Stone Duality for Stable Functions

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Introduction

The problem of finding an algebraic structure for stable open subsets of a suitable domain has been recently raised by several authors. Specifically, it could be useful to have, in the stable case, a notion similar to the one of "frame", in order to develop something similar to pointless topology, that is an algebraic insight of spaces.

In domain theory, an interesting application of frame theory is the logic of domains developped by Abramsky (see [Ab]) in the topological case.

The idea of constructing a logic for "stable properties" has been originated by Zhang (see [Zha]), who got very interesting results about "stable opens" of dI domains. However, his notions remain concrete, and it is not clear whether they give rise to a duality. In that sense, they lack the "localic" properties which justify the canonicity of Abramsky's approach.

We introduce the notion of S-Structure as the structure of the algebra of stable open sets intended to correspond to the concept of frame in the stable case. These S-structures have properties which are very similar to the ones of frames, from the point of view of duality. So we may hope to achieve a logic of domains as natural as Abramsky's one, but expressing properties of programs which are not captured by the continuous approach.

In this paper, we give the fundamentals of S-structures theory. We prove first general duality results which do not involve any domain theoretical assumtion about spaces. Actually we introduce the S-spaces which play wrt S-structures the same role as topological spaces wrt frames. These duality results can be specialized to the case of domains, and then we obtain a result similar to known Scott-topological duality in domain theory. The corresponding notion of domain widely subsumes the usual dI domains to which stabilty theory is usually restricted. Indeed these domains are the most general ones where stability makes sense. By duality method it is rather simple to treat function spaces and we retrieve cartesian closedness of the category of L-domains and stable functions which has been recently proved by P. Taylor.

1 Basic definitions and results

This section is intended to provide an abstract definition of the algebraic structure of stable open sets. Let us first give some intuition for the forthcoming definitions. Let us consider a dI domain D. We

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call stable open subset of D the inverse image of $\top$ under a stable map from D to the two points domain $\mathbf{O} = \{\bot < \top\}$. This is a natural definition since a Scott open subset is the inverse image of $\top$ under a continous function. The study of stable open sets has been initiated by Zang [Zha] and in this sense we follow his approach. As Zhang remarked stable opens are closed under finite intersection and disjoint union. Moreover they are closed under union of directed families, but not under union of arbitrary sets. In order to keep distributivity these kinds of join are the only ones we will consider.

Definition 1 *In a meet-semilattice with* 0*, we say that a subset* D *is disjoint-directed (dd for short) if for any* $u, v \in D$*, if* $u \wedge v \neq 0$ *then there exists a* $w \in D$ *such that* $u \leq w$ *and* $v \leq w$*.*

Lemma 1 *If* D *is dd not containing* 0*, then the binary relation* $\sim_D$ *defined by:* $u \sim_D v$ *iff there exists* $w \in D$ *such that* $u, v \leq w$ *is an equivalence relation the classes of which are directed.*

Proof: We have just to check transitivity of $\sim_D$; if $u \sim_D v$ and $v \sim_D u'$ then we have $w, w' \in D$ which verify $u, v \leq w$ and $v, u' \leq w'$; it follows that $0 \neq v \leq w \wedge w'$ so there exists w'' such that $u, u' \leq w''$. ∎

Definition 2 *A* S-space X *is a pair* $(X, \Omega_S(X))$ *where* X *is a set (the set of points) and* $\Omega_S(X)$ *is a subset of* $\mathcal{P}(X)$ *containing* $\emptyset$*,* X *and which is closed under finite intersections and dd unions (*$\Omega_S(X)$ *is the set of S-open subsets of* X*, a* S-topology *on* X*).*

If X*,* Y *are S-spaces, a function* $f : X \to Y$ *is a* S-map *if it preserves S-opens under inverse image.*

The category of S-spaces and S-maps is noted **Ssp***.*

Definition 3 *A meet-semilattice* $(\mathcal{L}, \wedge, 0)$ *is an S-structure if*

i) any dd subset D *of* $\mathcal{L}$ *has a lub noted* $\bigvee D$

ii) finite glbs distribute over dd lubs

iii) there is a top element noted 1*.*

If $\mathcal{L}$ *and* $\mathcal{M}$ *are two S-structures, a function* $f : \mathcal{L} \to \mathcal{M}$ *is an S-morphism if it preserves the structure.*

