.3 The family of functions $\text{Tr}_{A,B}^U$ defined in ?? forms a trace of the symmetric monoidal category $\text{NLRel}$.

Proof. We start the proof with the naturality of the trace. We treat only one of the three cases, as the two others could be dealt with along the same lines. Let $\mathcal{R} : A \otimes U \rightarrow B \otimes U$ and $Q : A' \rightarrow A$. Then one must check that:

$$\text{Tr}_{A,B}^U(\mathcal{R}) \circ Q = \text{Tr}_{A',B}^U(\mathcal{R} \circ (Q \otimes \text{id}_U)).$$

The left-hand-side term unfolds to:

$$\{u \uparrow A' \times B | u \in A' \times \hat{A} \times \hat{B} \times \hat{U}, u \uparrow A' \times \hat{A} \in Q, u \uparrow (\hat{A} \times \hat{U}) \times (\hat{B} \times \hat{U}) \in \hat{R}, u \uparrow \hat{U} \times \hat{U} \in \text{id}_U, u \uparrow \hat{A} \times \hat{B} \in A \times B\}$$

whereas the second can only be easily presented if we use the functoriality of $(\cdot)$, that is, $\hat{R} \circ Q = \hat{R} \circ Q$.

$$\hat{R} \circ Q = \{u \uparrow \hat{A} \times \hat{C} | u \in \hat{A} \times \hat{B} \times \hat{C} \uparrow \hat{A} \times \hat{B} \in \hat{R}, u \uparrow \hat{B} \times \hat{C} \in \hat{Q}\}$$

as well as $Q \star \text{id}_U = \hat{Q} \times \text{id}_U$, which follows from the monoidality of the functor $(\cdot)$ which is proven in proposition ???.

Using it, the second term unfolds to:

$$\{w \uparrow A' \times B | w \in (\hat{A} \times \hat{U}_1) \times (\hat{A} \times \hat{U}_2) \times (\hat{B} \times \hat{U}_3), w \uparrow A' \times \hat{A} \in \hat{Q}, w \uparrow (\hat{A} \times \hat{U}_2) \times (\hat{B} \times \hat{U}_3) \in \hat{R}, w \uparrow \hat{U}_1 \times \hat{U}_3 \in \text{id}_U, w \uparrow \hat{U}_2 \times \hat{U}_3 \in \text{id}_U, w \uparrow \hat{A} \times B \in A \times B\}$$

In the second term, as $w \uparrow \hat{U}_1 \times \hat{U}_3 \in \text{id}_U$, and $w \uparrow \hat{U}_2 \times \hat{U}_3 \in \text{id}_U$, one can devise that $u \uparrow \hat{U}_1 \times \hat{U}_2 = u \uparrow \hat{U}_2 \times \hat{U}_3 = u \uparrow \hat{U}_1 \times \hat{U}_3$. The two terms only differ by the presence, in the second one, of $U_1 \times U_2$, which is insignificant (as $w \uparrow \hat{U}_1 \times \hat{U}_2 \in \text{id}_U$), and the fact that $w \uparrow \hat{A} \times \hat{A} \in \hat{Q}$ whereas $u \uparrow A' \times A \in Q$ in the first one. At this stage, it is useful to note the following property. Let $\mathcal{P} : A \rightarrow B$ a nominal linear relation. Suppose $u \in \mathcal{P}$ and $u \uparrow \hat{A} \in A$. Then $u \in \mathcal{P}$. This is simply proven by noticing that as $u \uparrow \hat{A} \in A$, then no names have been merged. We can apply this property in our case, as $w \uparrow \hat{A} \in A$. As a result, $w \uparrow A' \times \hat{A} \in Q$ and the two sets are equal.

Next, we check the first part of vanishing property, namely that

$$\text{Tr}_{B,C}^U(\mathcal{R}) = \mathcal{R}$$

which translates into:

$$\{u \uparrow B \times C | u \in \hat{R} \times \text{id}_U, u \uparrow \hat{B} \times \hat{C} \in B \times C\} = \{u \in \mathcal{R}\}$$

The right to left inclusion is straightforward. For the left to right one, we use the property that $r \in \hat{R} \wedge r \uparrow \hat{B} \in B \Rightarrow r \in \mathcal{R}$, as explained above. This entails the left to right inclusion.

The second part of the vanishing property consists in proving that given $\mathcal{R} : A \otimes U \rightarrow V \rightarrow B \otimes U \otimes V$, then

$$\text{Tr}_{A,B}^{U,V}(\mathcal{R}) = \text{Tr}_{A,B}^U(\text{Tr}_{A,B}^V(\mathcal{R})).$$

This first term can be described by:

$$\{r \uparrow A \times B, r \in \hat{R}, r \uparrow A \times \hat{B} \in A \times B, r \in \hat{U} \times \hat{U} \times \hat{V} \times \hat{V} \in \text{id}_{\hat{R} \times \hat{V}}\}$$

The second term unfolds in two steps. First, we consider $\text{Tr}_{A,B}^V(\mathcal{R})$.

$$\text{Tr}_{A,B}^V(\mathcal{R}) = \{u \uparrow (A \star U) \times (B \star U) | u \in \hat{R}, u \uparrow (\hat{A} \times \hat{U}) \times (\hat{B} \times \hat{U}) \in (A \star U) \times (B \star U), u \uparrow \hat{V} \times \hat{V} \in \text{id}_V\}.$$
Let us remind that using a permutation $(A\ A')\#v (\hat{A}\times \hat{V}) \in id_U, u \uparrow \hat{A} \times \hat{B} \in A \times B]$ equal to $\Tr^{\nu}(\mathcal{R})$.

We now deal with superposing:

$$\Tr_{C_{\emptyset},\emptyset}^{U}(Q \otimes \mathcal{R}) = Q \otimes \Tr_{C_{\emptyset},\emptyset}^{U}(\mathcal{R}),$$

for $\mathcal{R} : A \otimes U \rightarrow B \otimes U$, and $Q : C \rightarrow D$. This translates into:

$$\{u \uparrow (C \star A) \times (D \star B) \mid u \in \hat{Q} \star \mathcal{R}, u \uparrow \hat{U} \times \hat{U} \in id_U, (C \times \hat{A}) \times (D \times \hat{B}) \in (C \star A) \times (D \star B)\}$$

The fact that the second set is included in the first one is the easiest inclusion to prove. It follows from $Q \star \mathcal{R} \subseteq \hat{Q} \star \mathcal{R}$. Let us prove the reverse inclusion. As $u \in Q \star \mathcal{R} \wedge u \uparrow C \times \hat{D} \in C \times D$ then $u \uparrow C \times D \in Q \wedge u \in id_C \times \hat{R}$, however, nothing imposes that $u \in id_C \star \mathcal{R}$, the names in the $id_C$ and $\hat{R}$ part of $u$ could be the same. However, as $u \uparrow C \times \hat{D} \in C \times D$, one gathers that $v(u \uparrow (\hat{A} \times \hat{U}) \times (\hat{D} \times \hat{U})) \cap v(u \uparrow C \times D) = v(u \uparrow \hat{U} \times \hat{U}) \cap v(u \uparrow C \times D)$. Let us suppose that this set is not empty, and let $a$ be a name appearing in it. Then, by definition, this name does not appear in $u \uparrow \hat{A} \times \hat{B}$. Therefore, by applying a permutation $(a, c)$ to $u \uparrow (\hat{A} \times \hat{U}) \times (\hat{B} \times \hat{U})$, where $c$ is fresh, we can remove the name $a$ from the intersection. That is, let $u'' = (u \uparrow C \times \hat{D}) \times (\hat{A} \times \hat{U} \times (\hat{B} \times \hat{U}))$. Then $u'' \uparrow (\hat{C} \times \hat{A}) \times (\hat{D} \times \hat{B}) = u \uparrow (\hat{A} \times \hat{D}) \times (\hat{B} \times \hat{D})$, $u'' \uparrow (\hat{A} \times \hat{U}) \times (\hat{B} \times \hat{U}) \in \mathcal{R}$, and $u'' \uparrow \hat{U} \times \hat{U} \in id_U$. Furthermore, $v(u'' \uparrow (\hat{A} \times \hat{D}) \times (\hat{B} \times \hat{D})) \cap v(u'' \uparrow C \times D) = v(u \uparrow (\hat{A} \times \hat{D}) \times (\hat{B} \times \hat{D})) \cap v(u \uparrow C \times D) \setminus \{a\}$. Since the support of $u$ is finite, we can repeat the procedure for every name in the intersection. Finally, we obtain that an element $u''$ such that $u'' \uparrow (\hat{C} \times \hat{A}) \times (\hat{D} \times \hat{B}) = u \uparrow (\hat{C} \times \hat{A}) \times (\hat{D} \times \hat{B})$, $u'' \in Q \star \mathcal{R}$, and such that $u''$ satisfies all the conditions of the second set. Hence the two sets are equal.

Finally, we must prove the yanking property:

$$\Tr_{A,\emptyset}^{A}(s_{A,A}) = id_A$$

Let us remind that $s_{A,A}$ is the isomorphism coming from the symmetry of the monoidal category, $s_{A,A} = \{(u,v,v,u) \mid u, v \in A, u \#v\}$. Hence

$$\Tr_{A,\emptyset}^{A} = \{(u,w) \in A \times A \mid \exists (v,x) \in \hat{A} \times A, (u,v,v,w) \in s_{A,A}, (v,x) \in id_A\}$$

As $(v,x) \in id_A, v = x$. Furthermore, as $(u,v,v,v) \in s_{A,A}, u = v$, and $v = w$. As a result:

$$\Tr_{A,\emptyset}^{A}(s_{A,A}) = \{(u,u) \mid u \in A\} = id_A$$

\[\square\]

A.1 Nominal polarised linear relations

As the category $\text{NLTRel}$ is monoidal traced, one can define a compact closed category by polarising the category as before. Let us expose that formally. The objects of $\text{NomLinRelPol}$ will be nominal sets of annotated, polarised and separated lists. Elements of the sets are defined from the following grammar.

$$L :: (a, p) \mid (\bullet, p) \mid \text{inl}(L) \mid \text{inr}(L) \mid L_1 \# L_2$$

where $a \in A$, and $p \in \{-1, 1\}$, and $L_1 \# L_2$.

The $p$ next to each atom is called the polarity of the atom. Given a list $L$, we define by $\text{Pos}(L)$ the restriction of $L$ to its atoms of positive polarity, and $\text{Neg}(L)$ its restriction to the negative ones, where $\text{pos}((\text{inl}(L))) = \text{inl}(\text{pos}(L))$, $\text{pos}((\text{inr}(L))) = \text{inr}(\text{pos}(L))$, and similarly for neg. We write $L_1 \#_{\text{pol}} L_2$ if $(\text{Pos}(L_1) \#_{\text{pol}} \text{Pos}(L_2)) \land (\text{Neg}(L_1) \#_{\text{pol}} \text{Neg}(L_2))$, and note $\#_{\text{pol}}$ the associated polarised separated concatenation operation:

$$A \#_{\text{pol}} B = \{L_1 \#_{\text{pol}} L_2 \mid L_1 \in A, L_2 \in B, L_1 \#_{\text{pol}} L_2\}.$$
We write $L_1 \ast_{\text{pol}} L_2$ as a shorthand for $L_1 \cup L_2$ knowing $L_i \#_{\text{pol}} L_2$. We furthermore define the operation $(\cdot)^\dagger$ on lists, that consists in inverting all polarities.

\[
(a, p)^\dagger = (a, -p) \quad (\cdot, p)^\dagger = (\cdot, -p)
\]

\[
\text{inl}(L)^\dagger = \text{inl}(L^\dagger) \quad \text{inr}(L)^\dagger = \text{inr}(L^\dagger)
\]

\[
(L_1 \cup L_2)^\dagger = L_1^\dagger \cup L_2^\dagger
\]

We extend this operation to sets : $A^\dagger := \{ L^\dagger \mid L \in A \}$. Just as its seminal category, the morphisms $\mathcal{R} : A \to B$ of the category are relations. More precisely a nominal linear polarised relation $\mathcal{R} : A \to B$ is a nominal relation $\mathcal{R} \subseteq A^\dagger \ast_{\text{pol}} B$ such that each element is linear between negative and positive elements. That is, $\forall x \in \mathcal{R}, \nu(\text{Pos}(x)) = \nu(\text{Neg}(x))$. The identity $\text{id}_A : A \to A$ is the identity relation $L^\dagger \subseteq A^\dagger \ast_{\text{pol}} A$. Note that the $\ast_{\text{pol}}$ extends directly to morphisms, looking at them as subsets. Given two relations $\mathcal{R}_1 : A_1 \to B_1$ and $\mathcal{R}_2 : A_2 \to B_2$, we define $R_1 \ast_{\text{pol}} R_2 : A_1 \ast_{\text{pol}} A_2 \to B_1 \ast_{\text{pol}} B_2$ by:

\[
\mathcal{R}_1 \ast_{\text{pol}} \mathcal{R}_2 = \{ L_{A_1}, L_{A_2}, L_{B_1}, L_{B_2} \mid L_{A_1}, L_{B_1} \in \mathcal{R}_1, L_{A_2}, L_{B_2} \in \mathcal{R}_2, L_{A_1} \#_{\text{pol}} L_{B_2} \}
\]

Note that actually, $L_i \#_{\text{pol}} L_j$, as $\nu(\text{Neg}(L_i)) = \nu(\text{Pos}(L_j))$ and similarly for $L_k$. Hence $R_1 \ast_{\text{pol}} R_2 = R_1 \ast R_2$.

Just as in the case of NLRel, the patterns inl, inr allow us to form coproducts, denoted by $\oplus$:

- $A \oplus B = \text{inl}(A) \cup \text{inr}(B)$.
- $\mathcal{R}_1 \oplus \mathcal{R}_2 = \text{inl}(\mathcal{R}_1) \cup \text{inr}(\mathcal{R}_2)$

where we remind that $\cup$ expresses the fact that the union is disjoint.

Finally, given a set $A$, we define the two following sets:

\[
\text{Pos}(A) = \{ \text{Pos}(L) \mid L \in A \}
\]

\[
\text{Neg}(A) = \{ \text{Neg}(L) \mid L \in B \}
\]

One can then easily notice that $A \subseteq \text{Neg}(A) \times \text{Pos}(A)$, although the inclusion is strict in general. That is, there is an injective function $A \to \text{Neg}(A) \times \text{Pos}(A)$, that maps a list $L$ to $(\text{Neg}(L), \text{Pos}(L))$. Through this inclusion, a nominal linear polarised relation $\mathcal{R} : A \to B$ lifts to a nominal linear relation $\mathcal{R}' : \text{Pos}(A) \ast \text{Neg}(B) \to \text{Neg}(A) \ast \text{Pos}(B)$. Indeed, given an element $L \in \mathcal{R}$, then $\nu(\text{Neg}(L)) = \nu(\text{Pos}(L))$, and $L \in A^\dagger \ast_{\text{pol}} B$. Hence $\text{Neg}(L) = \text{Pos}(L \upharpoonright A).\text{Neg}(L \upharpoonright B)$. $\text{Pos}(L) = \text{Neg}(L \upharpoonright A).\text{Pos}(L \upharpoonright B)$ and furthermore $\text{Pos}(L \upharpoonright A)\#\text{Neg}(L \upharpoonright B)$. $\text{Neg}(L \upharpoonright A)\#\text{Pos}(L \upharpoonright B)$.

Just as functions in NomLinPol did not compose as set-functions, nominal linear polarised relations do not compose as relations. The composition is defined via the trace, just as in the previous section \(?\). Let us consider two morphisms $\mathcal{R} : A \to B$ and $Q : B \to C$, seen as nominal linear relations $\mathcal{R}' : \text{Pos}(A) \ast \text{Neg}(B) \to \text{Neg}(A) \ast \text{Pos}(B)$ and $Q' : \text{Pos}(B) \ast \text{Neg}(C) \to \text{Neg}(B) \ast \text{Pos}(C)$. Their composition is defined by taking their trace along $\text{neg}(B) \times \text{pos}(B)$. 

\[
\mathcal{R}' \ast Q' = \text{Tr}_{\text{neg}(B) \times \text{pos}(B), \text{neg}(A) \ast \text{pos}(C)}(\text{id}_{\text{neg}(A) \ast \text{pos}(B), \text{neg}(B) \ast \text{pos}(C)} \circ (\mathcal{R}' \ast Q'))
\]

where we recall that $s$ is the swapping morphism coming from the symmetry. This unfolds to:

\[
\mathcal{R}' \ast Q' = \{ u \uparrow \text{Pos}(A) \ast \text{Neg}(C) \ast \text{Neg}(A) \ast \text{Pos}(C) \mid u \in \mathcal{R}' \times \mathcal{Q}' \}
\]

\[
u(\text{Neg}(B) \times \text{Pos}(B)) \times (\text{Pos}(B) \times \text{Neg}(B)) \in \mathcal{S}_{\text{pos}(B), \text{neg}(B)}
\]

\[
u(\text{Pos}(A) \ast \text{Neg}(C)) \times (\text{Neg}(A) \times \text{Pos}(C)) \in (\text{Pos}(A) \ast \text{Neg}(C)) \times (\text{Neg}(A) \ast \text{Pos}(C))
\]

\[
u(\text{Pos}(A) \ast \text{Neg}(C)) \times (\text{Neg}(A) \ast \text{Pos}(C)) \in (\text{Pos}(A) \ast \text{Neg}(C)) \times (\text{Neg}(A) \ast \text{Pos}(C))
\]

As $u \in \mathcal{R}' \times \mathcal{Q}'$, $u$ comes from the injection $D \to \text{Neg}(D) \times \text{Pos}(D)$, where $D = A^\dagger \times B \times B^\dagger \times C$. Therefore, it is possible to backtrack $u$ to an element of $D$. Doing so we get:

\[
\mathcal{R} \ast Q = \{ u \uparrow \ast_{\text{pol}} C \mid u \in \mathcal{R}' \times \mathcal{Q}' \}
\]

\[
\{ u \uparrow \mathcal{A} \ast \mathcal{B} \in \mathcal{R}, u \uparrow \mathcal{B} \ast \mathcal{C} \in \mathcal{Q}, (u \uparrow \mathcal{B})^\dagger = u \uparrow \mathcal{B}^\dagger, u \uparrow \mathcal{A} \times \mathcal{C} \in \mathcal{A} \ast_{\text{pol}} C \}
\]

\[
\{ u \in \mathcal{A} \ast_{\text{pol}} C \mid \exists r \in \mathcal{R}, r_2 \in \mathcal{Q}, (r_1 \uparrow \mathcal{B})^\dagger = r_2 \uparrow \mathcal{B}, r_1 \uparrow \mathcal{A} = u \uparrow A, r_2 \uparrow \mathcal{C} = u \uparrow C \}
\]

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Furthermore, as the composition of \( \mathcal{R}' ; \mathcal{Q}' \) corresponds to tracing nominal linear relations, it results in a nominal linear polarised relation. That is, for each element \( u \) of \( \mathcal{R}' ; \mathcal{Q}' \), we have that \( v(u \mid \operatorname{Pos}(A) \ast \operatorname{Neg}(C)) = v(u \mid \operatorname{Neg}(A) \ast \operatorname{Pos}(C)) \). Therefore, backtracking \( u \) into an element of \( A \ast \mathcal{Q} \), we get that \( v(\operatorname{Neg}(u)) = v(\operatorname{Pos}(u)) \), that is, the relation \( \mathcal{R}; \mathcal{Q} \) hence defined is linear polarised. The closure under permutation follows straightforwardly from the invariance under permutation of the definitions. Therefore \( \mathcal{R}; \mathcal{Q} \) is indeed a morphism \( A \to C \) of \( \text{NomLinRelPol} \).

To sum up, the paradigm for composition is: closure under strict substitutions + relational composition + projections on “good” elements. Just as above, we restrict the objects to the well-typed ones. We define \( \text{MLL} \), collapsed version of \( \text{NLRel} \).

\( \text{NomLinRelPol} \) is the category that has as objects the smallest set such that \( I \) on “good” elements. Just as above, we restrict the objects to the well-typed ones. We define \( \text{NomLinRelPol} \).

Since \( \mathcal{R} \) is a star-autonomous category with products, and therefore, a model of \( \text{MALL} \).

Hence objects of \( \text{NomLinRelPol} \) are nominal sets of polarised separated annotated lists. Morphisms \( A \to B \) of \( \text{NomLinRelPol} \) are nominal polarised linear relations \( A \ast \ast \mathcal{Q} \) as described above.

By construction, \( \text{NomLinRelPol} \) is a compact closed category, since it is equivalent to \( \text{Int(NLRel)} \). We furthermore show that given two objects \( A, B \), the object \( A \oplus B \) is their coproduct, and that \( 0 \) is the unit for it, that is, an initial object. Namely, we need to exhibit two morphisms \( \text{inl} : A \to A \oplus B \) and \( \text{inr} : B \to A \oplus B \), such that for every object \( C \), for every morphisms \( \mathcal{R}_A : A \to C \) and \( \mathcal{R}_B : B \to C \), there exists a unique nominal polarised relation \( \mathcal{Q} : A \oplus B \to C \) such that \( \text{inl} ; \mathcal{R}_A \) and \( \text{inr} ; \mathcal{Q} = \mathcal{R}_B \). The corresponding diagram is showcased in Figure A.1. The morphism \( \text{inl} : A \to A \oplus B \) is defined to be the nominal polarised relation \( \text{inl}_A = \{ u^* \cdot \text{inl}(u) \mid u \in A \} \) and \( \text{inr} : B \to A \oplus B \) is built in a similar fashion. Finally, we set \( \mathcal{Q} \) as follows:

\[
\mathcal{Q} = \{ \text{inl}(u) ; v \mid u, v \in \mathcal{R}_A \} \cup \{ \text{inr}(u) ; v \mid u, v \in \mathcal{R}_B \}
\]

Then one can straightforwardly check that \( \text{inl} ; \mathcal{Q} = \mathcal{R}_A \), and similarly for \( \text{inr} \). Furthermore, \( \mathcal{Q} \) is unique verifying this property. Hence \( A \oplus B \) is the coproduct of \( A \ast B \). 0 is initial since for any object \( A \), there is a unique morphism \( 0 \to A \), namely the empty relation. As the co-product distributes over the tensor it is a bi-product, as established in [3]. Hence we can conclude this paragraph with the following proposition.

**Proposition A.5** \( \text{NomLinRelPol} \) is a star-autonomous category with products, and therefore, a model of \( \text{MALL} \).

One could wonder what is the “canonical” logic associated with the nominal linear polarised relations. In other terms, what is the sequent calculus associated with compact closed categories with bi-products. Surely, one can present a collapsed version of \( \text{MLL} \), where \( \otimes = \mathcal{R} \) and \& = \&. However, as explained in [??], “it turned out that there is a counter-example to the cut-elimination”. On the other hand, in the context of strongly compact closed category, Abramsky and Duncan were able to define a notion of proof-net [?] that precisely define those morphisms arising from the categorical structure, although no presentation was given in terms of sequent calculus.

As a final remark, these relations are the relations that we should use in the second part of the thesis devoted to nominal games to tensorial logic. Before moving to the next chapter, we present a severe downside of nominal polarised relations, and show in Section ?? a model with hypercoherences that overcomes it.
B Proofs regarding Nominal Structures

In the following, we write A-nominal sets for nominal sets with support, and whose support belongs in A. Nominal sets as defined in the paper are then θ-nominal sets.