The category of S-structures and S-morphisms is noted **Sstr***.*

Definition 4 *An element* $u \neq 0$ *of an S-structure* $\mathcal{L}$ *is*

- connected *if for any* $v, w \in \mathcal{L}$ *such that* $v \wedge w = 0$*, if* $v \vee w \geq u$ *then* $v \geq u$ *or* $w \geq u$
- compact *if for any directed* $D \subseteq \mathcal{L}$*, if* $\bigvee D \geq u$ *then there exists a* $v \in D$ *such that* $v \geq u$
- dd-prime *if for any dd* $D \subseteq \mathcal{L}$*, if* $\bigvee D \geq u$ *then there exists a* $v \in D$ *such that* $v \geq u$*.*

Typographical convention: we use letters $u, v, w \ldots$ to denote arbitrary elements of $\mathcal{L}$, letters $a, b, c, \ldots$ to denote connected elements and $\alpha, \beta, \gamma \ldots$ to denote dd-prime elements. From now on, we shall not specify otherwise the nature of elements of $\mathcal{L}$.

Lemma 2 *The lub of any directed set of connected elements of an S-structure is connected.*

Proof: Let C be a directed set of connected elements, $w = \bigvee C$, and u, v such that $u \wedge v = 0$ and suppose $w \leq u \vee v$. If $a \in C$ then $a \leq u \vee v$, so $a \leq u$ or $a \leq v$; now if for $a, a' \in C$, $a \leq u$ and $a' \leq v$ then for $b \in C$ such that $b \geq a, a'$ if $b \leq u$ then $a' \leq u \wedge v$ and if $b \leq v$ then $a \leq u \wedge v$, both cases contradicting the hypothesis. We conclude that any $a \in C$ is less than u or any $a \in C$ is less than v, so $w \leq u$ or $w \leq v$. ∎

Definition 5 *A subset of a meet-semilattice with* 0 *is* disjoint *if the glb of any two different elements of this set is* 0*.*

Lemma 3 *Let* $D \subseteq \mathcal{L}$ *be a dd set (not containing* 0*). Then the set* $\{\bigvee \Delta \mid \Delta \in D/{\sim_D}\}$ *is disjoint.*

Proof: Let $\Delta, \Delta' \in D/{\sim_D}$. If $(\bigvee \Delta) \wedge (\bigvee \Delta') \neq 0$, there exist by distributivity a $u \in \Delta$ and a $u' \in \Delta'$ such that $u \wedge u' \neq 0$. Since D is dd, we have $u {\sim_D} u'$ and hence $\Delta = \Delta'$. ∎

Lemma 4 *An element* u *of an S-structure* $\mathcal{L}$ *is connected iff for any disjoint subset* E *of* $\mathcal{L}$*, if* $\bigvee E \geq u$ *there exists a* $v \in E$ *such that* $v \geq u$*.*

Proof: Let $\dot{\bigvee}$ denote the sup of a disjoint set, and suppose $u \leq \dot{\bigvee} E$; then, by distributivity, $u = \dot{\bigvee} A$ where A is $\{e \wedge u \mid e \in E\}$, hence there exists $e' \in E$ such that $0 \neq e' \wedge u \leq u$; by defining $E_0 = E - \{e'\}$ we get $u \leq \dot{\bigvee} E_0 \mathbin{\dot{\vee}} e'$; the fact that u is connected implies $u \leq \dot{\bigvee} E_0$ or $u \leq e'$; but if $u \leq \dot{\bigvee} E_0$ then $0 \neq u \wedge e' \leq \dot{\bigvee} \{e \wedge e' \mid e \in E_0\} = 0$, which is a contradiction, hence $u \leq e'$. ∎

Lemma 5 *An element* u *of* $\mathcal{L}$ *is dd-prime iff it is compact and connected.*

Proof: The "if" part is trivial.
In order to show the "only if" part we remark that from lemma 1 a dd-set D can be factorized in a family of disjoint directed sets $(\Delta_i)_{i \in I}$, such that $\bigcup_{i \in I} \Delta_i = D$; moreover by defining $\delta_i = \bigvee \Delta_i$ we have $\bigvee D = \dot{\bigvee}_{i \in I} \delta_i$; so if $u \leq \bigvee D = \dot{\bigvee}_{i \in I} \delta_i$ then $\exists i \in I (u \leq \delta_i)$ (by using lemma 4 because u is connected) and finally that $\exists v \in \Delta_i (u \leq v)$ because u is compact. ∎

Definition 6 *An S-structure* $\mathcal{L}$ *is* algebraic *iff for any* $u \in \mathcal{L}$ *the set* $\pi(u)$ *of prime lower bounds of* u *is dd and* $u = \bigvee \pi(u)$*.*

Lemma 6 *An element* u *of an algebraic S-structure is connected iff* $\pi(u)$ *is directed.*