**Lemma B.1** Given an edge $e : u \rightarrow v$ of an A-nominal graph, then $(v(e) \setminus A) = (v(u) \cup v(v)) \setminus A$.

**Proof.** We start by proving the left to right inclusion. Let us suppose there exists $v \in (v(u) \cup v(v)) \setminus A$, and let us consider a fresh $b$, $b \notin e, u, v, A$. Then we have $(a, b) \cdot e = e$, but $(a, b) \cdot u = u$ and $(a, b) \cdot v \neq v$. As the graph's map $f : E \rightarrow V \times V$ is injective, $f((a, b) \cdot e) \neq f(e)$. On the other hand, by nominality $f((a, b) \cdot u, (a, b) \cdot v) = (u, v) = f(e)$. Hence, we reach a contradiction and $(v(e) \setminus A) \subseteq (v(u) \cup v(v)) \setminus A$. The same reasoning works to prove the reverse inclusion.

**Proposition B.2** A-nominal trees are conservative. For any pair of nodes $u, v$ such that there exists an edge $e : u \rightarrow v$, then $v(u) \setminus A \subseteq v(v) \setminus A$. In particular, in a nominal tree $v(u) \subseteq v(v)$.

**Proof.** Suppose that there is an $a \in (v(u) \setminus v(v)) \setminus A$, and pick a fresh $b$ such that $b \notin u, v, A$. Then, as the graph is A-nominal, there is an edge $(a, b) \cdot e = e$, and $(a, b) \cdot u = u$ and $(a, b) \cdot v \neq v$. Therefore $v$ has two predecessors $(a, b) \cdot u$ and $a$. Now, taking a path from the root $r$ to $u$, and one from $r'$ to $(a, b) \cdot u$, we obtain two paths from $r$ to $v$, contradicting the definition of the graph being a directed tree.

**Proposition B.3** Let $T$ be an A-nominal tree. Then $T$ gives rise to an A-partially ordered set $(S, \leq)$ such that the down-closure of each element is well-ordered.

• Let $(S, \leq)$ be an A-partially ordered set such that the down-closure of each element is well-ordered. Then this one provides a description of an A-nominal tree.

**Proof.** We rely on the correspondence established within set theory. Let $T$ be an A-nominal tree. Forgetting about its nominal structure, it leads to a partially ordered set such that the down-closure of each element is well-ordered, that is, the restriction of the partial order to this subset is a total order. By definition, it set $V$ is A-nominal, and as the partial order relation is coming from $(E, f)$, which are A-nominal objects, it is A-nominal. Therefore, it gives rise to an A-nominal partially ordered set as expected. The reverse direction is proven on an equal footing.

The successor relation $\tau$ on $V$ is characterised by $u \tau v$ if there exists an $e \in E$ such that $u \overset{e}{\rightarrow} v$. In that case, we say that $u$ justifies $v$. Furthermore, we say that an element is initial if it is justified by the root. Finally, we notice that the proposition B.2 entails that any vertex in a nominal tree must remember the history of names leading to it. That is, if there is a name in the support of an element $w \leq v$, then it must be in the support of $v$ as well. Formally, for any vertex $v \in V$, we have $v(v \downarrow) \setminus A = v(v) \setminus A$.

B.1 Some more relations and properties on legal positions

To ensure that two positions are name compatible for union, we introduce a “post-compatible” relation $C_{\text{post}}$, that selects those legal positions whose joints are legal.

• $m \in C_{\text{post}} m'$ if $m C m' \wedge ((m \neq m') \Rightarrow (S^\#_{\text{cell}}S'^{\#}))$ where $m = (a, v, S)$ and $m' = (a', v', S')$
• $p C_{\text{post}} q$ if $\forall m \in p, \forall m' \in q. m C_{\text{post}} m'$.

As expected, $p, q \in \text{Legal}(A)$, $p C_{\text{post}} q \Rightarrow (p C q \wedge (p \uparrow \Rightarrow p \cup q \in \text{Legal}(A)))$. It is worth noticing that if $p, q \in \text{Legal}(A)$, and $p \uparrow q$, then there is a permutation $\pi$ of $A_{\text{cell}}$ such that $p C_{\text{post}} \pi \cdot q$, and $(p, \pi \cdot q)$ is bounded in $\text{Legal}(A)$. This is proven below. Stated otherwise, if two legal positions are compatible in $\text{Legal}(A)$, then it is possible to change their untyped names in such a way such that they remain compatible and their join becomes legal. We introduce the following terminology: we say that a move $m$ brings a name $a$ if, writing $(a, v, S)$ then $a \in v(v \downarrow S')$. Similarly, we say that an event $e$ brings a name $a$ if $e^\downarrow$ brings $a$.

**Lemma B.4** Let $p, q$ be two legal positions of $A$ such that $p C q$. Then there exists $p' \Rightarrow p$ such that $p' C_{\text{post}} q$

**Proof.** We split the names of $p \cap A_{\text{cell}}$ into two parts, calling them $p_1, p_2$. We set $p_1 = v(p \cap q) \cap A_{\text{cell}}$, and $p_2 = (v(p) \cap A_{\text{cell}}) \setminus p_1$. As $p$ is legal, $p_2$ corresponds precisely to the set of cell-names brought by the moves in $p \setminus q$, seeing $p, q$ as set of moves. Let $\pi$ a permutation such that $v(\pi) \subseteq p_2 \cup z$ where $z \notin q, p_1$, and such that $\forall a \in p_2. \pi \cdot a \in z$. Then $\pi \cdot p_2 \# q$ and by definition of $\pi$, $\pi \cdot p_1 = p_1$ as $v(p_1) \subseteq v(q)$. Furthermore, legal positions are stable under permutation. Therefore, $\pi \cdot p$ is legal, and $\pi \cdot p C_{\text{post}} q$.

**Remark B.5** All the moves within a legal position are automatically post-compatible to one-another.

Typed names are dealt with differently than untyped names. Indeed, the typed names will be repeated in the play, to incorporate the fact that the strategy will establish axiom links between literals of opposite polarities. Therefore, it does
not make sense to impose a condition similar as legality for typed names. To cope with the possibility that repetitions might occur, we will work up to closure under typed substitutions. \( \Xi_T \) denotes the set of strict substitutions of \( \mathcal{A}_T \), called typed substitutions. We therefore define a new relation, written \( \equiv \), called congruence.

\[
x \equiv y \iff \exists \pi \in \text{Perm}(A_{\text{cell}}), e \in \Xi_T, \pi \cdot (e \cdot x) = y.
\]

We create two new relations \( C_{\text{cell}} \) and \( C_{\text{post}, \text{cell}} \), that are the restrictions of \( C, C_{\text{post}} \) to cells that are untyped.

- \( m = (\alpha, v, S) C_{\text{cell}} m' = (\alpha', v', S') \) if \( \alpha = \alpha \Rightarrow \alpha = \alpha' \) and \( \forall c \in S, \forall c' \in S', \nu(c) \subseteq A_{\text{cell}}, \nu(c') \subseteq A_{\text{cell}} \Rightarrow c \subseteq c' \).
- \( p \ C_{\text{cell}} q \) if \( \forall m \in p, \forall m' \in q, m C_{\text{cell}} m' \).
- \( e C_{\text{post}, \text{cell}} m' \) if \( e C_{\text{cell}} m' \) and writing \( m = (\alpha, v, S), m' = (\alpha', v', S') \), then \( m \not\equiv m' \Rightarrow rS^{-1} \#_{\text{cell}} rS^{-1} \).
- \( p C_{\text{post}, \text{cell}} q \) if \( \forall m \in p, \forall m' \in q, m C_{\text{post}, \text{cell}} n, e C_{\text{post}, \text{cell}} e' \) if \( e C_{\text{cell}} e' \).

The extend straightforwardly to plays. Two plays \( s : * \rightarrow x, t : * \rightarrow y \) satisfy \( s C_{\text{cell}} t \) if \( x C_{\text{cell}} y \). We prove the following properties:

**Proposition B.6** • If \( p \uparrow q \) and \( p C_{\text{post}, \text{cell}} q \) then \( p C_{\text{post}} q \).

- If \( m \not\equiv n \) and \( m C_{\text{post}, \text{cell}} n \) then \( m C_{\text{post}} n \).

**Proof.** The first property is straightforward, since \( p \uparrow q \) entails \( p C q \). The second property follows from the fact that we consider \( m, n \) legal. Therefore, \( m \not\equiv n \) implies that they are not, essentially, the same move with different names. Hence, we automatically got \( m C n \).

In the case where we forgot the condition \( p \uparrow q \), then it becomes harder to create elements that become compatible. Finally, we prove that given two elements, we can use permutations to make them compatible regarding untyped cells, and use substitutions to make them post compatible. We rely on the lemma B.7.

**Lemma B.7** Let \( f \) be a bijection between two finite subsets of \( A \). Then \( f \) can be completed into a permutation of \( A \) of finite support.

**Proof.** Let us name \( X, Y \) the two subsets such that \( f \) sends \( X \) onto \( Y \). As \( f \) is a bijection, the cardinality of \( X \) and \( Y \) is the same, and so is the cardinal of \( Y \setminus (X \cap Y) \) and \( X \setminus (X \cap Y) \). So consider \( g \) a function such that \( g : Y \setminus (X \cap Y) \to X \setminus (X \cap Y) \) is a bijection. The union (in the sense of union of graphs) of \( f \) and \( g \) hence leads to a bijection \( X \cup Y \to X \cup Y \). We can simply complete it into a full permutation \( \pi \) of \( A \) by letting \( \pi \) acting like the identity outside \( X \cup Y \). Furthermore, as \( X \) and \( Y \) are finite, \( \pi \) has finite support.

At last, we present the last property that we will need regarding the \( C_{\text{post}} \) relation.

**Proposition B.8** • Let \( p, q \) be two legal positions. Then there exists \( p', q' \) such that \( p' \equiv p, q' \equiv q \) and \( p' C_{\text{post}} q' \).

- Let \( p, q \) be two legal positions. Then there exists a permutation \( \pi \) such that \( \pi \cdot p C_{\text{post}, \text{cell}} q \).

Given a move \( m = (\alpha, v, S) \) we write \( rS^{-1} (m)^\pi \) for \( rS^{-1} \).

**Proof.** We prove the two points at once. Let \( p_1 = \{ m \mid m \in p, \exists m' \in y, m \equiv m' \} \). As \( p \) is legal, every event in it brings different untyped names, distinct from the initial cell of \( p \), and similarly for \( q \). That is, given \( m_1, m_2 \in p, m_1 \not\equiv m_2 \Rightarrow rS^{-1} (m_1)^\pi \#_{\text{cell}} rS^{-1} (m_2)^\pi \). Let us note that either the initial move of \( p \) is in \( p_1 \), or it is empty. In the case where it is not empty, \( \nu(p_1) = [\alpha] \), and \( rS^{-1} (m_1)^\pi \), where \( \alpha \) is the name of the initial cell of \( p \). We can define a function \( f : \nu(p_1) \cap A_{\text{cell}} \to A_{\text{cell}} \) such that for all \( m \in p_1 \), given \( \pi \) exhibiting the equivalence \( \pi \colon rS^{-1} (m)^\pi \cap A_{\text{cell}} = rS^{-1} (m')^\pi \cap A_{\text{cell}} \) from the definition of \( p_1 \), \( f \mid \nu(rS^{-1} (m')) \cap A_{\text{cell}} = \pi \mid \nu(rS^{-1} (m')) \cap A_{\text{cell}} \). We complete it into \( f(\alpha) = \beta \), where \( \beta \) is the name of the initial cell of \( p_2 \). Furthermore, as \( y \) is legal, each \( m' \) in the definition \( p_1 \) brings different names as well, and hence \( f \) establishes a bijection between \( \nu(p_1) \cap A_{\text{cell}} \) and a subset of \( A_{\text{cell}} \). Therefore, by applying the above lemma, we get a permutation \( \pi \) such that \( \pi \cdot p C_{\text{post}} q \). Furthermore, doing the same reasoning as in the proof of B.4, we can find a \( \pi' \) such that \( \pi' \cdot (\pi \cdot p) C_{\text{post}, \text{cell}} q \). As a consequence of the axioms of group actions \( (\pi' \circ \pi) \cdot p C_{\text{post}, \text{cell}} q \). Now, let us take two typed substitutions \( e_1, e_1 \), such that, for all \( X \in \text{TVar}, e_1, e_2 \) send all names of \( p, q \) of type \( X \) to a unique name \( c_x \in A_X \). Then \( e_1 \cdot (\pi' \circ \pi) \cdot p C_{\text{post}} e_2 \cdot q \).

**C Proof on strategies**

**Remark C.1** In a typed coherent strategy, if we have two moves \( s.m.n_1, s.m.n_2 \) such that both belong to the strategy, then \( s.m.n_1 C_{\text{cell}} s.m.n_2 \) entails \( n_1 = n_2 \). Indeed, by nominal determinacy \( s.m.n_1 = s.m.n_2 \), and, as the two positions are
compatible with relation to untyped cells, $s.m.n_1 \equiv_T s.m.n_2$. Particularly, there exists a permutation $\pi$ of $h_T$ such that $\pi$ lets $s.m$ invariant, and $\pi \cdot n_1 = n_2$. As $s.m$ has strong support, this entails $\pi h(s.m)$. In particular, as $\nu_T(n_1) \sqsubseteq \nu_T(s.m)$, this leads to $\pi \cdot n_1 = n_1$ and $n_1 = n_2$.  

**Definition C.2** A strategy is positional if for all $s, s' : \star \to x \in \sigma$, such that $s \sim s'$, for all $t : x \to y$ such that $s \cdot t \in \sigma$, then $s'.t : \star \to y \in \sigma$. This is drawn as follows.

![Diagram](image)

**Proposition C.3** Every nominal innocent strategy is positional.

The rest of this section is devoted to proving this. To start, we introduce, for a strategy $\sigma$, the set $\sigma^*$ defined to be the set of positions it reaches:

$$\sigma^* = \{ x \in \text{Legal}(A) \mid \exists s : \star \to x \in \sigma \}.$$  

Second, let us note $\leq$ the relation on paths $s \leq t$ if there is a $s'$ such that $s.s' \sim t$. We furthermore refine the homotopy relation into a second one, called $\sim_{OP}$ that acts by permuting only pairs of OP moves, we construct $\sim_{OP}$ in a similar way as we constructed $\sim$, but this time focusing on permutations between alternating paths of length 4, permuting pairs of $O - P$ moves.

**Definition C.4** We define the $\circ_{OP}$ relation between co-initial and co-final paths of length 4 as follows:

$$m_1.n_1.m_2.n_2 \circ_{OP} m_2.n_2.m_1.n_1 \Leftrightarrow (m_1.n_1.m_2.n_2) \sim (m_2.n_2.m_1.n_1) \text{ and } \lambda(m_i) = -\lambda(n_i) = -1.$$  

We say that this permutation of moves correspond to a single OP-homotopy step. We then define the intermediate $\circ'_{OP}$ to be the augmented relation from $\circ_{OP}$ between paths of lengths more than four:

$$s.u.t \circ_{OP} v \iff u \circ_{OP} v$$

We define $\sim_{OP}$ as the reflexive, transitive, and symmetric closure of $\circ'_{OP}$.

As the sequences of the strategies are alternating, and strategies are deterministic, this seems to be an appropriate notion for dealing with homotopy between paths of a strategy. Note however that two alternating paths can be $\sim_{OP}$ homotopic without being $\sim_{OP}$ homotopic. One can draw a parallel between history freeness and the $\sim_{OP}$ relation: two $\sim_{OP}$ homotopic plays will have same player moves after same opponent moves. Finally, we write $s \leq_{OP} t$ if there exists $s'$ such that $s.s' \sim_{OP} t$.

**Proposition C.5** Let $\sigma$ be an innocent strategy and $s, t \in \sigma$ such that $s \leq t$. Then $s \leq_{OP} t$.

**Proof.** For a given path $u$, we write $u_{\geq n}$ for the pre-sequence of $u$ consisting of its $n$-first moves, and $u_{> n}$ for the path such that $u_{> n} = u$. Furthermore, let us write $s'$ for a path such that $s.s' \sim t$. We will prove the existence of a sequence of plays $l_0 \sim_{OP} l_1 \sim_{OP} l_2 \sim_{OP} l_3 \sim_{OP} l_4(l_2)$, such that $l_0 = t$, and $l_1 \in \sigma$, $l_2 \sim_{OP} t$, and satisfying for each $i$, $(l_i)_{2 \leq i} = s_{2 \leq i}$. This way, $l_{12}$ will satisfy the following equality $l_{12} = s.s' \sim_{OP} t$, entailing $s \leq_{OP} t$.

The required conditions are obviously satisfied for $l_0$, so let us assume they are true for $l_i$ and we try proving the existence of $l_{i+1}$, assuming $2 \cdot n < |s|$. To simplify notations, we write $u$ for the path $s_{2 \leq 2n}$, and $v$ for $l_{2n+2}$ (that is, such that $l_{2n+2} = l_{2 \leq 2n} = v$. Notably, as $s \leq_{OP} n_{\leq 2n} = u, v$ are co-initial. In particular, this entails $u, s' \sim v$ since these are both co-initial and co-final.

As $s, l_0$ are in $\sigma$, they are alternating. Let us write $m$ for the moves of $u$ and $n$ for the moves of $v$. Formally, $u = m_1...m_n$ and $v = n_1...n_k$. Let us consider the first opponent move $m_1$ of $u$. As $u.s' \sim v$, $m_1$ also a move of $v$. If it is the first one, then as $s_{2 \leq 2n} m_1 \in \sigma$, $s_{2 \leq 2n} n_1 \in \sigma$, $s_{2 \leq 2n}$ is deterministic, and $s_{2 \leq 2n} m_1 = l_{2 \leq 2n} n_1$, this entails $m_1 = n_1$. Now as $s.s'$ and $t$ reach the same position, they are in compatible mode for equality, that is, $\forall i, j. m_i = n_i \Rightarrow m_i = n_i$. So $n_2 = m_2$, and $l_{2n+1} = l_4$.  

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We deal with the case where \( m_1 \) is not the first move of \( v \). We write \( v = v_0 = v'.o_1.o_2.m_1.m_2.v'' \). As \( m_1 \) appears before \( o_1 \) in \( u.s' \), and after in \( v \), it entails \( m_1 \uparrow o_1 \). In particular, there is a tile \( m_1.o_1 \uparrow o_1.m_1 \). By backward consistency, \( t_{u,s' \rightarrow OP,v} \sim OP \) \( t_{u,s' \rightarrow OP,v_0} \sim OP \). So, by a sequence of backward consistency steps, we can push \( m_1.m_2 \) as the start of \( v \) as we did from \( v_0 \) to \( v_1 \). This is represented in figure C.1. This way, we obtain a \( v_{\text{final}} \) such that \( t_{u,s' \rightarrow OP,v_{\text{final}}} \sim OP \), and \( t_{u,s' \rightarrow OP,v_{\text{final}}} \sim OP t \). We deduce, as above, that the two first moves of \( v_{\text{final}} \) are equal to those of \( u \), and set \( t_{u+1} = t_{u,v_{\text{final}}} \). This concludes the proof.

We also need a second lemma before proving the proposition, that states that strategies are closed under \( \sim_{OP} \) homotopy.

**Lemma C.6** Let \( s \in \sigma \), \( \sigma \) innocent strategy, and \( s \sim_{OP} t \). Then \( t \in \sigma \). Equivalently, an innocent strategy is closed under \( OP \)-homotopy.

**Proof.** We do the proof by induction on the number of \( OP \)-homotopy steps needed to go from \( s \) to \( t \). If it is 0, then \( s = t \) and hence the property holds naturally. All we have to prove is that if \( s \) is in \( \sigma \), and \( s \rightarrow s' \), where \( \rightarrow \) is a single \( OP \)-homotopy step, then \( s' \) is in \( \sigma \).

Suppose \( s = s_1.m_1.n_1.m_2.n_2.s_2 \) and \( s' = s_1.m_2.n_2.m_1.n_1.s_2 \). Then by definition there is a tile \( m_1 \uparrow m_2 \) and \( n_1 \uparrow m_2 \), and using the backward consistency, we deduce that \( s.m_2.n_2.m_1.n_1.s_2 \) is in \( \sigma \). We are now in position to prove the positionality of the strategy, that is, proposition C.3.

**Proof.** Let \( s,s' \in \sigma \), \( s \sim s' : \star \rightarrow x \), and \( t : x \rightarrow y \), such that \( s.t \in \sigma \). Then, \( s' \preceq s \) and hence, as both plays belong to the strategy, \( s' \preceq_{OP} s \). Furthermore, the length of the two plays being the same \( s' \sim_{OP} s \). As this relation is closed under post-composition, \( s'.t \sim_{OP} s.t \) and finally, as \( \sigma \) is stable under \( \sim_{OP} \) homotopy \( s'.t \in \sigma \).

Given a subset \( X \subseteq \text{Legal}(A) \) we write \( X^\uparrow \) for the set of plays defined by \( X \).
A strategy is strongly positional (or relational) if $\sigma = (\sigma^*)^i$.

In other terms, a relational strategy is a strategy that is both static and dynamic. It can be characterised dynamically as a set of sequences, or statically as a set of positions.

Proposition C.8 Every innocent strategy is strongly positional.

Proof. The inclusion $\sigma \subseteq (\sigma^*)^i$ is clear, so we only need to focus on the reverse inclusion, that we prove by induction on the length of the plays. We pick a play $s \in (\sigma^*)^i$. If the play is of length 0, then there is nothing to prove. So imagine the length of $s$ is now $n + 2$, and all the plays of length $n$ of $(\sigma^*)^i$ have been proven to belong to $\sigma$. So $s$ can be decomposed as $s = s_1.m.n : \star \rightarrow y$, such that $s_1 : \star \rightarrow x$, and as $x \in (\sigma^*)^i$, together with $s_1$ is of length $n$, we already know that $s_1 \in \sigma$. As $y \in (\sigma^*)^i$, we know that there is a path $t : \star \rightarrow y \in \sigma$. In particular as $s_1.m.n$ and $t$ reach the same position, $s_1.m.n \sim t$ and $s_1 \leq t$. As $s_1$ and $t$ are in $\sigma$, we can use proposition C.5 and infer that $s_1 \sim_{OP} t$. In particular, there are $m', n'$ two moves such that $m'$ is an O-move, $n'$ a P-move and $s_1.m'.n' \sim_{OP} t$. As $m', n', n \in t \setminus s_1$, we can gather that $m = m'$, $n = n'$ and $s_1.m.n \sim_{OP} t$, and thus $s = s_1.m.n \in \sigma$ as $\sigma$ is closed under $\sim_{OP}$ homotopy. □

Definition C.9 A subset $X$ of $\text{Pos}(A)$ is definable if there exists a typed-coherent innocent strategy $\sigma$ such that $\sigma^* = X$.

The description of definable sets is laid down in the following theorem. We say that a position $y$ dominated by a set of positions $X$, if there is an $x \in X$, and a path $s : y \rightarrow x$. In this case we say that $y$ is dominated by $x$ (that is, $y \leq x$), or $x$ (and $X$) dominates $y$. Similarly, we speak of under-domination in the case where $\exists x \in X, x \leq y$ (that is, there exists a path $s : x \rightarrow y$).