Proof: By using lemma 1 and by definition of algebricity we have $u = \dot{\bigvee}_{i \in I} (\bigvee \Delta_i)$ where each Δ_i is a directed set, $i \neq j$ implies $\Delta_i \cap \Delta_j = \emptyset$ and $\pi(u) = \bigcup_{i \in I} \Delta_i$; let $u_i = \bigvee \Delta_i$; it follows that $\pi(u_i) = \Delta_i$ hence u_i is connected; so $u = \dot{\bigvee}_{i \in I} u_i$ and by lemma 4 there exists $i_0 \in I$ such that $u = u_{i_0}$. The "only if" part follows from lemma 2. ∎

Proposition 1 *The category* **Sstr** *of S-structures is algebraic.*

Proof: Let **Slat** be the category of meet-semilattices with 0 and 1. We have a forgetful functor F : **Sstr** → **Slat**. We define a left adjoint dd-Idl : **Slat** → **Sstr** to F as follows:

- If $A \in$ **Slat**, we define dd-Idl A as the set of non empty dd downwards closed subsets of A, ordered under inclusion. That is, $u \subseteq A$ is in dd-Idl A iff: $0 \in u$, if $a \in u$ and $b \leq a$ then $b \in u$, and last u is dd.
- If $\varphi : A \to B$ is a morphism in **Slat**, and $u \in$ dd-Idl A, then dd-Idl $(\varphi)(u) = \downarrow (\varphi)(u)$.

First, we prove that dd-Idl A is an S-structure. As 0, we take $\{0\}$, and as 1, we take A itself. Let $u, v \in$ dd-Idl A. Then $u \cap v$ is a lower set (clear). Let us check it is dd. Let $a, b \in u \cap v$ be such that $a \wedge b \neq 0$. Since u and v are dd, we can find $c \in u$ and $d \in v$ such that $c \geq a, b$ and $d \geq a, b$. But $c \wedge d \in u \cap v$ and we conclude. Let now $D \subseteq$ dd-Idl A be dd. $\bigcup D$ is clearly a lower set. Let $a, b \in \bigcup D$ be such that $a \wedge b \neq 0$. Let $u, v \in D$ be such that $a \in u$ and $b \in v$. Since $a \wedge b \in u \cap v$, and $a \wedge b \neq 0$, we have $u \cap v \neq 0$. Hence there is a $w \in D$ such that $u, v \subseteq w$. Now $a, b \in w$ and w is dd, so there is a $c \in w$ such that $c \geq a, b$ and we conclude. Distributivity is obvious, so we have defined an S-structure, the free S-structure over A. Checking that dd-Idl (φ) is a an S-morphism is a straightforward verification.

Now we check that dd-Idl is the left adjoint of F. For this, we define naturally an S-morphism $s : \text{dd-Idl}\,\mathrm{F}\mathcal{L} \to \mathcal{L}$. Let $I \in \text{dd-Idl}\,\mathrm{F}\mathcal{L}$. We set $s(I) = \bigvee I$, which is defined since I is dd. Now we prove that s is an S-morphism. Preservation of 0, 1 and all dd lubs are obvious. Let $I, J \in \text{dd-Idl}\,\mathrm{F}\mathcal{L}$. We have

$$\begin{aligned} s(I \cap J) &= \bigvee \{u \wedge v \mid u \in I,\ v \in J\} \\ &= (\bigvee I) \cap (\bigvee J) \end{aligned}$$

by distributivity. Let $f : \text{dd-Idl}\,A \to \mathcal{L}$. To conclude that the required djunction holds, we have to show that there exists a unique $\varphi : A \to \mathrm{F}\mathcal{L}$ such that $f = s \circ \text{dd-Idl}\,(\varphi)$. We set, for $a \in A$, $\varphi(a) = f(\downarrow(a))$ (there is no choice indeed since $s(\downarrow(a)) = a$). That φ preserves 0 and 1 is obvious. Furthermore

$$\varphi(a \wedge b) = f(\downarrow(a \wedge b)) = f(\downarrow(a) \cap \downarrow(b)) = f(\downarrow(a)) \wedge f(\downarrow(b))$$

and we conclude.

It remains to prove that the adjunction is monadic. This is a routine verification. ∎

Definition 7 *A dd-prime filter of an S-structure $\mathcal{L}$ is a subset p of $\mathcal{L}$ which is a filter and such that, for any $D \subseteq \mathcal{L}$ dd, if $\bigvee D \in p$ then $p \cap D \neq \emptyset$. We note $\mathrm{pt}(\mathcal{L})$ the poset of dd-prime filters of $\mathcal{L}$ ordered by inclusion.*

Remark that any dd-prime filter is the inverse image of $1 \in \mathcal{O}$ (the two elements S-structure $\{0, 1\}$) under a morphism of S-structure, and conversely.