Theorem C.10 A set $X$ of positions is definable if and only if:
(i) (Root): $\bot \in X$.
(ii) (Legality): $X \subseteq \text{Legal}(A)$.
(iii) (Nominal closure): $X$ is nominal closed, and closed under strict typed substitutions. That is, $\forall x \in X, \forall y \equiv x \Rightarrow y \in X$.
(iv) (Closure under intersection): $\forall x, y \in X, x \cap y \wedge x \rightarrow y \Rightarrow x \cap y \in X$.
(v) (Closure under union): $\forall x, y \in X, x \cup y \wedge x \rightarrow y \Rightarrow x \cup y \in X$.
(vi) (Preservation of compatibility): Let $x \in X$, and two moves $m, m'$ such that $x \xrightarrow{m} y$ and $x \xrightarrow{m'} y'$, satisfying $y \uparrow y'$, $y \cap y'$ are dominated in $X$. Then $y \cup y'$ is dominated in $X$.
(vii) (Forward confluence 1): For all $x \in X$, if there is an opponent move $m : x \rightarrow y$ and $y$ is dominated in $X$ then there is a unique $z \in X$, up to equivalence, such that there is a P-move $n$ satisfying $x \xrightarrow{m.n} z$, and furthermore $\nu_T(n) \subseteq \nu_T(y)$.
(viii) (Forward confluence 2) For all $x, y \in X$, if there is an opponent move $m : x \rightarrow y$ and $y$ is dominated in $X$ by $w$ then there is a unique P-move $n$, such that $x \xrightarrow{m.n} z$, and $z \in X, z \leq w$, and furthermore $\nu_T(n) \subseteq \nu_T(y)$.
(ix) (Mutual attraction): For all $x, y \in X$ such that $y$ dominates $x$, either $x = y$ or there is an opponent move $m : x \rightarrow x'$ and a player move $n : y \rightarrow y'$ such that $y'$ dominates $x'$.

Proof. We first show that there is an innocent strategy $\sigma$ such that $X = \sigma^*$. We then show that given any innocent strategy $\tau$, its set $\tau^*$ satisfies the properties of a definable set.

We start by showing that there is a set of plays $\sigma$, $(\sigma = X^\downarrow)$, such that $X = \sigma^*$. Formally, we need to prove that $(X^\downarrow)^i = X$. We will prove that $\sigma$ is an innocent strategy afterwards. By definition, $(X^\downarrow)^i \subseteq X$, as the target of every play of $X^\downarrow$ is an element of $X$. So it remains to prove the reverse inclusion. Let us prove that every element of $X$ is the target of an alternating path, such that each even-length subpath reaches an element of $X$. This is true for the root by (1). Let $x \in X$, and let us suppose that we proved the property for every element $y$ of $X$ with $y \leq x$, and we note that, as $\bot \downarrow X$, the set of such $y$ is never empty. Then let $y$ be maximal in $X$ under $x$. We apply the last property, mutual attraction (9), and hence know that there in an opponent move $m : y \rightarrow y'$ and a player move $n : x' \rightarrow x$ such that $x'$ dominates $y'$. Now either $x' = y'$, in which case we conclude using the induction hypothesis. Or, using forward confluence (2), there is a player move $n' : x' \rightarrow y''$, and $y'' < x$ is in $X$. This brings a contradiction as $y$ was supposed to be maximal in $X$ under $x$.

In this paragraph, we prove that the set of plays $X^\downarrow$ forms a typed-coherent strategy. As the root is an element of $X$, the empty play is part of $X^\downarrow$, as expected. Moreover, thanks to the definition of $\downarrow$, the closure under prefix follows. The closure under nominal permutations comes from the fact that $X$ is itself closed under permutations, and similarly for typed substitutions. The remaining property is nominal determinacy. Let $s : \star \rightarrow x, s' : \star \rightarrow x'$ two plays of $X^\downarrow$ such

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that \( s \equiv s', \) and hence \( x \equiv x'. \) Let \( m : x \times m \rightarrow y, \) \( m' : x' \times m' \rightarrow y' \) two opponent moves such that \( s.m = s'.m' \), and thus \( y \equiv y'. \)

Let \( m : y \rightarrow z, \) \( m' : y' \rightarrow z' \) such that \( s.m.n \ (\text{resp } s'.m'.n') \in X^1 \). We must prove that \( z \equiv z' \). Let us consider \( \pi \) a nominal permutation such that \( \pi \cdot s.m = s.m \). Then by forward consistency (1) there exists a unique, up to nominal permutation, \( z' \), such that there exists a \( \text{P-move } \theta' \) satisfying \( s.m.n' : \star \rightarrow \star' \in X \). In other terms, \( \pi \cdot z \equiv \pi' \cdot z' \) and therefore \( z \equiv z' \), that is, \( s.m.n' \equiv s.m.n \), and the strategy is nominal deterministic. Furthermore, \( \forall n' \subseteq \forall (s.m) \) entails that the strategy is semi-linear, and hence typed-coherent.

This leads us to prove that the strategy \( X^1 \) satisfies the 2 diagrams of innocence. We start with forward consistency.

Let \( x \in X \), and \( m_1 : x \rightarrow y_1 \) an opponent move, such that \( y_1 \) is dominated in \( X \). We also pick \( m_2 : x \rightarrow y_2 \) such that \( m_1 \uparrow m_1, m_2 \not\equiv m_1, y_2 \) is dominated in \( X \) and \( y_2 \subseteq \text{Post } y_1 \). As \( y_1 \) and \( y_2 \) are compatible and dominated in \( X \), is so their union \( y_1 \cup y_2 \) by (6)). So let us call \( w \) an element of \( X \) such that \( w \) dominates \( y_1 \cup y_2 \) and \( w \) minimal among those. Then \( y_1 \) is dominated by \( w \), and therefore there is a single move \( n_1 \) such that \( y_1 \nrightarrow z_1 \) and \( z_1 \) is dominated by \( w \). Similarly, we can conclude that there is a single move \( n_2 : y \rightarrow z_2 \), such that \( X \ni z_2 \subseteq w \). Furthermore, \( z_1 \) is dominated by \( w \), and so is \( z_1 \subseteq [m_1] \), and similarly for \( z_1 \subseteq [m_2] \). Therefore, we conclude the existence of two additional moves \( n'_1, n'_2 \) such that \( z_1 \subseteq [m_1] \cup [n'_1], z_2 \subseteq [m_2] \cup [n'_2] \) both lead to positions in \( X \) dominated by \( w \). As these positions are both dominated by \( w \), there are compatible, and, their intersection belong in \( X \). We deal with different cases:

- If \( n'_1 \not\equiv n_1, n'_2 \equiv n_2 \), then their intersection is \( y_1 \cup y_2 \), which cannot be in \( X \) since it is an unbalanced position.
- If \( n'_1 = n_1, n'_2 \not\equiv n_2 \), then their intersection is \( y_1 \cup y_2 \cup [n_1] \), which cannot be in \( X \) for the same reason as above.
- If \( n_1 \not\equiv n'_1, n'_2 = n_2 \), then this is a similar case as above.
- If \( n_1 = n'_1 = n_2 = n'_2 \), then similar case as above once again.

So we conclude that \( n_1 \equiv n'_1, n_2 \equiv n'_2, n_1 \not\equiv n_2, \) and furthermore \( n_1 \uparrow n_2 \), since they appear at different orders in different plays. At this stage we have not finished the proof since the \( n_1, n_2 \) were picked in accordance to \( w \), and we shall prove that the proof can be made to work for any \( n_1, n_2 \) chosen such that \( s.m_1 \cup m_2 \text{Post } s.m_2 \cup m_1, s.m_1, o \) and \( s.m_2, o \) reach positions in \( X \). By nominal determinacy, we automatically have \( s.m_1 \cup m_2 = s.m_1, o \) and similarly for \( o_2, m_2 \). We first do the proof in the “easy” case, where \( o_1 \not\equiv o_2 \) and \( o_1 \equiv o_2 \). Note that \( o_1 \not\equiv o_2 \) as \( o_1 \not\equiv o_2 \) by the relation \( \text{Post } \). In particular, as \( \forall (o_1) \subseteq \forall (n_1) \cup \forall (o_1) \), and similarly for \( o_2 \), we get \( \forall (n_1) \). We then have \( s.m_1, o \equiv s.m_1 \cdot w \), and \( s.m_2, o \equiv s.m_2 \cdot w \). Notably, as \( o_1, o_2 \) have disjoint support, \( (o_1 \not\equiv o_2) = (o_1 \not\equiv o_2) \cdot w \). And we have found a common bound for \( s.m_1, o \), namely \( \forall(n_1 \cup o_1) \cdot w \). We can then apply the proof as above. To finish, we need to tackle the “hard” case, where the propositions \( o_1 \not\equiv o_2 \) and \( o_2 \not\equiv o_1 \). Finally, we conclude the existence of \( o_1 \not\equiv o_2 \) by the relation \( \text{Post } \).

We prove the backward consistency. Let \( s.m_1, n_1, m_2, n_2, t : \star \rightarrow w \in X^1 \) such that \( \star \rightarrow x \rightarrow y \rightarrow z \rightarrow w \), and suppose that \( s.m_2 \) is legal, and \( m_2 \uparrow n_1 \). Furthermore, \( s.m_1 \) is dominated by \( z \in X \). Then let \( n' \) be the move such that \( s.m_2, n' \in X^1 \) and \( s.m_2, n' : \star \rightarrow y' \) is dominated by \( y' \). Therefore, \( n' = n_1 \) or \( n' = n_2 \). Note that \( y' \equiv y \) since they are both dominated by \( y' \). If \( n' = n_1 \), then as \( X \) is closed under compatible intersection, \( y' \subseteq y \) is in \( X \). However, in that case, \( y \cap y' = y \subseteq [n_1] \), which is a unbalanced position. Then \( n' = n_2 \), and \( s.m_2, n_2 \in X^1 \). Finally, we devise that \( s.m_2, n_2, n_1 \) is in \( X^1 \) by forward consistency. Finally, noticing that \( s.m_2, n_2, m_1, n_1 \) and \( s.m_1, n_1, m_2, n_2 \) reach the same position, we can conclude that \( s.m_2, n_2, m_1, t \in X^1 \) by definition of \( \equiv \), which concludes the proof.

At last, reverse consistency follows from the proposition ??

Subsequently, we now have to prove the reverse direction. Let \( \sigma \) be an innocent strategy, and consider the set \( \text{X = } \sigma * \). The condition (1) (\( \in X \)) comes from the fact that the empty play is always part of a strategy. The second condition (2) \((X \subseteq \text{Legal}(A)) \) is by definition and typed substitutions closure follows directly from the one of \( \sigma \). Harder to prove are the conditions of closure under intersection (4) and union (5). We start with the first.

Let \( x, y \in X \), such that \( x \cap y \) (and hence \( x \cap C Y \)), and let \( s : \star \rightarrow x \in \sigma \), and \( t : \star \rightarrow y \in \sigma \). Consider \( z = x \cap y \), our goal is to show that there is \( u : \star \rightarrow z \) such that \( u \subseteq \text{OP } s, t \). As \( u \) is closed under \( \text{OP } \), this will prove that \( u \in \sigma \) and hence \( \sigma \equiv \sigma \). We call \( M \) the set of moves that \( s, t \) have in common. \( M \) is the set of moves that appear in \( x \cap y \). As both \( s, t \) are plays, and hence closed under down-closure, so is \( M \). Therefore, the restriction of \( s \) to the moves that appear in \( M \) define a path (see proposition ??), in the sense that it defines a path starting from the root. We call it \( u \). We prove that \( u \subseteq \text{OP } s, t \), by induction on the pre-sequences \( v \) of \( u \) of even length. That is, we prove that for every pre-sequence of even length \( v \) of \( u, v \subseteq \text{OP } s, t \), and furthermore \( u \) is of even-length. Firstly, we notice that the empty sequence \( e \) obviously satisfies the property. For the next step, consider it true for an arbitrary pre-sequence of even length \( v \) of \( s' \). Then let \( m \) the move after \( v \) in \( s', \) and suppose that \( m \) is an \( O \)-move. As \( v \subseteq \text{OP } s, \exists m_1, v.m_1 \subseteq \text{OP } s, \) and, equivalently, \( \exists m_2, v.m_2 \subseteq \text{OP } t \). Now, by nominal determinacy, \( n_1 \equiv n_2, \) and as \( x \) and \( y \) are compatible for equality, \( n_1 = n_2 \). Hence \( v.m_1.n_1 \) is a prefix of \( u \)

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and \(v.m.n_1 \leq_{OP} s.t\). On the other hand, let us consider that the next move \(n\) after \(v\) in \(x\) is a \(P\)-move. Then there exists \(m_1, s_1, t_1\) such that \(v.s_1, m_1 \leq_{OP} s, v.t_1, m_2, n \leq_{OP} t\) and \(m_1 \neq m_2\). Now, we apply a permutation \(\pi\) of \(\mathcal{B}\text{cell}\) to \(x\) such that \(\pi \cdot x \in C_{post} y\). Let us note that this let \(x \cap y\) (that is, \(x \cap y = \pi \cdot x \cap y\)), invariant, and hence we assume without lost of generality that \(x \in C_{post} y\). As \(x \uparrow y, x \cap y,\) and \(t_1 \cap s_1 = \emptyset\) (in the sense that they have no common moves), we can apply forward consistency, and push \(s_1\) along \(t_1\), obtaining a play \(v.s_1.t_1 \in \sigma\). At this stage, we can perform a final step of forward consistency on \(v.s_1.t_1, m_1, n\) and \(v.s_1.t_2, m_2, n\), relying on \(m_1 \uparrow m_2, m_1 \neq m_2, m_1 \in C_{post} m_2\). This one tells us that the after \(m_1\) is non-congruent to the move after \(m_2\), that is \(n \neq n\). Hence this is a contradiction, and this case is excluded. As a result, for all pre-sequences (of even-length) \(v\) of \(u, v \leq_{OP} s\), and \(y\) of even-length. Consequently, \(u \leq_{OP} s\) and \(u \in \sigma\). As \(u\) reaches \(x \cap y\), this entails \(x \cap y \in \sigma^*\).

Let \(x, y\) be two post-compatible positions in \(X\) and \(s, t\) two plays of \(\sigma\) that target them. Let \(u\) be a play of \(\sigma\) that targets \(x \cap y\). We write \(s \sim_{OP} u.s'\) and \(t \sim_{OP} u.t'\) and do the induction on the length of \(s'\). If it is null, then \(x \leq y\) and \(t\) is \(\sigma\)-move and finishes with a \(P\)-move and \(t'\) is \(\sigma\)-move and finishes with a \(P\)-move. Therefore, by pushing the forward consistency diagram of innocence, \(u.m.n.t' \in \sigma\). Then let \(z\) such that \(\star \mapsto_{a.m.n.t'} z\). It is easily seen that \(z \geq y\), and \(z \cap x \leq y \cap x\). As a result, \(z \cap x = y \cap x\), and we can work with the new play \(t'' = u.m.n.t'\). Thus, we can intersect \(z\) and \(x\), and consider the play \(u' = u.m.n\) that targets this position in \(\sigma\). However, this time, we consider \(s''\) such that \(u'.s'' \sim_{OP} s\), and \(s''\) is two moves shorter than \(s'\). we apply the induction hypothesis on it.

The sixth condition, preservation of compatibility, is a simple consequence of the forward consistency of \(\sigma^*\). Let \(x\) be a position reached by \(\sigma\) (by a witness play \(s\), and two compatible opponent moves \(m, m'\) played from \(x, x \mapsto y, x \mapsto y', y \in C_{post} y',\) such that \(m, m', n' \in \sigma\). Then \(n, n'\) can be chosen such that \(n \cap n',\) entailing \(n \cap n',\) by forward consistency, \(m, m', n' \in \sigma\) and reaches a position that dominates \(y \mapsto y' = x \oplus \{m, m'\}\).

Finally, \(x\) satisfies the last property (9) of mutual attraction. Let \(x, y \in \sigma^*,\) such that \(x \leq y\). Then let \(u\) prove that there exists a path \(x \mapsto y\) such that \(s\) begins with an \(O\)-move. To do that, let consider a path \(s : x \mapsto y \in \sigma\), and let \(t : x \mapsto y \in \sigma\). As \(x \leq y\), then \(t \leq s\). Then by proposition C.5, \(t \leq_{OP} s\), and \(s \sim_{OP} t.s' \in \sigma\). Hence, \(s' : x \mapsto y\) is either empty, or starts with an \(O\)-move and finishes with a \(P\)-move, which finishes the last case of this proof.

\[\square\]

### C.1 Innocent strategies and weak sequentiality structures

We say that a cell \(\alpha\) justifies a move \(m,\) written \(\alpha \vdash m\), if \(m = (\alpha, v, S)\). We say that a cell \(\alpha\) is accessible from a position \(x \in \text{Legal}(A)\) in a dialogue game if:

\[\exists m \in x.m = (\beta, v, S), \alpha \in S, \text{and } \neg(\exists m' \in x.m' = (\alpha, v', S'))\]

In other terms, the cell \(\alpha\) justifies no move in the position. Recalling the fact that the dialogue game is almost the syntactical tree of the formula, the accessible cells correspond to those sub-formulas that are yet to be explored by the position. We denote \(A_{x}\) the set of accessible cells of the position \(x\). We divide \(A_{x}\) into two subsets, \(A_{x}^+\) the subset of \(A_{x}\) of cells of positive polarity and \(A_{x}^-\) those of negative polarity. Those of negative polarity are brought by a move of positive polarity and justify moves of negative polarities, and inversely for those of negative polarities.

**Definition C.11** Let \(\sigma\) be an innocent strategy, and \(x \in \sigma^*\). Let \(\alpha, \beta \in A_{x}^+\) two different opponent cells. Let \(\sigma \upharpoonright x\) be defined as follows:

\[\sigma \upharpoonright x = \{s : x \mapsto y | \exists s' : \star \mapsto x.s'.s' \in \sigma \land \forall m \in s.(\exists \alpha' \in A_{x}^- \{[\alpha], \alpha' \vdash m})\}\]

\(\sigma \upharpoonright x\) is the restriction of \(\sigma\) above \(x\).

\(\sigma \upharpoonright x\) is the part of the strategy above \(x\) that corresponds to a trigger by opponent of the cell \(\alpha\). Given a move \(m\), we say that \(m \in \sigma \upharpoonright x\), if there is a sequence \(s \in \sigma \upharpoonright x\), such that \(m\) is a move of \(s\).

**Lemma C.12** The following property holds:

\[m \in \sigma \upharpoonright \alpha \Rightarrow \forall n \in \sigma \upharpoontright \beta.m \neq n\]

This lemma is quite a strong property, also called the “separation of contexts”. It says that the strategy above two different cells explores two distinct sub-trees.

**Proof.** We consider a move \(\sigma\) of \(\sigma \upharpoonright \alpha\) (we use a Greek letter to distinguish it from the other moves), and let \(s \in \sigma \upharpoonright \alpha\)
such that \( \sigma \in s \). Now, let \( t \) a sequence of \( \sigma \upharpoonright | \beta \), with \( \beta \neq \alpha \). As we could consider \( s' \equiv s, t' \equiv t \) such that \( s' C_{\text{post}} t \), we assume without lost of generality that \( s C_{\text{post}} t \). In that case \( \sigma \equiv n \Rightarrow \sigma = n \). We will prove that for all moves \( m \in s, n \in t, m \parallel n \). As the first move \( m_1 \) of \( s \) is justified by \( \alpha \), and the first one \( n_1 \) of \( t \) by \( \beta \), we already know that \( m_1 \parallel n_1 \). By forward consistency \( \forall i, j \leq 2, m_i \parallel n_j \). So we proved the property for the paths of length 2. The proof is done by induction on the lengths of the paths \( s \) and \( t \).

We consider that the length of \( s \) is \( n + 2 \), \( t \) is of length \( m \), and that we proved the property hold for \( (n, m) \). We write \( s' \) for the subpath of \( s \) that consists of its \( n^{th} \) first moves. By using the forward consistent diagram of innocence and the inductive hypothesis, we can push the path \( t \) along the \( s' \). We now need to prove that \( m_{n+1} \parallel n \) for every move \( n \) of \( t \). As \( m_{n+1} \) is an opponent move, its departure cell is brought by a player move. By definition of \( s \), its departure cell cannot be a cell available at \( x \). Hence it is brought up by a player move in \( s' \). As \( s', t \) are independent (in the sense that \( \forall m \in s, n \in t, m \parallel n \), the move \( m_{n+1} \) is not related by the partial order, not in conflict with the moves of \( t \), and is post-compatible. Therefore, \( m_{n+1} \parallel t \) (meaning it is strongly compatible with every move of \( t \)). Hence, by repetition applications of forward consistency \( m_{n+1} \parallel t \), finishing the inductive case.

If \( t \) is of length \( m + 2 \), the induction is strongly similar, as the role being played by \( s, t \) are interchangeable. \( \square \)

There is a direct corollary to this lemma, that highlights why we speak about separation of contexts.

**Corollary C.13** Let \( x \) a position of \( \sigma^* \), and \( \alpha \in A^* \). Then there is a set, called \( \text{dominion}_*(\alpha) \subseteq A^* \) defined by:

\[
\text{dominion}_*(\alpha) = \{ \beta \in A^*_\upharpoonright | \exists m \in \sigma \upharpoonright, \alpha, \beta \parallel m \}
\]

such that \( \sigma \upharpoonright | \alpha \) takes place above \( \alpha \cup \text{dominion}_*(\alpha) \) and such that \( \alpha \neq \beta \Rightarrow \text{dominion}_*(\alpha) \cap \text{dominion}_*(\beta) = \emptyset \).

This corollary enables the following definition.

**Definition C.14** Given an innocent strategy \( \sigma \), there exists a family partial function \( \{ \phi_e : A^*_\upharpoonright \rightarrow A^* | x \in \sigma^* \} \), called weak sequentiality structure, such that:

\[
\phi_e(\alpha) = \beta \Leftrightarrow \exists m \in \sigma \upharpoonright, \alpha, \beta \parallel m.
\]

A function \( \phi_e \) is partial as it is undefined on cells without move above (that is, cells corresponding to \( \bot \)), or even on cells above whom the strategy does not operate. Given a cell \( \alpha \in A^* \), we say that \( \text{dominion}(\alpha) \) is the context captured by \( \alpha \). Using the correspondence between untyped cells and formulas of tensorial logic, this corollary is the game semantics counterpart of the following (wrong) logical rule:

\[
\frac{\Gamma_1 \vdash A \quad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}.
\]

Indeed, the formulas on the right hand side of the sequents correspond to negative cells, the ones on the right to positive ones. Now, the corollary tells us that the strategy splits in two independent parts, depending on the cell (formula) chosen by opponent. However, as one can clearly notice, the main problem is that the context is affine, there might be some parts of the context (that is some positive cells), that might not be explored. This will be targeted in Section E.2.

We introduce an invariant of the sequentiality structure.

**Lemma C.15** Let \( s : \star \rightarrow x \rightarrow y \in \sigma \), and \( \alpha \in A^*_\upharpoonright \cap A^* \). Then \( \text{dominion}_*(\alpha) = \text{dominion}_*(\alpha) \).

In other terms, let \( x, y \in \sigma^* \) such that \( x \leq y \). Then if \( \alpha \in A^*_\upharpoonright \cap A^*_\preceq \), it holds that \( \text{dominion}_*(\alpha) = \text{dominion}_*(\alpha) \).

**Proof.** We do the proof in the case where \( t = m.n \). This proof generalises straight away in the general case.

Consider the set of plays \( s \) that belongs to \( \sigma \upharpoonright | \alpha \). As \( \alpha \) is an accessible cell of both \( x \) and \( y \), it means that the play \( m.n \) belongs to \( \sigma \upharpoonright | \beta \) for \( \beta \neq \alpha \). We proved above (in the proof of C.12) that this entails \( m.n \parallel \sigma \upharpoonright | \alpha \). Consequently, the plays of \( \sigma \upharpoonright | \alpha \) can be pushed above \( m.n \) (assuming they satisfy the necessary conditions of \( C_{\text{post}} \)), and \( \sigma \upharpoonright | \alpha \upharpoonright | s \parallel s C_{\text{post}} y \). Given a path \( s \in \sigma \upharpoonright | \alpha \), there exists a path \( s' \) such that \( s' \equiv s, s' C_{\text{post}} y \) and \( s' \in \sigma \upharpoonright | \alpha \). Consequently, if a cell of \( x \) justifies a move in \( \sigma \upharpoonright | \alpha \), it also justifies one in \( \sigma \upharpoonright | \alpha \).

One of the key points of sequentiality structures is this simple property, that states that if \( \phi(\alpha) = \beta \), then \( \beta \) has appeared before \( \alpha \) in the sequence. This will turn to be a central point to later prove compositionality.

**Definition C.16** Let \( \sigma \) be an innocent strategy, \( s : \star \rightarrow x \rightarrow y \in \sigma \), and \( \alpha \in A^*_\upharpoonright \). We define \( \|s\|_\alpha \) as the length of the minimal prefix \( s' : \star \rightarrow y \) of \( s \) such that \( \alpha \in A^*_\upharpoonright \).

**Proposition C.17** Let \( \sigma \) be an innocent strategy, \( s : \star \rightarrow x \rightarrow y \in \sigma \), such that \( \alpha \in \text{dominion}(\beta) \). Then \( \|s\|_\beta \leq \|s\|_\alpha \).

Note that, one can equivalently write \( \alpha \in \text{dominion}_*(\beta) \) or \( \phi_e(\alpha) = \beta \). Hence \( \phi_e(\alpha) = \beta \Rightarrow \|s\|_\alpha \leq \|s\|_\beta \).
Proof. Let \( s, \alpha, \beta \) as before. Let \( s' : \ast \rightarrow y \) the smallest even prefix of \( s \) that reaches a position where \( \beta \) is available. Then, as \( \phi_c^{-1}(\beta) = \phi_c^{-1}(\beta) \), it entails that \( \alpha \in A^*_x \). Now, as \( \beta \) is a negative cell, is has been brought by a player move. This one has to be the last move of \( s' \). Similarly, as \( \alpha \) is a positive cell, it has been brought by an opponent move, and therefore has to be already available before the last move of \( s' \). That is, \( \|s'\|_p < \|s'\|_y \). Finally, as for any given cell \( \gamma \in A_x \cap A_y \), we naturally have \( \|s\|_y = \|s'\|_y \), we conclude. \( \square \)

For instance, let us suppose that \( \alpha \) appears at a position \( y \) in \( s \). As \( \alpha \in A^+_x \), it is introduced by an opponent move \( m \), and we consider the positions \( \sigma \upharpoonright x \overset{m}{\underset{n}{\rightarrow}} y \overset{n}{\rightarrow} z \in \sigma^* \) right below and above \( y \) in \( s \). Suppose that \( \phi_z(\alpha) = \beta \) points to a cell available at \( x \). This situation is pictured in the figure below:

This entails there is a sequence in \( \sigma \upharpoonright x, \beta \) that contains a move \( m \) above \( \alpha \). Consequently, there is a sequence in \( \sigma \upharpoonright x, \beta \) that contains a move above \( \lambda \) (as \( \alpha \) is above \( \lambda \)), which is in contradiction with the definition of \( \sigma \upharpoonright x, \beta \). Therefore, the cell that \( \phi(\alpha) \) points to appears after \( \alpha \); it is brought up by \( n \) in \( z \).

We end up this section with a final technical lemma.

Lemma C.18 Let \( \sigma \) an innocent strategy, and \( x, y \in \sigma^* \) such that \( x < y \). Let \( A_{cy} \subseteq A_x \) be the subset of cells \( \alpha \) available at \( x \) such that \( \exists m. \alpha \vdash m \) and \( x \overset{m}{\rightarrow} y \). Then \( \phi_x \upharpoonright A_{cy} \) is a total, surjective function \( A^+_x \rightarrow A^-_{cy} \).

Proof. We focus on totality. Let \( \alpha \in A^+_x \). Then let \( s' : x \rightarrow y \in \sigma \). This path has a move \( m \) such that \( \alpha \vdash m \). Therefore, \( \exists \beta \in A^-_{cy} \) such that \( m \in \sigma \upharpoonright \beta \). Then \( \phi_x(\alpha) = \beta \) and hence \( \phi \) is total. \( \square \)

D On composition of innocent strategies

D.1 Transverse strategies

We remind that our goal is to have a category, whose objects are the dialogue games, seen as arenas, and whose morphisms \( A \rightarrow B \) are strategies on \( A \upharpoonright B \). In order to make it work, we have to constrain ourselves to strategies that are transverse. They are those whose positions in \( A \upharpoonright B \) are transverse, as defined in section ??.

Definition D.1 • A play \( s \) of \( A \upharpoonright B = (A \otimes \neg B)^* \), of length more than 2 is transverse if the second move of \( s \) belongs in the arena \( B \).

• A strategy is transverse if every play in it is transverse.

Equivalently, a play is transverse if it reaches a transverse position, and a position is transverse if it is reached by a transverse play. Let us remind here the structure of \( A \upharpoonright B \), in the easy case where \( A, B \) are simple.
Then any transverse play of length greater than 2 starts with \(m.n\) (or any congruent moves). Finally, let us remind that in section ??, we established a bijection:
\[
\text{Trans}(A \triangleright B) \simeq \text{Pos}(A) \otimes \text{Pos}(B).
\]
However, if we restrict to legal positions, then:
\[
\text{Legal(Trans}(A \triangleright B)) \neq \text{Legal}(A) \otimes \text{Legal}(B).
\]
since in the right hand side term, a name of \(A_{\text{cell}}\) might belong to both \(A\) and \(B\). Hence, the bijection is as follows:
\[
\text{Legal(Trans}(A \triangleright B)) \simeq \{(x, y) \mid (x, y) \in \text{Legal}(A) \otimes \text{Legal}(B), x \neq A_{\text{cell}}\}.
\]

D.2 Relational and sequential compositions

We are in possession of two descriptions of innocent strategies. One is based on the set of positions they reach, the second on the set of sequences that realise them. We now address composition, and prove that relational and sequential composition are equivalent.

D.2.1 Sequential composition

We consider interaction of sequences. Given \(\sigma, \tau\) define \(\sigma \mid \tau\) by:
\[
\sigma \mid \tau = \{s \in \text{Legal}(A \triangleright B \triangleright C) \mid s \uparrow A \triangleright B \in \sigma, \text{ and } s \uparrow B \triangleright C \in \tau\}.
\]
We might want to prove the projection of \(s \uparrow A \triangleright C\) leads to a legal, alternating sequence. Unfortunately, it does not necessarily hold, as the play might be non-alternating. Indeed, consider the following case, where we write \(O/P\) to indicate that the polarity of the move is \(O\) from the left arena point of view, and \(P\) from the right arena point of view:

\[
\begin{array}{ccc}
A & B & C \\
O & P/O & \\
P & P/O & O/P \\
O/P & O/P & P/O \\
O & \\
\end{array}
\]

By projecting this play on \(A \rightarrow C\), we reach a non-alternating \(O - P - O - O - P - P\) sequence. So we have to select the alternating plays, leading to the following definition.

**Definition D.2** Given two strategies \(\sigma : A \rightarrow B\) and \(\tau : B \rightarrow C\) we define their sequential composition by:
\[
\sigma; \tau = \{s \in \text{Alt}(A \rightarrow C) \mid \exists \rho \in \sigma \mid \tau. \ t \uparrow A \rightarrow C = s\}.
\]
D.2.2 Relational Composition

In this paragraph, we briefly recall the definition of relational composition. Given a transverse innocent strategy \( \sigma : A \rightarrow B \), its set of positions \( \sigma^* \) forms a subset of \( \text{Legal}(A) \bowtie \text{Legal}(B) \). Similarly, if we consider a second transverse innocent strategy \( \tau : B \rightarrow C \), its set of positions \( \tau^* \) forms a set of positions in \( \text{Legal}(B) \bowtie \text{Legal}(C) \). Using the fact that \( \text{Legal}(A) \bowtie \text{Legal}(B) \subseteq \text{Legal}(A) \times \text{Legal}(B) \), one can see the set \( \sigma^* \times \tau^* \) as a relation \( \text{Legal}(A) \rightarrow \text{Legal}(B) \times \text{Legal}(C) \). We are interested in making use of the relational composition. The only property that needs to be checked is that, given \( \sigma \) and \( \tau \) as above, does \( \sigma^* \times \tau^* \subseteq \text{Legal}(A) \times \text{Legal}(C) \) correspond to a subset of positions of \( \text{Legal}(A \rightarrow C) \). Unfortunately, this is not the case, as the name separation condition is not invariant under composition. For instance, if we compose \( a \rightarrow b \) with \( b \rightarrow a \), we obtain \( a \rightarrow a \), where the name \( a \) is reused across the \( \rightarrow \) operator.

To cope with these minor difficulties, given two subsets \( R \subseteq A \times B \) and \( R' \subseteq B \times C \) we define a new \( \cdot \text{rel} \) for the remaining of this section as follows:

\[ R \cdot \text{rel} R' = \{ (x, z) \mid x \# \text{cell} \subseteq \exists y, (y, z) \in R, (y, z) \in R' \} \]

Finally, we write \( R \mid \text{rel} R' \) for the subset of \( A \times B \times C \) defined below:

\[ R \mid \text{rel} R' = \{ (x, y, z) \mid (x, y) \in R, (y, z) \in R', x \# \text{cell} \subseteq \} \]

Finally, given a position \( (x, y) \) of \( \text{Legal}(A) \rightarrow \text{Legal}(B) \), such that \( x \# \text{cell} \subseteq y \), we will indifferently deal with it as an element of either the original set or \( \text{Legal}(A \rightarrow B) \). Sometimes, as analogy with the theorem C.10 we will write \( X \) for a definable set of positions.

D.2.3 The correspondence

Proposition D.3 Let \( (x, y, z) \in \sigma^* \mid \text{rel} \tau^* \). Then there is a sequence \( s : \star \rightarrow (x, y, z) \in \sigma \mid \text{rel} \tau \).

In the sequel, we use conflict-freeness for the property that given a play \( s : \star \rightarrow x \in \sigma \), if there exists a \( y \in \sigma^* \) such that \( y \geq x \), given any opponent move \( m : x \rightarrow x' \) such that \( x' \leq y \), there exists a path \( s \cdot m \cdot s' : \star \rightarrow y \). This is a reformulation of forward confluence, and hence every innocent strategy is indeed conflict-free. We also write \( (\sigma \mid \tau^*) \) for the set of positions reached by sequences in \( (\sigma \mid \tau) \). Moreover, we shall write \( P\text{-cell}, P\text{-move} \) for proponent cell, proponent move, and respectively for \( O \) and opponent.

Proof. Suppose this is not the case and let \( s : \star \rightarrow (x', y', z') \in \sigma \mid \tau \) be an even alternating sequence that reaches a maximal position under \( (x', y', z') \in (\sigma \mid \tau)^* \). Then \( s \cdot \hat{A} \rightarrow B \cdot s \in \sigma \), and, as \( \sigma \) is conflict free, let \( s' \) be a sequence such that \( (s \cdot \hat{A} \rightarrow B) \cdot s' : \star \rightarrow (x, y, z) \). Furthermore, let \( t' \) be its counterpart in \( B \rightarrow C \). Now, if either \( s' \) or \( t' \) has its first move in either \( A \) or \( C \) (let us say, \( s' \)), then we add to \( s \) the two first moves of \( s' \). Either the two are in \( A \), in which case we have reached a higher position of \( (\sigma \mid \tau)^* \) under \( (x, y, z) \). Or, one is in \( A \) and the other in \( B \). In this case, the move in \( B \) appears as opponent move for \( \tau \). Furthermore, the \( B \rightarrow C \) part is still under \( y \rightarrow z \), hence by conflict-freeness, we can complete the sequence by the reaction from \( \tau \). Then if this move is in \( C \), we stop, and have reached a new position of \( \sigma^* \mid \text{rel} \tau^* \). If it is in \( B \), then by conflict-freeness, we can extend the sequence with a reaction from \( \sigma \) and so on. This whole process terminates, as any position of \( B \) can only have a finite number of moves. Furthermore, the newly created play has alternating projection on \( A \rightarrow C \). Therefore, in the case where either \( s' \) or \( t' \)'s first move in \( A \) or \( C \), then we can reach a higher position under \( (x, y, z) \) contradicting the maximality of \( (x', y', z') \).

So let suppose that for all sequences \( s', t' \) as above, the first moves of \( s' \) and \( t' \) are in \( B \). By conflict-freeness of \( \sigma \) and \( \tau \), it means that every path \( x' \rightarrow x \) (respectively \( z' \rightarrow z \)) starts with a \( P \)-move, and hence there are only \( P \)-cells below \( x \) in \( x' \) (respectively below \( z \) in \( z' \)). That is, all paths from \((x', y', z') \) to \((x, y, z) \) in \( \sigma \) (respectively from \((y', z', z) \) to \((y, z, z) \) in \( \tau \)) begin with an \( O \)-move from \( y' \). We will see that there is a contradiction.

Let \( \phi \) be the sequentiality structure associated with \( \sigma \), and \( \psi \) the sequentiality structure associated with \( \tau \). Consider \( a \) a available cell of \( A_{y' < z'} \) that has been introduced last in \( s \). Then suppose that \( a \) is an \( O \)-cell of \( B \), then it is a \( P \)-cell of \( B \rightarrow C \). Then as \( a \in A_{y' < z'} \), \( \psi(y', z') (a) \) is well defined. Furthermore, as only \( P \)-cells are available at \( z' \), \( \beta = \psi(y', z') (a) \) is a cell of \( y' \). But then \( \beta \) must have been introduced after \( a \), by proposition C.17. Furthermore, \( \beta \) belongs in \( A_{y' < z'} \) by C.18. However, \( a \) was, by definition, a latest cell of \( A_{y' < z'} \) to be introduced, we reach a contradiction. Thus, we can conclude that \( a \) is a \( P \)-cell of \( B \). In that case, we can repeat the reasoning using \( \psi \) instead of \( \psi \), and get a similar contradiction.

Overall, we reach the conclusion that there exist sequences \( s' \) or \( t' \) that start with an \( O \)-move in \( A \) or \( C \), and \((x', y', z') \) is not the highest position under \( (x, y, z) \). And therefore, by contradiction, there is a sequence \( s : \star \rightarrow (x, y, z) \in \sigma \mid \text{rel} \tau \). Furthermore, as the position is legal, the sequence leading to it is legal.

The reverse direction is straightforward. If there is a sequence \( s : \star \rightarrow (x, y, z) \) then \((x, y, z) \in \sigma^* \mid \text{rel} \tau^* \), as \((x, y) \in \sigma^*, (y, z) \in \tau^* \), and legal interaction sequences lead to legal positions. Therefore, there is a correspondence between the two compositions. We now investigate briefly some properties about the witness of interaction.

D.2.4 Some additional properties

Before studying more in depth the relation between sequential and relational composition, we prove this simple proposition.

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Proposition D.4 Let \((x_1, z_1)\) and \((x_2, z_2)\) in \(X_{\text{Rel}}\) \(Y\), such that \((x_1, z_1) C_{\text{post.cell}} (x_2, z_2)\). Then, there exists \(y_1, y_2\) such that \(X_{\text{Rel}} Y \ni (x_1, y_1, z_1) C_{\text{post.cell}} (x_2, y_2, z_2) \in X_{\text{Rel}} Y\).

**Proof.** Given any position \((x, y)\) in \(X\) in \(\text{Pos}(A) \sqsupset \text{Pos}(B)\), then the position \((x, y) \in X\) is legal and therefore \(x \#_{\text{cell}} y\). Therefore, picking \(\pi \in \text{Perm}(A_{\text{cell}})\) such that \(\pi \#_{\text{cell}} x\) then \(\pi \cdot (x, y) = (x, \pi \cdot y)\). Furthermore, as \(X\) is closed under permutation, \((x, \pi \cdot y) \in X\). So let us consider \((x_1, y_1, z_1), (x_2, y_2, z_2) \in X_{\text{Rel}} Y\) witnessing \((x_1, z_1), (x_2, z_2)\) respectively. Applying appropriate permutations of \(A_{\text{cell}}\) if necessary, we assume \(y_1, y_2 \#_{\text{cell}} x_2, x_1, z_1\). Then we simply apply the proposition B.8 on the positions \(y_1, y_2\) to conclude the existence of \(\pi \in \text{Perm}(A_{\text{cell}})\) such that \(\pi \cdot y_1 C_{\text{post.cell}} y_2\). Furthermore, \(\pi\) could be chosen such that \(\pi \# x_1, z_1\), and \(\pi \cdot y_1 \#_{\text{cell}} x_1, z_1\). This way, we obtain \(\pi \cdot (x_1, y_1, z_1) = (x_1, \pi \cdot y_1, z_1)\), and \((x_1, y_1, z_1) C_{\text{post.cell}} (x_2, y_2, z_2)\) as expected. □

The properties proven in this section will be useful in order to prove that the composition of innocent strategies results in innocent strategies. The first goal is to establish that there is a unique witness of interaction, and then to prove that this one behaves well with regard to compatibility. We prove below that the strategy "preserves the compatibility": the incompatibility between different positions of the strategy has to originate from the opponent. That is, it must be opponent that introduces first a move incompatible with future positions.

Recall that we write \((m \prec n)\) for the conflict relation, defined by \(m \prec n \iff (m \uparrow n)\). Given a path \(t = n_1, \ldots, n_k\) and a move \(m\), we write \(m \sim t\) if \(\exists m \in t . m \prec n\). This extends to paths: \(s \sim t\) if \(\exists m \in s, n \in t . m \text{conflict}\). Similarly, given a position \(x\), we write \(m \sim x\) if, seeing \(x\) as a set of moves, \(\exists m \in x . m \prec n\).

**Lemma D.5** Let \(x, y \in \sigma^*\) such that \(x C_{\text{post.cell}} y\) and \(x \sim y\), then given any path \(\sigma \ni s : \star \rightarrow x\), the first move \(m\) of \(s\) such that \(m \sim y\) is an opponent move.

**Proof.** Let \(x'\) be the greatest member of \(\sigma^*\) under \(x\) such that \(x' \uparrow y\). Namely, we define:

\[
x' = \{z \in \sigma^* | z \leq x, z \sim y\}.
\]

By noticing that the set \(\{z \in \sigma^* | z \leq x, z \sim y\}\) is stable under union (since \(\sigma^*\) is closed under compatible union \(C\), we conclude that \(x'\) belongs to it. Similarly, we define \(y'\) for \(y\). As \(\sigma^*\) is closed under compatible union, \(x' \cup y', x' \cup y' \in \sigma\). Now let us consider two paths \(\sigma \ni x : x' \cup y' \rightarrow x \cup y\) (meaning that there exists \(x' : \star \rightarrow x' \cup y'\) such that \(s', s \in \sigma\) and \(\sigma \ni t : x' \cup y' \rightarrow x' \cup y\). Such paths exist since \(x' \cup y\) is a point of view and \(\sigma\) is conflict-free. These paths are non-empty, as \(x \sim y\) so \(x' \neq x\) and \(y' \neq y\). Let us name the moves of \(s (s = m_1.m_2,...,\ldots)\) and \(n\) the moves of \(t (t = n_1.n_2,...,\ldots)\). Then we shall have \(m_1.m_2 \sim n_2.n_2\), otherwise this would contradict the maximality of \(x'\) and \(y'\) as parts of \(x, y\) that are not in conflict. It should be noted that if \(m_1 \uparrow n_1\) (respectively \(m_1 = n_1\)), then, by the forward consistency of innocence, (respectively by nominal determinacy and compatibility) \(m_1.m_2 \uparrow n_1.n_2\) (respectively \(m_1.m_2 = n_1.n_2\)), and therefore, \(m_1 \sim n_1\).

So let us consider a play and a \(m\) such as in the lemma, and assume for contradiction that it is a player move. Then, in the play, it belongs to an OP-pair \(m', m\), and this one does not belong in \(x'\). Therefore, there is a path of the strategy from \(x' \cup y' \rightarrow x \cup y\) that starts with \(m'.m\), and \(m' \sim y\) as explained above, contradicting that \(m\) is the first move of the play to be in conflict with \(y\) . □

The previous lemma will help us prove the following proposition, stating the unicity of the witness of interaction.

**Proposition D.6** Let \((x, y, z) \in \sigma^* |_{\text{Rel}} \tau^*\), and \((x, y', z') \in \sigma^* |_{\text{Rel}} \tau^*\), such that \(y C_{\text{cell}} y'\), then \(y = y'\).

**Proof.** We already proved earlier in lemma D.5 that if there were two distinct \(y\), such that \(y C_{\text{post.cell}} y'\), then they must be in conflict. According to the previous lemma, from the \(\sigma\) point of view, opponent must initiate the conflict in \(B\), that is, protonent from the \(\tau\) point of view. On the other hand, from the \(\tau\) point of view, it must be the opponent that initiates conflict, that is, himself from the \(\sigma\) point of view. This is a dead-end, and there is a unique \(y\).

Finally we note that requiring \(y C_{\text{cell}} y'\) instead of \(y C_{\text{Post.cell}} y'\) is enough. Indeed, let us pick two \(y, y'\) such that \(y C_{\text{cell}} y'\), and \(\pi\) such that \(\pi \cdot y C_{\text{Post.cell}} y'\). Then, in the lemma above, this would imply \(\pi \cdot y = y'\). Therefore \(y \equiv y'\), and, as \(y C_{\text{cell}} y, y = y'\). In particular, given \((x, y, z), (x, y', z') \in \sigma^* |_{\text{Rel}} \tau^*\), then taking \(\pi \in \text{Perm}(A_{\text{cell}})\) such that \(\pi \cdot y C_{\text{cell}} y'\) entails \(\pi \cdot y = y'\). That is, \(y \equiv y'\). □

The witness of interaction preserves compatibility, as formulated in the below proposition. This lemma is a consequence of the preservation of compatibility by the strategy.

**Proposition D.7** Let \((x_1, z_1), (x_2, z_2) \in \sigma^* |_{\text{Rel}} \tau^*\) such that \((x_1, z_1) \uparrow (x_2, z_2)\). Let \(y_1, y_2\) such that \(y_1 C_{\text{Post.cell}} y_2\) and \((x_1, y_1, z_1), (x_2, y_2, z_2) \in \sigma^* |_{\text{Rel}} \tau^*\). In that case \(y_1 \uparrow y_2\), and, in particular, \(y_1 C_{\text{Post}} y_2\).

**Proof.** Suppose \(y_1 \sim y_2\). Let \(\sigma \ni t : \star \rightarrow (x_1, y_1, z_1)\), and let \(n\) be the first move of \(s\) such that \(n \sim y_2\). Then by the lemma above D.5 it has to be an opponent from the \(\sigma\) point of view, and an opponent from the \(\tau\) point of view. So we reach a contradiction, and \(y_1 \uparrow y_2\). In particular, by proposition B.6, \(y_1 C_{\text{Post}} y_2\). □
D.3 Innocent strategies are stable under composition

We find it easier to handle composition through the relational composition of strategies as sets of positions. Our goal is to prove that the composition of two innocent strategies leads to an innocent strategy. For sets of positions, this translates into the following proposition.

Proposition D.8 The relational composition of two definable sets leads to a definable set.

Proof. To begin, let us properly define the notations used along the process. We consider two definable sets \( X \subseteq \text{Legal}(\text{Trans}(A \triangleright B)) \) and \( Y \subseteq \text{Legal}(\text{Trans}(B \triangleright C)) \). If needed, we might consider the strategies \( \sigma \) and \( \tau \) associated respectively to \( X \) and \( Y \). We look at \( X \) and \( Y \) as sets of positions in \( \text{Legal}(A) \otimes \text{Legal}(B) \) and \( \text{Legal}(B) \otimes \text{Legal}(C) \).

The first condition (1) to check is \( \bot \in X_{\text{Rel}} \). This comes from \( (\bot, \bot) \in X, (\bot, \bot) \in Y \), and hence \( \bot = (\bot, \bot) \in X_{\text{Rel}} Y \), as \( (\bot, \bot) \in \text{Rel}(A \triangleright B) \triangleq (\bot, \bot) \in \text{Rel}(A \triangleright B) \).

Next, we need to ensure (2) that \( X_{\text{Rel}} Y \in \text{Legal}(A \triangleright B \triangleright C) \). This follows from the definition of \( \text{Rel} \), that ensures that the resulting positions are indeed legal.

We now investigate the nominal closure (3) of \( X_{\text{Rel}} Y \). Given an element \( x, z \in X_{\text{Rel}} Y \), and \( y \in B \) such that \( (x, y, z) \in X_{\text{Rel}} Y \), then for all \( \pi \in \text{Perm}(A) \), \((\pi \cdot x, y, z) \in X_{\text{Rel}} Y \) as \( X, Y \) are closed under permutation. Therefore \( \pi \cdot (x, y, z) = (\pi \cdot x, y, z) \in X_{\text{Rel}} Y \). Closure under typed substitutions is dealt with on a equal footing.

The two following properties are closure under compatible intersection (4) and union (5). We tackle both at once. Let \( (x_1, z_1), (x_2, z_2) \in X_{\text{Rel}} Y \) such that \( (x_1, z_1) \triangleq (x_2, z_2) \). For the union, we also assume that \( (x_1, z_1) \in \text{Post}_c (x_2, z_2) \). Then let \( y_1, z_2 \) such that \( (x_1, y_1, z_1) \in \text{Cell} (x_2, y_2, z_2) \in X_{\text{Rel}} Y \) for the union. As proven in the lemma D.7, \( y_1 \not\rightarrow y_2 \), and consequently \( (x_1, y_1, z_1) \triangleq (x_2, y_2, z_2) \). In particular, \( y_1 \in \text{Cell} (x_2, z_2) \). As \( X, Y \) are closed under compatible union and intersection, the element \( (x_1 \triangleleft y_1, y_1 \triangleleft z_1, z_1 \triangleleft z_2) \) belongs in \( X_{\text{Rel}} Y \), just as \( (x_1 \triangleleft y_1, y_1 \triangleleft z_1, z_1 \triangleleft z_2) \) (in the case where they are post compatible). So \( (x_1 \triangleleft y_1, z_1 \triangleleft z_2) \in X_{\text{Rel}} Y \), \( (x_1 \triangleleft y_1, z_1 \triangleleft z_2) \in X_{\text{Rel}} Y \), and therefore \( X_{\text{Rel}} Y \) is closed under compatible union and intersection.

The sixth property (6) focuses on conservation of composition after two opponent moves. This corresponds to the first part of forward consistency, namely that given two opponent moves \( m, m' \) starting from a position \( x \) of an innocent strategy and such that \( m \models m' \), if the strategy reacts by playing two moves \( n, n' \) then, taking \( n, n' \) such that \( n \in \text{Post}_c (m' \triangleleft x) \), it entails \( n \models n' \). So let \( (x, y, z) \) be a position of \( \sigma \models \tau \). Let \( m, m' \) two moves starting either from \( x \) or \( z \), \((x_1, y_1, z_1) \) (respectively \((x_2, y_2, z_2)\)) two positions in \( X_{\text{Rel}} Y \) such that \( \sigma \models \tau \). Then, taking \( (x_1, y_1, z_1) \) \( \triangleq (x_2, y_2, z_2) \) and \( \sigma \models \tau \), we have that \( (x_1, y_1, z_1) \) \( \triangleq (x_2, y_2, z_2) \). In particular, using \( \text{Perm} (A \triangleleft x) \), then \( (x_1, y_1, z_1) \) \( \triangleq (x_2, y_2, z_2) \). Moreover, by preservation of \( \text{Cell} \), \( (x_1, y_1, z_1) \) \( \triangleq (x_2, y_2, z_2) \). As \( x_1 \models x_2 \), \( y_1 \models y_2 \), and \( z_1 \models z_2 \), then \( (x_1, y_1, z_1) \) \( \triangleq (x_2, y_2, z_2) \). Therefore, \( X_{\text{Rel}} Y \) is closed under compatible union and intersection.

Before proving the next properties, we introduce the following remark. Let \( x = (x_A, x_C), y = (y_A, y_C) \) two positions of \( X_{\text{Rel}} Y \) such that \( x < y \), and let us take two positions \( x' = (x_A, x_B, y_C) \) and \( y' = (y_A, y_B, y_C) \) in \( X_{\text{Rel}} Y \) such that \( x' \in \text{Cell} \ y' \). Then \( x' \not\rightarrow y' \). Indeed, by preservation of compatibility, \( x' \not\rightarrow y' \) and in particular \( x' \in \text{Cell} \ y' \). By stability under compatible permutation, \( (x_1 \triangleleft y_1, y_1 \triangleleft y_2, y_2 \triangleleft y_B, y_B \triangleleft y_C) \in X_{\text{Rel}} Y \). Moreover, \( y_B \in \text{Cell} \ y_B \) by definition. Consequently, by proposition D.6, \( y_B \models y_B \), and hence \( x' \not\rightarrow y' \).

On the seventh point, we tackle the stability of forward consequence (7)/(8). Let \( x \in X_{\text{Rel}} Y \) such that there is an opponent move \( m \models y \) and \( y \) is dominated in \( X_{\text{Rel}} Y \) by \( w \). One needs to prove that there is a unique P-move \( n \) such that \( x \not\rightarrow z \in X_{\text{Rel}} Y \) and furthermore \( \gamma_C (n) \subseteq \gamma_C (y) \cup \gamma_C (m) \). We suppose (without loss of generality) that \( m \) is \( A \). We use the unique property of interaction D.6. Let \( x = (x_A, x_C) \), then there is a unique (up to cells permutation) \( y_B \) such that \( (x_A, x_B, y_C) \in X_{\text{Rel}} Y \). As \( w \in X_{\text{Rel}} Y \), there exists a unique, up to cells permutation, \( w' \in X_{\text{Rel}} Y \) such that \( w' \not\rightarrow A \not\rightarrow C = w \). Furthermore, \( w' \) can be chosen such that \( w' \not\rightarrow A \not\rightarrow C = (x_A, x_B, y_C) \). Suppose that \( x \not\rightarrow z \). As \( X \) answers to \( m \) in \( A \), then we conclude easily. In the case where \( X \) answers \( m \) with a move \( n_1 \) in \( B \), then it will answer it uniquely such that it reaches a position dominated by \( w' \). Furthermore, this move will respect the type condition on its support. Now, however, it is Y’s turn, and for the same reason, \( Y \) will answer uniquely by a \( n_y \) whose target position that will remain dominated by \( w' \). Furthermore, \( \gamma_C (n) \subseteq \gamma_C (y) \cup \gamma_C (m) \) and \( \gamma_C (n) = \gamma_C (y) \cup \gamma_C (m) \). This way we can build an alternated sequence of moves alternating between \( \sigma \) and \( \tau \), until one of the strategy moves into either \( A \) or \( C \). Then the position reached still remains dominated by \( w' \). Projecting it on \( A \not\rightarrow C \), we get a desired P-move reaction in \( A \not\rightarrow C \). Furthermore, following the uniqueness of the sequence that leads to this P-move, one can gather that this P-move is unique satisfying the conditions. Finally, the type condition follows from \( \gamma_C (y) = \gamma_C (x) \), since each move in \( B \) is a P-move from either the left or the right point of view. (7) is proven on an equal footing.

Remaining is the proof of mutual attraction (10). Let \( s : x' \not\rightarrow y' \in \sigma \models \tau \) and \( t : x' \not\rightarrow y' \in \sigma \models \tau \), such that \( x' \not\rightarrow y' \). We set \( x' = (x_A, x_B, y_C), y' = (y_A, y_B, y_C) \) and \( y = y' \not\rightarrow A \not\rightarrow C = (y_A, y_C) \). As above. Let \( s' = s \setminus x' \) defined by induction as follows:

\[
\begin{align*}
\text{if } s = \emptyset & \Rightarrow s' = \emptyset \\
\text{if } s = m.s'' \land m \in x' & \Rightarrow s' = s'' \setminus x \\
\text{if } s = m.s'' \land m \notin x' & \Rightarrow s' = m.s'' \setminus x
\end{align*}
\]
The resulting \( s' \) is simply a sequence of moves. It is not a play, as it does not necessarily start from the root. We consider the sequence \( t.s' \). Our goal is to show that it is in \( \sigma \text{ in } \tau \). First of all, we need to check that \( t.s' \) is a play. We check the 3 conditions of proposition ??.

If \( x = \perp \), this is straightforward. In the case where \( x \neq \perp \), the first move of \( t \) is an initial move, and this is the only one of \( t.s' \). Furthermore, no moves are repeated in \( t.s' \). Finally, we need to check the down-closure. Let \( m \) be a move in \( t.s' \). Either this move belongs to \( t \), and, as \( t \) is a play, the predecessor of \( m \) is in \( t \). Either it belongs to \( s' \).

Then either its predecessor belongs in \( s' \) as well, and, as \( s \) is a play, it appeared before it in the sequence. Or it does not belong to \( s' \), in which case we can conclude it belonged to \( s' \). In that case, it appeared in \( t \), and hence before it. So this sequence forms a play. Furthermore, it reaches the position \( y' \), which is legal. Therefore the sequence is legal.

Now, let us look at \( t.s' \uparrow A \rightarrow B \). We already know that \( t \uparrow A \rightarrow B \in \sigma \). We will prove that \( t.s' \uparrow A \rightarrow B \in \sigma \) by even induction on the length of \( t' \).

Indeed, every position \( (x, y) \) is a copycat. In order to conclude that the innocent, transverse, typed-coherent strategies form a category, we simply need to present \( \text{INN} \).

D.3.1 The category \( \text{INN} \)

Innocent strategies form the ground structure behind denotation of tensorial logic proofs. However, they do not form a fully complete model. We will have to select those strategies that satisfy additional conditions to achieve the final result. Before presenting those, we examine how the weak sequentiality structures that come with innocent strategies behave with composition.

Corollary D.9 Innocent, transverse, typed coherent strategies are stable under composition.

Innocent strategies form the ground structure behind denotation of tensorial logic proofs. However, they do not form a fully complete model. We will have to select those strategies that satisfy additional conditions to achieve the final result. Before presenting those, we examine how the weak sequentiality structures that come with innocent strategies behave with composition.

D.3.1 The category \( \text{INN} \)

In order to conclude that the innocent, transverse, typed-coherent strategies form a category, we simply need to present strategies that act as identity morphisms. These are called copycat strategies.

Given an arena \( A \), we define the transverse, innocent strategy typed-coherent copycat\(_{A} \) : \( A \rightarrow A \) as follows, where we tag the two occurrences of \( A \) in \( A \rightarrow A \) as \( A_{1}, A_{2} \) respectively.

\[
\text{copycat}_{A} = \text{Alt}(\text{Legal}(\{ s \in \text{Play}(A_{1} \rightarrow A_{2}) \mid s \uparrow A_{1} \simeq_{\text{cel}} s \uparrow A_{2})\})
\]

where we recall that we write \( \simeq_{\text{cel}} \) to signify that there must exist \( \pi \in \text{Perm}(A_{\text{cel}}) \) such that \( \pi \cdot s \uparrow A_{1} = s \uparrow A_{2} \). The copycat strategy can also be described through its set of positions:

\[
\text{copycat}_{A}^{*} = \text{Legal}(\{(x,y) \in (\text{Trans}(A_{1} \rightarrow A_{2})) \mid x \simeq_{\text{cel}} y\})
\]

Indeed, every position \( (x, y) \) such that \( x \simeq y \) can be reached by an alternating sequence.

Proposition D.10 copycat\(_{A} \) is a transverse, typed-coherent, innocent strategy.

Proof. We work with copycat\(_{A} \) as a set of sequences. By definition copycat\(_{A} \) is closed under nominal equivalence and strict substitutions. Furthermore, given \( s \in \text{copycat}_{A} \), and \( m \) in, let us say, \( A_{1} \) such that \( s.m \) is legal, then \( s.m.n \in \text{copycat}_{A} \) with \( n \simeq_{\text{cel}} m \) in \( A_{2} \). Hence, \( s.m.n \) is legal, and copycat\(_{A} \) is closed by even prefix obviously. Furthermore, the condition \( s \uparrow A_{1} \simeq_{\text{cel}} s \uparrow A_{2} \) ensure that all the sequences in it are even-length, and that the strategy is nominal deterministic.

To simplify things, given a move \( m \) in \( A_{1} \), we write \( m' \) for its equivalent one in \( A_{2} \), and respectively. That is, \( (m'y) = m \). Finally, given \( s \in \text{copycat}_{A} \), and \( m_{1}, m_{2} \) such that \( m_{1} \uparrow m_{2} \) and \( s.m_{1}, s.m_{2} \) are legal and post compatible, then \( s.m_{1}.(m_{1} \cdot m_{1}).m_{2}(m_{2} \cdot m_{2}) \in \text{copycat}_{A} \) (where \( \pi_{1}, \pi_{2} \in \text{Perm}(A_{\text{cel}}) \) are picked such that the sequence is legal), and therefore the strategy is forward consistent.

Similarly, if a sequence \( s.m_{1}, m_{2}, n_{1}, m_{2}, t \) is in copycat, then so is \( s.n_{1}, n_{2}, m_{1}, m_{2}, t \) and therefore the strategy is backward and forward consistent, that is, innocent.
Finally, it is straightforward, especially by looking as copycat as a set of positions, that copycat acts as the identity: \( \sigma; \) copycat = \( \sigma \) and copycat; \( \sigma \) = \( \sigma \). So overall, we got this final proposition.

**Proposition D.11** \( \text{INN} \) is a category,

- whose objects are positive dialogue games that arise as denotations of formulas of tensorial logic.
- whose morphisms are transverse, innocent, typed-coherent strategies of negative dialogue games of the form \( A \to B \).

### D.4 Composition of weak sequentiality structure

We now address the composition of weak sequentiality structure. Let \( (x, y) \) be a position of \( \text{Trans}(A \to B) \), seen as \( \text{Pos}(A) \otimes \text{Pos}(B) \). A weak sequentiality structure \( \varphi_{(x,y)} \) is a partial function \( \varphi_{(x,y)} : A^*_x \cup B^*_y \to A^*_x \cup B^*_y \) where we remind that \( A^*_x \) are the \( O \)-cells of \( A \) available at \( x \) (and respectively for the other ones). Then, given two innocent transverse strategies \( \sigma : A \to B \) and \( \tau : B \to C \), let assume \( \varphi \) and \( \psi \) are their assigned weak sequentiality structure. Then one might wonder if the weak sequentiality structure of \( \sigma ; \tau \) can be computed directly from \( \varphi, \psi \).

Let \( w = (x, y, z) \in (\sigma^* \text{rel} \tau^*) \), then to compute \( (\varphi; \psi)_{(x,z)} : A^*_x \cup C^*_z \to A^*_x \cup C^*_z \), we can rely on:

\[
\begin{align*}
\varphi_{(x,y)} & : A^*_x \cup B^*_y \to A^*_x \cup B^*_y \\
\psi_{(y,z)} & : B^*_y \cup C^*_z \to B^*_y \cup C^*_z \\
\end{align*}
\]

We prove the following theorem.

**Proposition D.12** Let \( \varphi, \psi, w, x, y, z \) as before. Then the sequentiality structure of \( \sigma ; \tau \) at \( w \) is a subset of:

\[
\begin{align*}
\text{Tr}^{B^*_y \cup B^*_y}_{A^*_x \cup C^*_z, A^*_x \cup C^*_z}((\varphi_{(x,y)} \cup \psi_{(y,z)}); (\id_{A^*_x \cup s_{B^*_y \cup B^*_y} \cup \id_{C^*_z}})) : A^*_x \cup B^*_y \cup B^*_y \cup \id_{C^*_z} \to A^*_x \cup B^*_y \cup B^*_y \cup C^*_z
\end{align*}
\]

where the trace is taken from the traced symmetric monoidal category \( (\text{pSet}, \cup, \emptyset) \) of sets and partial functions, with monoidal product the disjoint union of sets, and unit the empty set. The morphism \( s_{B^*_y \cup B^*_y} : B^*_y \cup B^*_y \to B^*_y \cup B^*_y \) is the morphism coming from the symmetry. That is, denoting \( x, y \) the weak sequentiality structure of \( \sigma ; \tau \),

\[
(\varphi_{(x,y)} \cup \psi_{(y,z)}; (\id_{A^*_x \cup s_{B^*_y \cup B^*_y} \cup \id_{C^*_z}}))(\alpha) \Rightarrow \beta
\]

We remind below what is the canonical trace in the category of partial functions. Given \( f : A \times C \to B \times C \), we compute \( \text{Tr}^{C \times C}_{A \times B}(f) \) as follows:

- if \( i \in A \) and \( f(i) \) is defined, \( f(i) \in B \) then \( \text{Tr}^{C \times C}_{A \times B}(f(i)) = f(i) \).
- if \( i \in A \), and \( f(i) \) is undefined, then \( \text{Tr}^{C \times C}_{A \times B}(f(i)) \) is undefined.

- If \( i \in A \), \( f(i) \) is defined and \( f(i) \in C \) then \( \text{Tr}^{C \times C}_{A \times B}(f(i)) = \text{feedback}(f(i)) \).

- Given \( i \in C \) we get \( \text{feedback}(f(i)) = \begin{cases} 
\text{feedback}(f)(f(i)) & \text{if } f(i) \text{ is defined and } f(i) \in C \\
(\text{feedback}(f))i & \text{if } f(i) \text{ is defined and } f(i) \in B \\
\text{is undefined} & \text{if } f(i) \text{ is undefined}
\end{cases} \).

**Proof.** Let \( \alpha \in A^*_x \cup C^*_y \), such that there exists \( \beta \in A^*_x \cup C^*_y \) satisfying \( \alpha \in \text{dominion}_{\sigma;\tau}(\beta) \). Suppose without loss of generality that \( \beta \) is in \( A^*_x \), the case where \( \beta \) is in \( C^*_y \) is deal with on an equal footing. Let \( w = (x, y, z) \) witnessing \( (x, z) \) and \( \sigma \in \text{dominion}_{\sigma;\tau}(\beta) \) means that there exists a path \( \lambda : w \to w' \in \sigma \) rel \( \tau \), with \( x \uparrow \lambda \to B \sigma \in \sigma , x \uparrow \lambda B \tau \in \tau \), \( s \) starts from the cell \( \beta \), with \( a + m \in s \), and such that \( \forall y \in (A^*_x \cup C^*_y) \in \beta \Rightarrow \exists m \in x, y + m \). Suppose that \( \alpha \in C^*_y \), the case where \( \alpha \in A^*_x \) being dealt with on an equal footing. Then, by the fact that \( x \uparrow B \tau \neq C \) is above no \( O \)-cell of \( z \), we can deduce the fact that \( \psi_{(y,z)}(\alpha) = \alpha \) is in \( B^*_y \). Now, the \( O \)-move of \( \text{Tr}^{C \times C}_{A \times B}(f) \) is actually \( \alpha \) from the \( \sigma \) point of view. This \( O \)-move has been triggered by a \( O \)-move from the cell \( \varphi_{(x,y)}(\alpha) = \alpha \). Now \( \alpha \) is either a \( A^*_x \), \( A^*_y \), \( \text{dominion}_{\sigma;\tau}(\beta) \) then \( \alpha \) is explored by \( s \). If in \( B^*_y \), \( s \) we redo the same reasoning, and conclude that this is triggered from the \( \tau \) point of view from a \( O \)-move above \( \psi_{(y,z)}(\alpha) \) and \( \psi_{(y,z)}(\alpha) \). Going on like this we obtain a sequence:

\[
\begin{align*}
\alpha_1 &= \alpha \\
\alpha_{i+1} &\equiv \psi(\alpha_i) \text{ if } i \text{ is odd} \\
\alpha_{i+1} &\equiv \psi(\alpha_i) \text{ if } i \text{ is even}
\end{align*}
\]

sequence that stops when \( \varphi_{(x,y)}(\alpha) = \beta \). Now, let \( f : A_m \to \mathbb{N} \) defined by \( f(\alpha) = \|s\|_w \). Then \( f \) is strictly decreasing along \( \psi_{(x,y)} \) and \( \psi_{(y,z)} \). That is, given \( \alpha \) such that \( \psi_{(x,y)}(\alpha) \) is defined then \( f(\varphi_{(x,y)}(\alpha)) < f(\alpha) \) (and equivalently for \( \varphi_{(y,z)} \)). This ensures us that the sequence above is finite and eventually stops. So we can conclude that \( f = \beta = \text{Tr}^{B^*_y \cup B^*_y}_{A^*_x \cup C^*_z, A^*_x \cup C^*_z}((\varphi_{(x,y)} \cup \psi_{(y,z)}; (\id_{A^*_x \cup s_{B^*_y \cup B^*_y} \cup \id_{C^*_z}}))(\alpha) = \beta \). □
Unfortunately the reverse inclusion might not be true. Suppose that there exists a cell $\alpha$ in $A^n$ such that $\alpha$ has three different cells in its $\text{dominion}_{\alpha}(A^n)$: two in $A^+$ and only one in $B^*$. The strategy reacts to a move in $\alpha$ by playing a player move in $B^*$, and depending on the reaction of the opponent, it explores one of the two cells of $\text{dominion}_{\alpha}(A^n)$ located in $A$, or the other. Then as the context explored depends on the reaction of opponent, when post-composed with another strategy, it will never explore one the cells that was in $\text{dominion}_{\alpha}(A^n)$, (as the reaction of the opponent in $B$ is now encoded into $\tau$, that is deterministic). Hence the dominion of the composite strategy is now restricted to one of the two original cells of $A$.

This is related to the difficulty of modelling the sequence $\otimes \Rightarrow \&$ by a strategy in linear logic. For instance, consider the sequent $\vdash \Gamma, (A \& B) \otimes \top$. The strategy, playing the tensor $\otimes$, will bring two cells; one corresponding to the $\&$ on the left hand side (call it $\alpha$), and one corresponding to the $\top$ on the left hand side (call it $\sigma$), and depending on the reaction of the opponent, it explores one of the two cells of $\alpha$, or, equivalently

$$\vdash \Gamma', A \& B \vdash \bot$$

$$\vdash \Gamma, (A \& B) \otimes \top$$

However, the problem is that the context explored might depend whether the strategy decides to explore the left hand side, or the right hand side of the formula $A \& B$. That is, the context explored by the set of plays of the strategy whose first moves (at this stage) belong in $A$ might be different to the one explored by the set of plays of the strategy whose first moves (at this stage) lie in $B$. That would lead to the following sequent deduction,

$$\vdash \Gamma'_1, A \vdash \bot$$

$$\vdash \Gamma'_2, B \vdash \bot$$

$$\vdash \Gamma', A \& B \vdash \bot$$

where $\Gamma' = \Gamma'_1 \cup \Gamma'_2$, or, equivalently

$$\vdash \Gamma'_1, A \vdash \bot$$

$$\vdash \Gamma'_2, B \vdash \bot$$

$$\vdash \Gamma', A \& B \vdash \bot$$

in tensorial logic. However, in both logics, it is required that $\Gamma_1 = \Gamma_2$.

E Refining innocent strategies

Innocent strategies suffer defects that prevent them from forming a fully complete for tensorial logic. First, they are affine, meaning that a part of the context might not be explored and therefore might be discarded. For instance, there is an innocent strategy $A \otimes A \Rightarrow A$. This problem is also apparent in the confusion between $1$ and $\top$. From the game perspective, they are equal (both accepts only one move), whereas they fundamentally differ from the logical point of view. The second problem comes from the fact that on every type there is a strategy, namely, the trivial one whose only play is the empty play. In order to exclude those cases we must focus on strategies that are able to answer every opponent query. Those are called total.

Therefore, we add two properties to our strategies. They shall be total, and with strong sequentiality structures that prevent them from being affine and that encode well the structure of the atomic types.

E.1 Totality

A strategy is total if it can always answer an opponent move.

**Definition E.1** A strategy is **total** if for all $s \in \sigma$ and for all $m$ such that $\lambda(m) = -1 \land s.m \in \text{Legal}(A)$ then $\exists n. s.m.n \in \sigma$.

Similarly, this property can be encoded on definable sets.

**Definition E.2** A definable set $X$ is **total** if $\forall x \in X. \forall m. (\lambda(m) = -1 \land x \uplus [m] \in \text{Legal}(A)) \Rightarrow x \uplus [m]$ is dominated in $X$.

There is a straightforward equivalence between the sequential definition of totality and its static one. More important is to prove that the total strategies are stable under composition. This relies on our arenas being finite in the sense that any play on them can only have a finite number of moves.

**Proposition E.3** The total strategies of $\text{INN}$ form a sub-category of $\text{INN}$. That is, the composite of two total strategies is total and the identity strategy is total.

**Proof.** Consider two strategies $\sigma : A \Rightarrow B$ and $\tau : B \Rightarrow C$ such that both are total. Let $s : A \Rightarrow C \in \sigma; \tau$. Let $m$ an $O$-move of
\( A \to C \) such that \( s.m \) is legal. Consider \( t \) a sequence of \( \sigma \vdash \tau \) such that \( t \vdash A \to C = s \). Suppose, without lost of generality, that \( m \in A \). Then, as \( \sigma \) is total, there is a \( m_2 \) such that \( t \vdash A \to B.m, m_2 \in \sigma \). If \( m_2 \in A \), we conclude that \( s.m, m_2 \in \sigma \vdash \tau \). Otherwise, \( m_2 \in B \), where from \( \tau \) point of view, it appears as an opponent move. Then \( \tau \) reacts to it with \( m_3 \). If \( m_3 \in C \), the \( s.m., m_3 \in \sigma \vdash \tau \). Otherwise, it is in \( B \) where it appears as an \( O \)-move from \( \sigma \) point of view. Following this reasoning, we obtain a sequence \( m_1, \ldots, m_n \) of moves in \( B \), until one of the strategies decides to play in \( A, C \). As there are only finite chains in \( B \), such a sequence always terminates, and one of the strategy \( \sigma \vdash \tau \) always reacts in \( A, C \) with \( m_{n+1} \) leading to a play \( s.m., m_{n+1} \in \sigma \vdash \tau \). Finally, the \text{copycat} strategy is obviously total. \( \square \)

Therefore, total strategies form a sub-category of \( \text{INN} \).

**Definition E.4** The category \( \text{TotINN} \) has same objects as \( \text{INN} \) and morphisms total, typed-coherent, transverse, innocent strategies.

It should however being noted that totality and maximality are two different notions. A total strategy might not reach a maximal position. For instance, let us consider the arena associated with the sequent \( \neg 1 \vdash \neg 0 \), whose simplified tree is displayed below:

![Tree diagram](image)

This sequent has a proof, namely:

\[
\frac{0, \neg 1 \vdash \bot}{-1 \vdash \neg 0} \quad \text{Right } \neg
\]

This proof corresponds to the total strategy \( \{ 0, m_1, n_1 \} \), where the \( n_1 \) move is the denotation of the triggering of the 0-rule. However, this strategy does not reach the maximal position \( \{ m_1, n_1, n_1 \} \). In linear logic, this would correspond to a proof of the sequent \( \vdash 1, \top \).

**E.1.1 On frugality**

A play is frugal if opponent never introduces twice the same name. In particular, it never introduces twice the same typed name. As typed cells are maximal (they never justify another move), this can be formalised by focussing on cells available at the target of the play and of positive polarities.

**Definition E.5** A play \( s : \star \to x \) is frugal if it is legal and furthermore:

\[ \forall \alpha, \alpha' \in A^\sigma_* \alpha \neq \alpha' \Rightarrow \alpha \neq \alpha'. \]

This notably entails:

\[ \forall m_i, m_j \in s.(i \neq j \land \lambda(m_i) = \lambda(m_j) = -1) \Rightarrow \tau(S(m_i)) \neq \tau(S(m_j)). \]

Concerning the cell names, the legality condition already ensures that each move by opponent brings different names. So the frugality only constraints the strategy regarding typed names. Given a strategy \( \sigma \), we call \text{frugal}(\sigma) \) its set of frugal sequences.

\[ \text{frugal}(\sigma) = \{ s : \star \to x \mid \forall \alpha, \alpha' \in A^\sigma_*, \alpha \neq \alpha' \Rightarrow \alpha \neq \alpha' \} \]

Equivalently, we say that a position is frugal if it is the target of a frugal sequence. Given \( X = \sigma^* \), we write \text{frugal}(X) \) for its set of frugal positions.

Just as we can project a proof of tensorial logic into a proof of linear logic, we would like to project our strategies to morphisms coming from a categorical model of linear logic. More specifically, we would like to project our strategies to the linear polarised nominal relational model. In it, the tensor is modelled by the polarised separated product, that does not let the same name appears twice in two different occurrences of atomic type of same polarity.
The reason why we do not enforce frugality at the level of strategies is that it does not compose well. This has to be put in relation with section A.1, where we showed that the composition of separated polarised relations could not be defined as the usual relational composition. As example, let us consider the composition of the strategies corresponding to the following cut.

First, let us display the structure of the dialogue game corresponding to \( \neg(X \otimes \neg X) \).

where \( \lambda, \chi \) have type \( A_X \). The idea is that the proponent will copy the name introduced by opponent to display a copycat link. So if we look from the \( 1 \vdash \neg(X \otimes \neg X) \) point of view, that corresponds to the left hand side of the cut, player will simply copy the \( \lambda \) passed by opponent. On the other hand, in the arena \( \neg(X \otimes \neg X) \vdash \neg((X \otimes \neg X)) \), as player will also follow a copy-cat strategy, it will copy back and forth the \( \lambda \). So from its point of view, it seems like opponent has played twice the same name, hence breaking the frugality condition. The play is displayed below, where the arrow between move emphasize the relation \( \vdash \) between them, and where the typed names are bold and underlined.

\[
1 \quad \neg(X \otimes \neg X) \quad \neg((X \otimes \neg X))
\]

\[
(\mu, *, \alpha), O \\
(\alpha, *, [\beta, \alpha']), P \mid O \\
(\beta, *, [\gamma]), O \mid P \\
(\gamma, *, \lambda, \gamma'), P \mid O \\
(\alpha', *, \beta'), P \\
(\beta', *, [\gamma', \lambda], O)
\]

To solve this issue, we have adopted the same technique as for linear relations: we have considered from the start
strategies that are closed under typed nominal substitutions. Given \( \sigma \) be a set of plays, we define \( \hat{\sigma} \) by:

\[
\hat{\sigma} = \{ e \cdot s \mid e \in \Xi_T, \ s \in \sigma \}
\]

and equivalently, given \( X \) a set of positions, we define:

\[
\hat{X} = \{ e \cdot x \mid e \in \Xi_T, \ x \in X \}.
\]

One can easily check that \( (\hat{\sigma})^* = (\hat{\sigma^*}) \). The first important lemma of this section states that our strategies are well-defined by their subset of frugal plays. We remind that a play is semi-linear if proponent never introduces a typed name in it, and a strategy is semi-linear if all the plays in it are. Semi-linearity is a sub-property of typed-coherency \( \text{6.2.} \)

Proposition E.6 Let \( \sigma \) be an innocent, typed-coherent, total strategy. Then \( \sigma = \frugal(\hat{\sigma}) \).

Proof. We start by the left to right inclusion. The proof is done by induction on the lengths of the plays \( s \) of \( \sigma \). Suppose \( s = s \cdot m \cdot n, \) and \( s' = e_1 \cdot t', \) where \( t' \) is frugal(\( \sigma \)), and the only action of \( e_1 \) is to merge names of \( t' \). That is \( v(s') \subseteq v(t') \), and \( v(e_1) \subseteq v_T(t') \). Furthermore, let us assume that \( (v_T(t') \setminus v_T(s')) \cap v_T(m) = \emptyset \) without loss of generality (one can always do this assumption since if there were some names in this set, one could get rid of them by applying a typed permutation to \( t' \)). In the case where \( t' \cdot m \cdot n \) is frugal, then, as the only action of \( e_1 \) is to map names of \( v_T(t') \) to names of \( v_T(s') \), and \( (v_T(t') \setminus v_T(s')) \cap v_T(m) = \emptyset \), we can conclude that \( e_1 \cdot m \cong m \). Furthermore, by typed-coherence, \( v_T(n) \subseteq (v_T(s') \cup v_T(m)) \). Therefore, \( e_1 \cdot v_T(m) = n \). Finally, let us assume that \( t' \cdot m \cdot n \) is not frugal. Then this entails that \( m \) brings names appearing in \( t' \), or brings several cells filled with the same name. In either case, we pick a \( m' \) such that \( m' \) is frugal as a move, \( m' \neq_T s \cdot t \), and such that there is a typed substitution \( e_2 \) such that \( e_2 \cdot m' = m \), and \( v(e_2) \subseteq v_T(m) \cap v_T(m') \). Finally, let us consider a \( n_1 \) such that \( m' \cdot n_1 \in \sigma \). Such a move \( n_1 \) exists since \( \sigma \) is total. Now, \( e_1 \cdot e_2 \cdot (t' \cdot m' \cdot n_1) = s \cdot m_1 \cdot e_2 \cdot n_1 \cong s \cdot m 
\]

The reverse inclusion is automatic, since \( \frugal(\hat{\sigma}) \subseteq \sigma \), and \( \sigma \) is closed under typed substitutions. \( \Box \)

One now can define a category with nominal positive games as objects and innocent, semi-linear, total, frugal strategies as morphisms, where the composition is defined through closure under typed substitutions as in section 22. Given a typed-coherent innocent strategy, its frugal restriction is not innocent anymore, but almost. To remedy for that, we introduce frugal innocence.

Definition E.7 A strategy is \textit{frugal forward consistent}, if \( \forall s \in \sigma \) and \( m_1, m_2, n_1, n_1 \) such that \( s \cdot m_1 \cdot n_1, \ s \cdot m_2 \cdot n_2 \in \sigma, \ m_1 \neq m_2, \ m_1 \uparrow m_2 \) then \( n_1 \neq n_2 \). Moreover, if \( n_1, n_2 \) are such that \( s \cdot m_1 \cdot n_1 \cong_{\text{post}} s \cdot m_2 \cdot n_2 \) and \( \tilde{m}_1, n_1 \# \# \tilde{m}_2, n_2 \) then we have \( s \cdot m_1 \cdot n_1 \uparrow s \cdot m_2 \cdot e_1 \in \sigma \) and \( s \cdot m_1 \cdot n_1, m_2, n_2 \in \sigma \). A strategy is \textit{frugal innocent} if it is frugal forward consistent and backward consistent.

We then have the following straightforward property.

Proposition E.8 A strategy \( \sigma \) is typed coherent innocent if and only if \( \frugal(\hat{\sigma}) \) is semi-linear frugal innocent. Furthermore, \( \sigma \) is transverse, total if and only if \( \frugal(\hat{\sigma}) \) is.

The proof is obvious. This allows us to define the following category.

Definition E.9 Frugal is the category with objects positive dialogue games and morphisms frugal innocent, semi-linear, total, transverse, frugal strategies. The composition of morphisms is defined as follows:

\[
\sigma; \tau = \frugal(\hat{\sigma}; \hat{\tau})
\]

or, equivalently,

\[
\hat{X}; \hat{Y} = \frugal(\hat{X}; \hat{Y})
\]

For completeness, we prove below that the above definition makes sense.

Proposition E.10 Frugal is well-defined, that is, if \( \sigma, \tau \) are frugal innocent, semi-linear, total, transverse, frugal strategies, then \( \hat{\sigma}; \hat{\tau} \) are morphisms of \text{TotINN}. Therefore, \( \hat{\sigma}; \hat{\tau} \) is a morphism of \text{TotINN} and \( \frugal(\hat{\sigma}; \hat{\tau}) \) is a frugal innocent, semi-linear, total, transverse, frugal strategy. \( \Box \)
Associativity of composition follows from the one of typed-coherent innocent strategies, and the identity morphisms are defined to be \( \text{id}_x = \text{frugal}(\text{copycat}_x) \). Equivalently, a direct method can be proposed, by relating arenas to lists, strategies with semi-linear nominal relations, and the closure under nominal permutations with tracing of partial injective functions. More about this is given in the appendix ??.

Therefore, in the next paragraph, we focus on strategies of Frugal.

### E.2 Innocent strategies and strong structures of sequentiality

The key to definability for strategies lies in their sequentiality structures. Indeed, each move by player will correspond to sequence \( \bigoplus \bigotimes \) of global positive connectives, and in particular, the \( \bigotimes \) splits the formulas on the left hand side of the sequent into a partition. The division of context taking place at the level of formulas in the proof corresponds to the division of context taking place at the level of cells by the strategy. However, the problem with innocent strategies is that their sequentiality structures are weak: their functions are partial. Notably, given a cell with no move above, (corresponding to a unit), there is no way to establish what part of the context will be captured by it. Another way to state it, is that there is no difference between \( \neg 0 \) and \( 1 \) from the strategy point of view, both are interpreted by a single player move. There is no indication for the fact that \( \neg 0 \) might capture a context, while \( 1 \) cannot. Equivalently, we have to enforce the fact that a proponent move in \( X \) can only capture a context of type \( X \). In order to solve that, we extend the notion of innocent strategy to equip them with a “strong” sequentiality structure, that is not partial anymore. If the innocence of the strategy captures the underlying structure of the proof, then the sequentiality structure makes sure it stays logical, and, in particular, captures the structure of the leaves (axiom and 0-rule). Note that sequentiality structures have been introduced in [?].

Let us recall that we work within the category Frugal, as defined in the above section. Therefore, the strategies are not closed under typed substitutions anymore.

**Definition E.11** A strategy with sequentiality structure \( (\sigma, \phi) \) is an innocent strategy \( \sigma \) together with a family of sequentiality functions \( \phi = \{ \phi_x : A^* \to A^* \} \) such that:

- \( \phi \) is closed under permutations: \( \phi_{\sigma,(\cdot)} = \pi \cdot \phi_{\cdot} \).

- \( \phi_{\cdot} \) is a total function.

- \( \forall s : \star \mapsto x \overset{m}{\mapsto} y \in \sigma, \text{ for all } \alpha \in A^*_\gamma \cap A^*_\delta, \phi^{-1}_\gamma(\alpha) = \phi^{-1}_\delta(\alpha) \).

This definition of strong sequentiality structure is coherent with the definition of weak sequentiality structure given in the context of innocent strategies. We recall that the action of name permutations on functions is defined by:

\[
(\pi \cdot \phi)(\alpha) = \pi \cdot (\phi(\pi^{-1}(\alpha)))
\]

**Proposition E.12** Let \( (\sigma, \phi) \) be an innocent strategy with sequentiality structure, and \( \psi \) the weak sequentiality structure canonically associated to \( \sigma \). Then \( \psi \preceq \phi \) (where the inclusion takes place at the level of partial functions).

**Proof.** Let \( x \) be a position of \( \sigma^* \), \( \alpha \) a negative cell available at \( x \), and \( m \) an opponent move above \( \alpha \). Moreover, let us consider \( \beta \), a positive cell available at \( x \), such that there is a play in \( \sigma \mid_x \alpha \) that contains a move \( n \) above \( \beta \). That is, \( \psi(\beta) = \alpha \). Let \( s = m.s'n \) a play of \( \sigma \mid_x \alpha \) that starts with \( m \) and finishes with \( n \). Consider all the negative cells available at \( x \) but \( \alpha \). By definition of \( \sigma \mid_x \alpha \), they are not explored by the play \( s \), and therefore remain available all along it. Consequently, the set of cells \( \phi^{-1}_\gamma(A^*_\gamma \setminus \alpha) \) is still present at \( y \), the target position of \( m.s'n \). As \( n \) explores above \( \beta \), \( \beta \) is not available at \( y \) and therefore \( \beta \notin \phi^{-1}_\delta(A^*_\delta \setminus \alpha) \). As the function is total, this entails \( \beta \in \phi^{-1}_\gamma(\alpha) \), that is, \( \phi_{\cdot}(\beta) = \alpha \). \( \square \)

Since our new sequentiality structure is an extension of the canonical one imposed by the innocence of the strategy, it approximately satisfies the same properties. Among them, Proposition C.17 remains true. That is, given \( s : \star \mapsto x \in \sigma \), if \( \phi_{\cdot}(\alpha) = \beta \), (where \( \alpha, \beta \in A^*_\gamma \)), then \( ||s||_{\gamma} < ||s||_{\beta} \). The proof is exactly the same as the one of the original proposition C.17.

We furthermore need to make sure that the context captured by a cell is coherent. That is, an untyped cell can capture any context, but a typed cell can only capture a context of the same type, and, by linearity, only a single cell. We therefore impose the following conditions.

**Definition E.13** Given \( (\sigma, \phi) \) a strategy with sequentiality structure, we say that \( \phi \) is well-typed if:

- \( \forall x \in \sigma^* \), \( \forall \alpha \in A^*_\gamma, \nu(\alpha)^\gamma \in A_{\gamma} \Rightarrow (\forall \pi \in \text{Perm}(A_{\gamma}) \nu(\phi_{\cdot}(\pi \cdot \alpha)^\gamma) = \pi \cdot \nu(\phi_{\cdot}(\alpha)^\gamma)). That is \( \nu(\phi_{\cdot}(\alpha)^\gamma) \cap A_{\gamma} \subseteq \nu(\phi_{\cdot}(\alpha)^\gamma) \cap A_{\gamma} \).

- \( \forall x \in \sigma^* \), \( \forall \alpha \in A^*_\gamma, (\nu(\alpha)^\gamma \cap A_{\gamma} \neq \emptyset \Rightarrow |\phi^{-1}_\gamma(\alpha)| = 1) \).

One important consequence of this definition is that player cannot play a move in an atomic formula \( X \) if opponent has not played in another \( X \) before. Then, player will play the same name \( \alpha \) as opponent played before, establishing a copy-cat link. In other terms, the copy-cat links are oriented, from negative to positive literals. The strategies with well-typed sequentiality structures are automatically semi-linear. Indeed, if player plays a cell with a name of \( A_{\gamma} \), then from the first condition, it points to a positive cell of same type through \( \phi \), and this cell has the same name. Finally, as we know that sequentiality structures are such that if \( \phi(\alpha) = \beta \) then \( \alpha \) appears before \( \beta \), then we can devise that if proponent
plays a typed name, then this one was brought by opponent before. Finally, as the strategy is frugal, opponent never plays twice the same name. Now, if player would want to play twice the typed same name, then it would mean that there would be two positive cells that point to it through $\phi$. However, that is prevented thanks to the second condition.

Note that, as in a vertex $v$, only the final cell $\Gamma^T_v$ can be a typed cell, then it is equivalent to require $\nu(\Gamma^T_v) \cap A_T \neq \emptyset$ and $\nu(\alpha) \cap B_T \neq \emptyset$. This entails this small simplification.

**Lemma E.14** The well-typed conditions are equivalent to the following ones:

- $\forall x \in \sigma^* \forall \alpha \in A^*_T(x), \nu(\phi_\alpha(x)) \cap A_T \subseteq \nu(\alpha) \cap A_T$.
- $\forall x \in \sigma^* \forall \alpha \in A^*_T, (\nu(\alpha) \cap \bar{A}_T \neq \emptyset \Rightarrow |\phi_\alpha^{-1}(\alpha)| = 1)$.

### 2.2 Composition of sequentiality structures

To start this section, let us recall that weak sequentiality structures compose through tracing, but not exactly. That is, the composition of two weak sequentiality structures $\phi$ and $\psi$ along a position of interaction $(x, y, z)$ is defined at the level of the strategy, and is included in:

$$\mathsf{Tr}^R_{A',C';A,C}(\phi_{(x,y,z)} \cup \psi_{(y,z)}; (\mathsf{id}_{A'} \cup s_{B'_C,B_C} \cup \mathsf{id}_{C'}) : A'_x \cup B'_y \cup B'_z \cup C'_y \rightarrow A'_x \cup B'_x \cup B'_y \cup B'_z \cup C'_z)$$

the inclusion being, in the general case, strict. After a brief discussion, it was highlighted that the reason for this strictness might be pointed to a wrong logical behaviour with regard to the &-rule by innocent strategies: the inclusion being, in the general case, strict. After a brief discussion, it was highlighted that the reason for this strictness might be pointed to a wrong logical behaviour with regard to the &-rule by innocent strategies:

$$\begin{array}{c}
\Gamma_1 \vdash A \\
\Gamma_2 \vdash B
\end{array} \quad \Gamma_1 \cup \Gamma_2 \vdash A \& B$$

in linear logic, or

$$\begin{array}{c}
\Gamma_1, A \vdash \bot \\
\Gamma_2, B \vdash \bot
\end{array} \quad \Gamma_1 \cup \Gamma_2, A \oplus B \vdash \bot$$

in tensorial logic.

This behaviour follows from the fact that a definition of sequentiality structure only through "what is explored" is not strong enough. Our new definition does not encounter this problem, what is explored by the strategy on the left or on the right might differ, but this is not taken into account by the structure.

As expected, composition of sequentiality structures is defined through tracing. One already knows that this is coherent with their restriction to weak sequentiality structure. That is, given $\phi, \psi$ sequentiality structures associated with two strategies $\sigma : A \to B$ and $\tau : B \to C$, we already know that the weak sequentiality structure associated with $\sigma; \tau$ is a family of sub-functions of $\mathsf{Tr}(\phi \cup \psi; \mathsf{id} \cup s \cup \mathsf{id})$.

**Definition E.15** Let $(\sigma, \phi) : A \to B$ and $(\tau, \psi) : B \to C$ two strategies with sequentiality structures. We define their composition $(\sigma; \phi) \circ (\tau; \psi)$ by $(\sigma; \phi) \circ (\tau; \psi) = (\sigma\tau; \phi \circ \psi)$, where given a legal position $(x, y, z)$ of $\sigma^* \cap \tau^* \subseteq \mathsf{Pos}(A \to B \to C)$:

$$(\phi \circ \psi)_{(x,y,z)} = \mathsf{Tr}^R_{A';C';A,C}(\phi_{(x,y,z)} \cup \psi_{(y,z)}; (\mathsf{id}_{A'} \cup s_{B'_C,B_C} \cup \mathsf{id}_{C'}) : A'_x \cup B'_y \cup B'_z \cup C'_y \rightarrow A'_x \cup B'_x \cup B'_y \cup B'_z \cup C'_z)$$

We need to prove that this leads to a sequentiality structure, that is, that the thus defined function is total. Furthermore, we prove that if the two sequentiality structures are well-typed, then so is their composition. Finally, it is necessary to check that two distinct witnesses of interaction lead to the same final sequentiality structure.

**Lemma E.16** The function

$$\mathsf{Tr}^R_{A';C';A,C}(\phi_{(x,y,z)} \cup \psi_{(y,z)}; (\mathsf{id}_{A'} \cup s_{B'_C,B_C} \cup \mathsf{id}_{C'}) : A'_x \cup B'_Y \cup B'_z \cup C'_y \rightarrow A'_x \cup B'_x \cup B'_y \cup B'_z \cup C'_z)$$

is total.

**Proof.** We simply need to show that the partial function $\phi_{(x,y,z)} \cup \psi_{(y,z)} \uparrow B'_y \cup B'_z \cup s_{B'_C,B_C}$ is nilpotent. It is true since, given $s : \star \to (x, y, z)$, and the function $|||\|$ that to each cell $a$ in $B'$ gives the length of the minimal subsequence $s' \leq s$ $\uparrow B'$ such that $s'$ introduces $a$, then $||\phi_{(x,y,z)}(a)|| < ||a||$, and equally for $\psi$. Therefore, given any cell of $B'_y$, there is no infinite chain following repetitive applications of the function $\phi_{(x,y,z)} \cup \psi_{(y,z)} \uparrow B'_y \cup B'_z \cup s_{B'_C,B_C}$. \(\square\)

**Lemma E.17** If $\phi, \psi$ are well-typed, then so is their composition.

**Proof.** We only need to show that the two properties defining well-typedness are stable under iterated applications of $\phi$ and $\psi$ (as given any cell $\alpha \in A'_x$, there is a sequence of alternating $\phi, \psi$ such that $(\phi \circ \psi \circ \ldots \circ \alpha)(\alpha)$). Let us assume without loss of assumption that the last function of the previous sequence is $\phi$. Then $\nu(\alpha) \cap \bar{A}_T \supseteq \nu(\phi_{(x,y,z)}(\alpha)) \cap
Lemma E.18 Let \( y \neq y' \) such that \((x, y, z), (x, y', z) \in \mathfrak{e}^* \models \tau^* \). Then:

\[
\begin{align*}
\text{Tr}^{R_{1,2,3}}_{\mathcal{A}_1;\mathcal{C}_1;\mathcal{C}_1}((\phi_{1,2} \otimes \psi_{y,z}); (\text{id}_{\mathcal{A}_1} \uplus s_{B_2^*;B_3^*} \uplus \text{id}_{\mathcal{C}_1}) ) \\
= \text{Tr}^{R_{1,2,3}}_{\mathcal{A}_1;\mathcal{C}_1;\mathcal{C}_1}((\phi_{1,3}; \psi_{y',z}); (\text{id}_{\mathcal{A}_1} \uplus s_{B_2^*;B_3^*} \uplus \text{id}_{\mathcal{C}_1}) )
\end{align*}
\]

Proof. We know that there is a unique witness of interaction, up to cell nominal permutations. Hence \( y \equiv_{\text{cell}} y' \). Furthermore, as \((x, y)\) is legal, \( x \#_{\text{cell}} \), and similarly \( y \#_{\text{cell}} \). So it follows that there exists a permutation \( \pi \) such that \( \pi \cdot (x, y, z) = (x, y', z) \). Then the lemma follows from \( \phi, \psi \) being nominal, as well of the trace operator. That is, given \( f : y \times x \rightarrow z \times x \) a generic function:

\[
\begin{align*}
\text{Tr}^\pi_{1,2}(f)(\alpha) &= (\pi \cdot \text{Tr}^\xi_{1,2}(f))(\alpha) \\
&= \pi \cdot (\text{Tr}^{\pi^{-1}\cdot \pi\cdot f}_{\pi^{-1}\cdot \pi\cdot f})(\pi^{-1} \cdot \alpha) \\
&= \pi \cdot (\text{Tr}^{\pi^{-1}}_{\pi^{-1}}(\pi^{-1} \cdot f)(\pi^{-1} \cdot \alpha))
\end{align*}
\]

In our case,

\[
\begin{align*}
f &= \phi_{1,3} \otimes \psi_{y,z}; (\text{id}_{\mathcal{A}_1} \uplus s_{B_2^*;B_3^*} \uplus \text{id}_{\mathcal{C}_1}) \\
\pi^{-1} \cdot f &= \phi_{\pi^{-1}(1,3)} \otimes \psi_{\pi^{-1}(y,z)}; (\text{id}_{\mathcal{A}_{\pi^{-1}}^*} \uplus s_{B_2^*;B_3^*} \uplus \text{id}_{\mathcal{C}_{\pi^{-1}}^*})
\end{align*}
\]

So let us pick \( \pi \) such that \( \pi \cdot (x, y, z) = (x, y', z) \). Then as \( \alpha \) is a cell of \((x, z)\) and \( \pi \cdot (x, z) = (x, z) \), we got \( \pi^{-1} \cdot \alpha = \alpha \). Finally, to simplify, writing \( \text{Tr}^\pi_{1,2}(\phi_{1,3} | \psi_{y,z}) \) for \( \text{Tr}^{R_{1,2,3}}_{\mathcal{A}_1;\mathcal{C}_1;\mathcal{C}_1}((\phi_{1,3}; \psi_{y,z}); (\text{id}_{\mathcal{A}_1} \uplus s_{B_2^*;B_3^*} \uplus \text{id}_{\mathcal{C}_1}) ) \) we get:

\[
\begin{align*}
\text{Tr}^\pi_{1,2}(\phi_{1,3} | \psi_{y,z})(\alpha) &= \text{Tr}^\pi_{1,2}(\phi_{1,3}; \psi_{y,z})(\pi \cdot \alpha) \\
&= \text{Tr}^\pi_{1,2}(\pi \cdot \phi_{\pi^{-1}(1,3)} \mid \psi_{\pi^{-1}(y,z)})(\pi \cdot \alpha) \\
&= \pi \cdot (\text{Tr}^\pi_{1,2}(\phi_{\pi^{-1}(1,3)} \mid \psi_{\pi^{-1}(y,z)})(\alpha)) \\
&= \text{Tr}^\pi_{1,2}(\phi_{1,3} \mid \psi_{y,z})(\alpha)
\end{align*}
\]

where the last equality holds since \( \text{Tr}^\pi_{1,2}(\phi_{1,3} \mid \psi_{y,z})(\alpha) \) is a cell of \((x, z)\). □

So finally we can conclude that we have a category and transverse, innocent, total, frugal strategies with well-typed sequentiality structures as morphisms. These strategies will prove to be sound and fully complete for tensordial logic. We name this category GameSeq.

Definition E.19 GameSeq is the category with objects positive dialogue games and morphisms transverse, total, semilinear, frugal, innocent strategies with well-typed strong sequentiality structures on negative dialogue games \( A \vdash B \). We will call such strategies brave.

We will explore this category in the next section. We will notably present the associated copycat morphisms, and will expose its categorical structure.

F Proof of full-completeness

The goal of this section is to prove that the category GameSeq is the free dialogue category with products on Var. This is the content of the proposition below.

Proposition F.1 There is a correspondence between the equivalence classes of proofs of \( \Gamma \vdash A \) and the morphisms of GameSeq(\([\Gamma], [A]\))

This entails strong completeness, and more. That is, the functor from the proof invariants of tensordial logic to GameSeq is not only full, but also faithful. The proof of the proposition relies on the following lemma.

Lemma F.2 Let \( \neg B_1 \vdash \neg B_2 \vdash \ldots \vdash \neg B_k \vdash \neg A \) a sequent. Then there is a one to one correspondence between:

- The equivalence classes of proofs of the sequent focussing on one \( B_i \), and the transverse strategies of \( (\otimes \neg B_1) \otimes (\otimes \neg B_2) \otimes \ldots \otimes (\neg B_k) \otimes [A] \).

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The equivalence classes of proofs of the sequent and the strategies of $(\langle B_1 \rangle \otimes \langle \neg B_2 \rangle \otimes \langle \neg B_n \rangle \otimes \langle A \rangle)^*$.

What we meant by focussing on one $B_i$ is that, taking a focalised proof in the equivalence class, it will behave by focussing on one of the $B_i$ at the beginning of its first synchronous phase, as displayed in the proof below.

$$
\begin{align*}
\neg B_1, \ldots, \neg B_{i-1}, \neg B_{i+1}, \ldots, \neg B_n, A_1', \ldots, A_m' \vdash B_i & \Rightarrow \\
\neg B_1, \neg B_2, \ldots, \neg B_n, A_1', \ldots, A_m' \vdash \perp & \Rightarrow \\
\neg B_1, \neg B_2, \ldots, \neg B_n, A \vdash \perp & \Rightarrow \\
\neg A \vdash \neg (B_1 \otimes \ldots \otimes B_n)
\end{align*}
$$

Now, there is a syntactic equivalence between the focalised proofs of $B_1, \ldots, B_n \vdash A$ and the focalised proofs of $\neg A \vdash \neg (B_1 \otimes \ldots \otimes B_n)$, that will be focussing on $A$, as displayed below:

$$
\begin{align*}
\pi & \Rightarrow \\
\neg A, B'_1, \ldots, B'_n, \vdash \perp & \Rightarrow \\
\neg A, B_1' \otimes \ldots \otimes B_n', \vdash \perp & \Rightarrow \\
\neg A \vdash \neg (B_1 \otimes \ldots \otimes B_n)
\end{align*}
$$

Therefore, following the lemma, there is a one-to-one correspondence between the equivalence classes of proofs of $\neg A \vdash \neg (B_1 \otimes \ldots \otimes B_n)$ and the transverse strategies of $(\langle B_1 \otimes \ldots \otimes B_n \rangle \otimes \langle \neg A \rangle)^*$ that is, the transverse strategies of $\langle \Gamma \rangle \rightarrow \langle A \rangle$, where $\Gamma = B_1, \ldots, B_n$. Hence proving the lemma F.2 entails proving the proposition F.1.

We work with proofs in $\text{TENS}_{\text{loc.glob}}$. We remind that within this fragment, two proofs are equivalent if and only if they are equal.

**Proof.** [Proof of lemma F.2.]

We tackle the two points of the lemma at once. The proof is done by induction on the maximal length of the sequences of $\sigma$. Let us note that since $\sigma$ is equivariant and frugal, the names chosen on the moves of the sequent we pick to deal with the induction case will not matter. So let us start with the case where $\sigma$ simply has the empty sequence. Then, as the strategy is total, this implies that the whole pre-arena has no moves. Hence the pre-arena’s dialogue game consists of a set of untyped cells, and hence is 0. So we need to solve the following arena equation:

$$(-B_1 \otimes \ldots \otimes B_n \otimes A) = 0,$$

that has solutions $(n+1)$-tuples $(B_1, \ldots, B_n, A)$ of the form $(B_1, \ldots, B_n, 0)$ for any $B_1, \ldots, B_n$. This corresponds to a sequent:

$$\neg B_1, \ldots, \neg B_m \vdash \neg 0$$

which has the unique following proof:

$$\text{Right 0} \ \ \ \ |$$

$$\neg B_1, \ldots, \neg B_m, 0 \vdash \perp$$

$$\neg B_1, \ldots, \neg B_n \vdash \neg 0$$

This settles the base case.

We now move to the inductive case. So suppose $s \in \sigma$ is a sequence of maximal length, $s = m.n.s'$. We adopt the following notation convention: given two indices $i, j$, we write $i \mid j$ to say that the set from which $i$ ranges depends of the index $j$, and we will explicit the ranging sets only when necessary. For instance, we would write $\bigoplus_i \otimes \bigotimes_j$ for $\bigoplus_{i \in I} \otimes \bigotimes_{j \in J}$. Furthermore, as indices range through downward closed sets of positive natural numbers, we write with an uppercase the upper bound to which they range. That is, $j$ ranges from 1 to $J$.

Each $B_i$ is either isomorphic to 0, or a sum $B_i = \bigoplus_X \bigotimes_{i \mid j} \neg A_{i,j} \bigotimes_{i \mid j} \times_{i \mid j} Y_{o,i,j} \bigotimes_{i \mid j} I$ where $Y_{o,i,j}$ are atomic types, and $A = \bigoplus_X \bigotimes_{i \mid j} \neg A_{i,j} \bigotimes_{i \mid j} X_{i,j} \bigotimes_{i \mid j} I$, or is isomorphic to 0. The totality of the strategy prevents the case where there is a unique $B$, and this one is 0, as player would not be able to answer to the opponent move. This would correspond to the case $\neg 0 \vdash \neg A$, and, in essence, to $A \vdash 0$. That is, totality of the strategy is the counterpart of the absence of right induction rule for 0. In the following, to make it more readable, we denote each cell by the sub-formula it encompasses.

The structure of the two first moves of $s$ will be as follows, in the transverse case:
A of the cells \( B_\alpha \) that he would have chosen, and writing
\[
\bigoplus \quad \text{cannot capture any context. This would then be an application of the right unit rule.}
\]

\( \alpha \)

\[
\text{axiom-links to the proponent move} \quad \text{from the root (Right corresponds to a set of global focalised rules in the proof, sets out in Figure F, where the four first negative rules starting}
\]

\[
\text{pick. He will furthermore play a sequentiality function from the opponent cells just introduced to his own cells. This}
\]

\[
\text{can match a cell coming from} \quad X_k, \quad = j
\]

\[
\text{As the two cases are similar, we focus on the transverse one.}
\]

\[
\text{However, in the non transverse case, the move} \quad n_l \quad \text{can also be above one of the} \quad A_{ji}. \quad \text{Then, writing} \quad A_{ji} = \bigoplus_m (\bigotimes_m \neg C_{n,m} \bigotimes_{n,o} Y_{o,n}), \quad \text{we can get two moves as displayed in the drawing below.}
\]

\[
\text{Γ}_n \quad \text{corresponds to the axiom rule. Note that if one of the} \quad B_l \quad \text{is 1, then it cannot capture any context. This would then be an application of the right unit rule.}
\]
By the lemma of separation of contexts C.12, one can now focus on each of the branch individually. Furthermore, one can see that every sequence above the cells \( \alpha \cup \text{dominion}(\alpha) \) corresponding to \( \Gamma_\alpha = \lnot C_1, ..., \lnot C_b, X_1, ..., X_\ell \vdash B_{n,m,l} \) in the strategy can be faithfully translated as a sequence in \( \lnot C_1 \otimes \cdots \otimes C_b \otimes X_1 \otimes \cdots \otimes X_\ell \otimes B_{n,m,l} \), and hence can be seen as an interpretation of a proof \( \lnot C_1 \otimes \cdots \otimes C_b \otimes X_1 \otimes \cdots \otimes X_\ell \otimes B_{n,m,l} \). Indeed, let us look at the structure of the first move of the arena \( \lnot C_1 \otimes \cdots \otimes C_b \otimes X_1 \otimes \cdots \otimes X_\ell \otimes B_{n,m,l} \). To simplify things, we assume \( B_{n,m,l} = \bigoplus (\bigotimes D_{i,j}) \).

Then the sequences of \( \sigma_i \) can translated as sequences in \( \lnot C_1 \otimes \cdots \otimes C_b \otimes X_1 \otimes \cdots \otimes X_\ell \otimes B_{n,m,l} \), with the first move of opponent in \( \lnot C_1 \otimes \cdots \otimes C_b \otimes X_1 \otimes \cdots \otimes X_\ell \otimes B_{n,m,l} \) filling the cells of \( C_1, \ldots, C_b, X_1, \ldots, X_\ell \) with the cells of \( \text{dominion}(\beta) \).

Therefore, we can apply the induction hypothesis, and conclude that the sequences of \( \sigma \) define a unique global focalised proof, that is, a proof of \( \text{TENS}_{\text{lec-glob}} \).

Overall, we have obtained a perfect abstract representation and characterisation of the proofs of tensorial logic through nominal strategies in sequential, asynchronous games.

### G The case for MALL

#### G.1 Interpretation

We remind here that every proof of tensorial logic can be translated into a proof of linear logic, and reversely. Namely, we remind the proposition ?? below. In this property, \( \Pi \) denotes a set of positive formulas of linear logic, \( \Pi \) a set of negative ones, and \( \Xi \) a set of negative atomic formulas.

**Proposition G.1** Every proof of \( \vdash \Pi, \Pi, \Xi, \Xi \) (respectively \( \vdash \Pi, \Pi, \Xi, \Xi, \Xi \) \( P \)) in weakly focussed linear logic induces a proof of the sequent \( \vdash (\Pi)^F, (\Pi)^+, \Xi^+, \Xi^+ \vdash (\Pi)^F, (\Pi)^+, \Xi^+, \Xi^+ \vdash (\Pi)^F \) in tensorial logic, and reciprocally, where \( \Pi \) is the subset of positive formulas of \( \Gamma \), \( \Xi \) its subset of negative atomic formulas, \( \Xi^+ \) its subset of negative formulas that are not in \( \Xi \). Furthermore \( (\Pi)^F = \Xi \) if \( (\Pi) \) empty, \( (\Pi)^F \) in the case where \( (\Pi) = \Pi \) is positive, and \( \lnot (\Pi)^F \) the case where \( (\Pi) = \Pi \) is negative.

To simplify things, given a weakly focalised linear logic sequent \( \Gamma; \Pi \), we write \( (\Pi)^F \vdash (\Pi)^F \) for the appropriate translation into tensorial logic, where \( \Pi \) is either a single positive formula or the empty sequent. Then let \( \pi \) be a proof of linear logic. We can translate it into a proof \( (\pi)^F \) of tensorial logic. This proof \( (\pi)^F \) can be given a interpretation \( \llbracket (\pi)^F \rrbracket \in \text{GameSeq} \). However, we remark that this interpretation is not a categorical functor, as we might have two proofs \( \pi, \pi' \) of linear logic such that \( \pi \sim \pi' \), but \( \llbracket (\pi)^F \rrbracket \neq \llbracket (\pi')^F \rrbracket \).

Therefore, the right translation from linear to tensorial logic should be along the following lines:

\[
\llbracket \pi \rrbracket = \llbracket (\pi')^F \rrbracket \text{GameSeq} \mid \pi \sim (\pi')^F
\]

Therefore, one needs to define an equivalence relation of strategies of tensorial logic, relating strategies that denote the same proof of linear logic.

**Definition G.2** Two strategies \( \sigma, \sigma' \) : \( A \) are equivalent, written \( \sigma \sim_1 \sigma' \) if there exists \( \pi, \pi' : (A)^F \) such that \( \llbracket (\pi)^F \rrbracket \text{GameSeq} = \sigma', \llbracket (\pi')^F \rrbracket \text{GameSeq} = \sigma' \) and \( \pi \sim (\pi')^F \).

At this point, one should look for an invariant, that is, a function \( f \) together with a set \( S \), such that the image of \( f \) lies in \( S \), and \( \pi \sim_1 \pi' \Rightarrow f(\llbracket (\pi)^F \rrbracket \text{GameSeq}) = f(\llbracket (\pi')^F \rrbracket \text{GameSeq}) \). More precisely, we look for a categorical invariant, that is, a functor \( F \) from the category \( \text{GameSeq} \) to a star-autonomous category, such that \( (\sigma \sim_1 \sigma') \Rightarrow F(\sigma) = F(\sigma') \). Such a functor yields the ground of a fully complete denotational semantics of proofs of linear logic.

#### G.2 About the quotient

The difference between tensorial and linear logic lies in the non-involutive negation. As each negation is interpreted by a move in the category of games, one would like to project moves onto a flat domain. The basic idea is to project onto maximal positions. Indeed, one can notice that, given a strategy \( \sigma : F \), and the same strategy double-lifted with two negation moves \( \neg\neg \sigma : \neg\neg F \), then the two will reach the same maximal positions. This is coherent with the fact that the
negation is involutive in linear logic, thus \((\neg\neg F)^I = F\), and the strategy \(\neg\sigma\) and \(\sigma\) should correspond to the same proof of linear logic. That is \((\neg\neg F)^I = (\sigma)^I\).

However, we present below an example highlighting why we sometimes should identify strategies that do not reach the same set of maximal positions. For instance, let us consider the following two proofs of linear logic:

\[
\vdash \top, \top, A, A \otimes B \otimes \bot \vdash 1
\]

and

\[
\vdash A^I, A \vdash \top \otimes 1, A^I, A \otimes B^I \vdash 1
\]

Let us display below the underlying structure of the dialogue-game associated with the conclusion formula.

\[
\begin{array}{c}
\top \\
\downarrow \\
P, n_\top \\
\downarrow \\
O, m_1 \\
\downarrow \\
A \\
\downarrow \\
P, n_A \\
\downarrow \\
O, m_A \\
\downarrow \\
P, n_B \\
\downarrow \\
O, m_B \\
\downarrow \\
B \\
\end{array}
\]

The strategy corresponding to the second proof has two maximal plays associated with it, namely \(m_1, n_{\top}, m_A, n_A, m_B, n_\top\) and \(m_1, n_{\top}, m_B, n_\top, m_A, n_A\). These two reach the same maximal position. On the other hand, the strategy associated with the first proof has a unique maximal play \(m_1, n_\top\). As a result, we have two different strategies reaching two different sets of maximal positions. However, they correspond to two proofs that are equivalent.

Indeed, if we suppose that they are not equivalent, this would imply that there are (at least) two morphisms \(1 \to (\top \otimes 1) \otimes A \otimes A \otimes B\). But as \(\top \otimes 1 = \top\), and, for any formula \(F\), \(\top \otimes F = \top\), this would imply that there are (at least) two distinct morphisms \(1 \to \top\). Or, as \(\top\) is terminal, there is only one.

The reason behind it is to be looked for in the time where the proof decides to use the \(\top\) rule. In tensorial logic, as the model is dynamic, the moment when we decide to use it makes a difference. On the other hand, in linear logic, the model being flat, these two proofs will be conflated. Therefore, we have to focus only on those maximal positions that are significant.

**Definition G.3** An **external position** is a position such that no untyped cells are available.

Therefore, no move can happen from an external position, and it is maximal. Given a strategy \(\sigma\) of \(\text{GameSeq}\), we write \(\sigma^{\text{external}}\) for its set of external positions.

**Definition G.4** We define the equivalence relation \(\sigma \sim_2 \tau\) between strategies by relating strategies having the same set of external positions:

\[
\sigma \sim_2 \tau \iff \sigma^{\text{external}} = \tau^{\text{external}}
\]

The \(\text{proj}\) function is designed with external positions in mind. Indeed, \(\text{proj}\) is undefined on \(\top, 0\), and, by extension, on maximal non-external positions. This has to be put in relation with relations, where the unit is also interpreted by \(\emptyset\), and hence any multiplicative formula with an additive unit in it has denotation the empty-set. Likewise, for every set of positions \(x\), \(\text{proj}(\top).\text{proj}(x) = \emptyset\) (where here \(\top\) denotes the set of maximal positions of the dialogue game interpreting \(\top\)). Hence, the above equation translates formally as \(\text{proj}(\sigma) = \text{proj}(\tau)\). That is:

\[
\sigma \sim_2 \tau \iff \text{proj}(\sigma) = \text{proj}(\tau)
\]
The purpose of the next section G.3 is to make sure that this invariant is a sound one. That is, \( \sigma \sim_1 \tau \Rightarrow \sigma \sim_2 \tau \).

G.3 Quotient and star autonomy

**Proposition G.5** Given a formula \( \Lambda \) of linear logic, the function \( \text{proj}_A : \llbracket (A)^F \rrbracket_{\text{GameSeq}} \rightarrow \llbracket A \rrbracket_{\text{NLRel}} \) defined in section ?? is an invariant of the interpretations of proofs. That is, the following diagram commutes.

\[
\begin{array}{c}
\pi : A \\
\downarrow \llbracket \text{NomLinRelPol} \rrbracket \\
\llbracket \pi \rrbracket_{\text{NomLinRelPol}}
\end{array}
\quad
\begin{array}{c}
\langle \cdot \rangle^F \\
\downarrow \llbracket \text{GameSeq} \rrbracket \\
\llbracket \langle \cdot \rangle^F \rangle_{\text{GameSeq}}
\end{array}
\quad
\begin{array}{c}
(\pi)^F : (A)^F \\
\downarrow \llbracket \text{NomLinRelPol} \rrbracket \\
\llbracket (\pi)^F \rrbracket_{\text{NomLinRelPol}}
\end{array}
\quad
\begin{array}{c}
\text{proj}_A \\
\downarrow \\
\text{proj}_A
\end{array}
\]

The demonstration is a proof by induction on the last rule of \( \pi \). This could also have been proven by categorical means, relying on \( \text{proj} \) forming a functor of dialogue categories with sums. Indeed the translation \( \langle \cdot \rangle^F \) has been conceived such that for every functor of dialogue categories \( F : \text{Dial} \rightarrow \text{Star} \), where \( \text{Star} \) is a star-autonomous category seen as a dialogue category, for every denotation function \( \llbracket \cdot \rrbracket : \text{Tens} \rightarrow \text{Dial} \), then \( F \circ \llbracket (\cdot)^F \rrbracket : \text{MALL} \rightarrow \text{Star} \) is a denotation function for linear logic proofs, that is, a star-autonomous functor. The proof is along the same lines as the proof of the proposition, and this property entails \( \sigma \sim_1 \tau \Rightarrow \sigma \sim_2 \tau \).

**Proof.** We start the proof by treating the leaves cases. We first tackle the axiom, then the \( \bot \)-rule straight after. At last, we present the \( \top \)-rule.

The axiom proof of \( \pi \vdash X^+; X \) is sent to the relation \( F \( (a, -1), (a, 1) \mid a \in \mathcal{A} \) \). On the other hand \( (\pi)^F \) is the axiom proof of tensorial logic \( (\pi)^T : X \vdash X \). It is interpreted as in section ??, and one can clearly see that \( \text{proj}(\llbracket (\pi)^F \rrbracket) = \llbracket (a, -1), (a, 1) \mid a \in \mathcal{A} \rrbracket \), as expected.

If the proof only consists of a \( \bot \)-rule, introducing \( \pi \vdash I \), then it is translated as the relation \( (\cdot, -1)(\cdot, 1) : \llbracket I \rrbracket \rightarrow \llbracket I \rrbracket \). This proof is translated into tensorial logic as the proof \( \bot \vdash I \) as well, that is interpreted as the game of figure ??, reaching the unique maximal position \( x \) of \( I \vdash \neg I \). Hence, this position is sent by \( \text{proj} \) onto \( (\cdot, -1)(\cdot, 1) \).

The other multiplicative unit is \( \bot \). Let us consider a proof \( \pi \) whose last rule is a \( \bot \)-rule introduction:

\[
\frac{\pi' \vdash \Gamma : A}{\pi \vdash \Gamma, \bot : A}
\]

Then the interpretation of \( \pi \) is \( \llbracket x_T : (\cdot, -1).\mathcal{A} \mid x_T.\mathcal{A} \in \llbracket \pi' \rrbracket_{\text{NLRel}} \). Now let us consider the translation of \( \pi \) into tensorial logic.

\[
\frac{(\pi')^T}{(I)^F \vdash (\Pi)^F}
\]

The first move of the strategy \( \sigma \) interpreting \( \pi \) is the same as the strategy \( \sigma' \) interpreting \( \pi' \), but the projection now differs, and takes the left \( I \) into account. That is, we now have \( \text{proj}(\llbracket (\pi')^F \rrbracket_{\text{GameSeq}}) = \llbracket (x_T : (\cdot, -1).\mathcal{A} \mid x_T.\mathcal{A} \in \llbracket (\pi')^F \rrbracket_{\text{NLRel}} \) as expected.

If the last rule of the proof of \( \pi \) is a Foc rule:

\[
\frac{\pi' \vdash \Gamma : P}{\pi \vdash \Gamma, P, \text{Foc}}
\]

The proof \( \pi \) is translated into the same relation as \( \pi' \), plus a atom on the right hand side that would correspond to the unit of the \( \gamma \), that is \( \bot \); \( \llbracket \pi \rrbracket_{\text{NLRel}} = \llbracket \pi' \rrbracket_{\text{NLRel}}(\cdot, -1) \). Then it is translated into:

\[
\frac{(\pi')^F}{\Gamma^F \vdash (P)^F \vdash \bot}
\]

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Hence, written \( \sigma = \llbracket \pi \rrbracket_{GameSeq} \) and \( \sigma' = \llbracket \pi' \rrbracket_{GameSeq} \), \( \text{proj}(\sigma^*) = \text{proj}(\sigma^*)(\ast, -1) \), as we could see from the interpretation of the negation in terms of strategies, given in Section ??.

The case for unfoc is similar to the foc case.

\[
\frac{\pi'}{\Gamma, M \vdash \text{unfoc}}
\]

The proof of \( \pi \) is denoted as follows:

\[
\llbracket \pi \rrbracket_{NLRel} = \{(x_T,x_M) | (x_T,x_M,(\ast,-1)) \in \llbracket \pi' \rrbracket_{NLRel}\}
\]

It is translated by \((\cdot)^F\) into:

\[
\frac{(\pi')F}{\Gamma^F, (M^+)^F \vdash \bot} \quad \text{Right } \neg
\]

whose interpretation is again given in section ??.

Then again, one can notice that \( \text{proj} \) acts as follows:

\[
\text{proj}(\sigma^*) = \{(x_{\Pi},x_{\Pi\Pi\Pi},x_{\Pi\Pi\Pi\Pi}) | (x_{\Pi},x_{\Pi\Pi\Pi\Pi},x_{\Pi\Pi\Pi\Pi\Pi},(\ast, -1)) \in \text{proj}(\sigma^*)\}
\]

hence making the diagram commutes.

If the last rule of \( \pi \) is a \( \otimes \) on the left hand side, then its interpretation as a sequent remains unchanged. That is, \( \llbracket \pi \rrbracket = \llbracket \pi' \rrbracket \). On the other hand, it is translated as the application of a left \( \otimes \)-rule on \((\pi')^F\), that also leaves the strategy, and the arena, unchanged. So \( \sigma = \sigma' \) and hence \( \text{proj}(\sigma) = \text{proj}(\sigma') \). We remind the rule and its translation in the table below.

\[
\begin{array}{cc}
\frac{\pi'}{\Gamma, M, N; \Pi} & \frac{(\pi')^F}{\Gamma^F, (M^+)^F \vdash (N^+)^F \vdash (\Pi)^F} \quad \text{Left } \otimes \\
\frac{\Gamma, M \otimes N; \Pi}{\Pi} & \text{since } ((M \otimes N)^+)^F = (M^+ \otimes N^+)^F = ((M^+)^F \otimes (N^+)^F))
\end{array}
\]

We now deal in the case where the last rule of \( \pi \) is a \( \otimes \) rule. Then it is invariably translated by \((\cdot)^F\) into a \( \otimes \)-rule. We remind here the rule and its translation:

\[
\begin{array}{cc}
\frac{\pi_1}{\Gamma; Q} & \frac{\pi_2}{\Gamma; P} & \frac{(\pi_1)^F}{\Gamma^F, (P)^F} \quad \frac{(\pi_2)^F}{\Gamma^F, (Q)^F} \quad \text{Right } \otimes \\
\frac{\Gamma, P \otimes Q; \Pi}{\Pi} & \text{since } (P \otimes Q)^F = P^F \otimes Q^F.
\end{array}
\]

\[
\begin{array}{cc}
\frac{\pi_1}{\Gamma; M} & \frac{\pi_2}{\Gamma; N} & \frac{(\pi_1)^F}{\Gamma^F, (\Delta)^F} \quad \frac{(\pi_2)^F}{\Gamma^F, (\Delta)^F} \quad \text{Left } \otimes \\
\frac{\Gamma, M \otimes N; \Pi}{\Pi} & \text{since } (M \otimes N)^F = (\neg M)^F \otimes (\neg N)^F).
\end{array}
\]

Therefore, the interpretation of \( \pi \) is \( \llbracket \pi \rrbracket \otimes \llbracket \pi' \rrbracket \). That is:

\[
\llbracket \pi \rrbracket = \{(x_T,x_{\Pi\Pi\Pi\Pi},x_{\Pi\Pi\Pi\Pi\Pi}) | (x_{\Pi\Pi\Pi\Pi},x_{\Pi\Pi\Pi\Pi\Pi\Pi}) \in \llbracket \pi_1 \rrbracket \}, (x_T,x_{\Pi\Pi\Pi\Pi\Pi\Pi}) \in \llbracket \pi_2 \rrbracket \}, (x_T,x_{\Pi\Pi\Pi\Pi\Pi\Pi\Pi}) \text{pol}(x_{\Pi\Pi\Pi\Pi\Pi\Pi\Pi}, x_{\Pi\Pi\Pi\Pi\Pi\Pi\Pi})
\]
Furthermore, the strategy $\sigma = \sigma_1 \otimes \sigma_2$, will reach the maximal positions $\sigma^* \starPol \tau^*$, as $\sigma^* \simeq \Frugal(\sigma_1^* \otimesPol \sigma_2^*)$, therefore:

$$\proj(\sigma) = \{ (x_T, x_A, x_P, x_Q) \mid (x_T, x_P) \in \proj(\sigma_1), (x_A, x_Q) \in \proj(\sigma_2), (x_T, x_P) \#Pol (x_A, x_Q) \}$$

The case where the formula on the right hand side is $M, N$ is dealt on an equal footing. We now treat the case where the last rule of $\pi$ is a $\&$.

$$\frac{\pi_1 \vdash \Gamma, P; \Pi \quad \pi_2 \vdash \Gamma, Q; \Pi}{\pi_1 \vdash \Gamma, P \& Q; \Pi} \quad \text{Left } \&$$

$$\frac{(\Gamma)^F, \neg(P^F) \vdash (\Pi)^F}{(\Gamma)^F, \neg(P^F) \oplus \neg(Q^F) \vdash (\Pi)^F} \quad \text{Left } \oplus$$

$$\frac{(\Gamma)^F, (\Pi)^F}{(\Gamma)^F, (\Pi)^F} \quad \text{since } ((P \& Q)^\dagger)^F = (P^\dagger \oplus Q^\dagger)^F = \neg(P^F) \oplus \neg(Q^F)$$

$$\frac{\pi_1 \vdash \Gamma, M; \Pi \quad \pi_2 \vdash \Gamma, N; \Pi}{\pi_1 \vdash \Gamma, M \& N, \Sigma; \Pi} \quad \text{Left } \&$$

$$\frac{(\Gamma)^F, (M^\dagger)^F \oplus (\Pi)^F}{(\Gamma)^F, (M^\dagger)^F \oplus (\Pi)^F} \quad \text{Left } \oplus$$

$$\frac{(\Gamma)^F, (N^\dagger)^F \vdash (\Pi)^F}{(\Gamma)^F, (M^\dagger)^F \oplus (N^\dagger)^F \vdash (\Pi)^F} \quad \text{since } ((M \& N)^\dagger)^F = (M^\dagger \oplus N^\dagger)^F = (M^\dagger)^F \oplus (N^\dagger)^F$$

Then the proof of $\pi_1$ and $\pi_2$ is interpreted as the union between both:

$$\llbracket \pi_1 \& \pi_2 \rrbracket = \{ (x_T, \text{inl}(x_P), x_{\Pi_1}) \mid (x_T, x_P, x_{\Pi_1}) \in \llbracket \pi_1 \rrbracket \}$$

$$\cup \{ (x_T, \text{inr}(x_Q), x_{\Pi_1}) \mid (x_T, x_Q, x_{\Pi_1}) \in \llbracket \pi_2 \rrbracket \}$$

Similarly, looking at the strategy $\sigma$ interpreting the proof $\pi$, then $\sigma^* \simeq \text{inl}(\sigma_1) \cup \text{inr}(\sigma_2)$. Depending on which side of the $\oplus$ the opponent is going to play its first move, the strategy is going to react according to $\sigma_1$ or $\sigma_2$. Therefore:

$$\proj(\sigma^*) = \{ (x_T, \text{inl}(x_P), x_{\Pi_1}) \mid (x_T, x_P, x_{\Pi_1}) \in \proj(\sigma_1^*) \}$$

$$\cup \{ (x_T, \text{inr}(x_Q), x_{\Pi_1}) \mid (x_T, x_Q, x_{\Pi_1}) \in \proj(\sigma_2^*) \}$$

Finally, we address the $\oplus$-rule. If the last rule of $\pi$ is an $\oplus$-rule, then it is interpreted in $(\pi)^F$ as an $\oplus$-rule as well. We treat the case where the last rule of $\pi$ is $\oplus_1$. For instance, we present the translation where the formulas are both positive or negative.

$$\frac{\pi' \vdash \Gamma, P}{\pi' \vdash \Gamma; P \oplus Q} \quad \oplus_1$$

$$\frac{(\pi')^F}{(\Gamma)^F \oplus P^F \oplus Q^F} \quad \text{Right } \oplus_1$$

$$\frac{(\pi')^F}{(\Gamma)^F \oplus P^F \oplus Q^F} \quad \text{since } (P \oplus Q)^F = P^F \oplus Q^F$$

$$\frac{\pi' \vdash \Gamma, M}{\pi' \vdash \Gamma; M \oplus N} \quad \oplus_1$$

$$\frac{(\pi')^F}{(\Gamma)^F \oplus \neg(M^\dagger)^F \oplus \neg(N^\dagger)^F} \quad \text{Right } \oplus_1$$

$$\frac{(\pi')^F}{(\Gamma)^F \oplus \neg(M^\dagger)^F \oplus \neg(N^\dagger)^F} \quad \text{since } (M \oplus N)^F = (M^\dagger)^F \oplus (N^\dagger)^F$$

The nominal relation interpreting $\pi$ will be the left injection of the one interpreting $\pi'$, that is:

$$\llbracket \pi \rrbracket_{\text{NLRel}} = \{ (x_T, \text{inl}(x_P)) \mid (x_T, x_P) \in \llbracket \pi' \rrbracket_{\text{NLRel}} \}$$

Similarly, the strategy $\sigma$ interpreting $(\pi)^F$ will act as $\sigma' \simeq \llbracket (\pi')^F \rrbracket_{\text{GameSeq}}$, but going on the left branch of the $\oplus$ in its first P-move. Therefore, the following holds:

$$\proj(\sigma) = \{ (x_T, \text{inl}(x_P)) \mid (x_T, x_P) \in \proj(\sigma') \}$$

This is coherent with the interpretation of $\pi$ through $\llbracket \pi \rrbracket_{\text{NLRel}}$, as it satisfies the same equality. □
We remind the $\llbracket\cdot\rrbracket_{\text{NLRel}}$ is a functor, that is, respects the equivalence of proofs of linear of linear logic ($\pi \sim \pi' \Rightarrow \llbracket\pi\rrbracket_{\text{NLRel}} = \llbracket\pi'\rrbracket_{\text{NLRel}}$). Therefore, the interpretation:

$$\begin{array}{ccc}
\pi \mapsto (\pi)^F & \llbracket\cdot\rrbracket_{\text{GameSeq}} & \llbracket(\pi)^F\rrbracket_{\text{GameSeq}} \\
\pi \mapsto \text{proj} & \llbracket(\pi)^F\rrbracket_{\text{GameSeq}} & \llbracket(\pi)^F\rrbracket_{\text{GameSeq}} \\
\end{array}$$

is a denotation function that respects the equivalence of proofs of linear logic. That is, if $\pi \sim \pi'$ then $\llbracket(\pi)^F\rrbracket_{\text{GameSeq}} = \llbracket(\pi')^F\rrbracket_{\text{GameSeq}}$. Furthermore, one should ensure that it acts as a functor. That is:

$$\text{proj}\llbracket(\pi;\pi')^F\rrbracket_{\text{GameSeq}} = \text{proj}\llbracket(\pi)^F\rrbracket_{\text{GameSeq}} \llbracket(\pi')^F\rrbracket_{\text{GameSeq}}$$

If follows from $\text{proj}\llbracket(\pi)^F\rrbracket_{\text{GameSeq}} = \llbracket\pi\rrbracket_{\text{NLRel}} = \llbracket\pi'\rrbracket_{\text{NLRel}}$, $\llbracket\pi'\rrbracket_{\text{NLRel}}$.

**G.4 Full completeness for linear logic**

We are now in position of presenting the full completeness result for linear logic. We work within the category of nominal annotated polarised separated relations. Given some objects $A$, $B$, where $A$, $B$ are seen as formulas of linear logic, we can translate them as formulas of tensorial logic $(A)^F$, $(B)^F$. Now each strategy in $\text{GameSeq}(A)^F, (B)^F$ is a denotation of a proof of tensorial logic $\pi : (A)^F \vdash (B)^F$. To this one corresponds a proof $(\pi)^F : (A)^F \vdash (B)^F$, which has, as denotation $\text{proj}(\pi)^F$. Therefore, we can select precisely those nominal relations that do correspond to proofs, and obtain a full completeness result.

**Definition G.6** The category $\text{NomMall}$ is the star-autonomous category that has same objects as $\text{NomLinRelPol}$ and morphisms nominal linear polarised relations that arise as projections of strategies of $\text{GameSeq}$.

**Proposition G.7** $\text{NomMall}$ is fully complete for multiplicative additive linear logic.

In $\text{NomMall}$, a map $A \rightarrow B$ is a nominal polarised relation $\mathcal{R}$ such that there exists $\sigma \in \text{GameSeq}(A^F, B^F)$, with $\mathcal{R} = \text{proj}(\sigma)$. We proved in the section above that this forms a category, and, by definition, each morphism in it is the denotation of a proof of linear logic. $\text{NomMall}$ is precisely the sub-category of $\text{NomLinRelPol}$ that corresponds to the image of the functor $\llbracket\cdot\rrbracket : \text{MALL} \rightarrow \text{NomLinRelPol}$. As this functor is a star-autonomous one, so is $\text{NomMall}$. More precisely, $\text{NomMall}$ is a sound and fully complete model of MALL.

Finally, one might wonder, if, as in the case of $\text{GameSeq}$, the category obtained is the free star-autonomous category with products. Unfortunately, the answer is negative. A simple counter-example is formed by these two proofs together with their denotations:

$$\begin{array}{ll}
\begin{array}{c}
\text{\vdash} \top \\
\end{array} & \begin{array}{c}
\text{\vdash} I_1 \\
\end{array} & \begin{array}{c}
\text{\vdash} I_2 \\
\end{array} \\
\begin{array}{c}
\text{\vdash} \bot_1, I_1 \\
\end{array} & \begin{array}{c}
\text{\vdash} \bot_1, I_2 \\
\end{array} & \begin{array}{c}
\text{\vdash} \bot_2, I_1 \\
\end{array} \\
\begin{array}{c}
\text{\vdash} \bot_1 \otimes I_2 \\
\end{array} & \begin{array}{c}
\text{\vdash} \bot_2 \otimes I_1 \\
\end{array} & \begin{array}{c}
\text{\vdash} \top_1 \otimes I_2 \\
\end{array} \\
\begin{array}{c}
\text{\vdash} \top_1 \otimes \bot_2, I_2 \\
\end{array} & \begin{array}{c}
\text{\vdash} \top_2 \otimes \bot_1, I_1 \\
\end{array} & \begin{array}{c}
\text{\vdash} \top_1 \otimes \bot_2, I_1 \\
\end{array} \\
\begin{array}{c}
\text{\vdash} \top_1 \otimes \bot_2, I_1 \\
\end{array} & \begin{array}{c}
\text{\vdash} \top_2 \otimes \bot_1, I_2 \\
\end{array} & \begin{array}{c}
\text{\vdash} \top_1 \otimes \bot_2, I_1 \\
\end{array} \\
\begin{array}{c}
\text{\vdash} \top_1 \otimes \bot_2, I_1 \\
\end{array} & \begin{array}{c}
\text{\vdash} \top_2 \otimes \bot_1, I_2 \\
\end{array} & \begin{array}{c}
\text{\vdash} \top_1 \otimes \bot_2, I_1 \\
\end{array} \\
\end{array}$$

Let us name $\pi_1$ the left one, $\pi_2$ the other. Then $\neg (\pi_1 \sim \pi_2)$. This is notably proved in [?]. On the other hand, they are both denoted by the same relation:

$$\llbracket\pi_1\rrbracket = \llbracket\pi_2\rrbracket = \{(\bullet, -1), (\bullet, -1), (\bullet, 1), (\bullet, 1)\}$$

This mismatch proves that the the nominal relations are too simple, too flat, to fully distinguish between distinct proofs of linear logic. Furthermore, this full completeness is obtained through the medium of tensorial logic: we do not have a direct characterisation of nominal relations that are denotations of proofs of linear logic.