Let us now define the cpo's on which we shall define S-topologies.

Definition 8 *A cpo $(X, \leq)$ is said to be a* meet-cpo *if it has bounded binary glbs and if these glbs ditribute over directed lubs.*

Now we define stable open subsets of such cpo's. The word "stable" has here to be understood as "conditionally multiplicative", in the terminology introduced by Berry (see [Ber]).

Definition 9 *A subset U of a meet-cpo X is a* stable open *subset if it is Scott-open and closed under bounded binary glbs. We note $\Omega^{\wedge}_{\mathrm{S}} X$ the poset of stable open subsets of X ordered by inclusion.*

Definition 10 *A Scott-continuous function $f : X \to Y$ between two meet-cpos is said to be* stable *iff it preserves binary bounded glbs.*

Last, we recall Berry's stable ordering for stable functions.

Definition 11 *Let $f, g : X \to Y$ be two stable functions. One says that f is stably less than g and write $f \sqsubseteq g$ iff for all $x, x' \in X$ such that $x \leq x'$ one has*

$$f(x) = f(x') \wedge g(x)\ .$$

2 Duality results

The aim of this section is to establish some connection between categories of S-structures and S-spaces.

2.1 Basic duality

The first result we prove is the existence of an adjunction between the categories **Ssp** and $\mathbf{Sstr}^{\mathrm{op}}$.

We can always define a map $\varphi : \mathcal{L} \to \mathcal{P}(\mathrm{pt}(\mathcal{L}))$ by

$$\varphi(u) = \{p \in \mathrm{pt}(\mathcal{L}) \mid u \in p\}$$

We will write sometimes $\varphi_{\mathcal{L}}$ instead of φ in order to stress the dependence of φ wrt $\mathcal{L}$.

Conversely, for a S-space X it is possible to define a S-map $\Phi : X \to \mathrm{pt}(\Omega_{\mathrm{S}}(X))$ by

$$\Phi(x) = \{U \mid x \in U\} \ .$$

For the same purpose we will write sometimes Φ_X instead of Φ.

Lemma 7 *For any S-structure $\mathcal{L}$ there exists an S-space $(\mathrm{pt}(\mathcal{L}), \Omega^{\circ}_{\mathrm{S}}\mathcal{L})$ which is the least (wrt inclusion) among all S-topologies on $\mathrm{pt}(\mathcal{L})$ containing $\varphi(\mathcal{L})$.*

Furthermore, $\varphi_{\mathcal{L}}$ is an S-morphism from $\mathcal{L}$ to $\Omega^{\circ}_{\mathrm{S}}\mathcal{L}$.

Proof: Simlpy take for $\Omega^{\circ}_{\mathrm{S}}\mathcal{L}$ the intersection of all S-topologies on $\mathrm{pt}(\mathcal{L})$ which contain $\varphi(\mathcal{L})$. Remark that there is at least one of such S-topologies, namely $\mathcal{P}(\mathrm{pt}(\mathcal{L}))$. However for technical reasons we give an ordinal construction of $\Omega^{\circ}_{\mathrm{S}}\mathcal{L}$ by setting $L_0 = \varphi(\mathcal{L})$,

$$L_{\eta+1} = L_\eta \cup \{U \cap V \mid U, V \in L_\eta\} \cup \{\bigcup D \mid D \subseteq L_\eta \text{ and } D \text{ is dd}\}$$

$$L_\lambda = \bigcup_{\eta<\lambda} L_\eta \text{ for } \lambda \text{ limit}$$

It is easily verified that this increasing sequence becomes stationary for some ordinal ρ and that $\Omega^{\circ}_{\mathrm{S}}\mathcal{L} = L_\rho$.

The last statement of the lemma is clear. ∎

Lemma 8 *If $f : \mathcal{L} \to \mathcal{M}$ is a S-morphism then $f_* : \mathrm{pt}(\mathcal{M}) \to \mathrm{pt}(\mathcal{L})$ is a S-map wrt the S-topologies $\Omega^{\circ}_{\mathrm{S}}\mathcal{M}$ and $\Omega^{\circ}_{\mathrm{S}}\mathcal{L}$.*

Proof: We have to show that ${f_*}^{-1}(U) \in \Omega^{\circ}_{\mathrm{S}}\mathcal{M}$ for all $U \in \Omega^{\circ}_{\mathrm{S}}\mathcal{L}$. Let (L_η) (resp. (M_η)) denote the ordinal construction of $\Omega^{\circ}_{\mathrm{S}}\mathcal{L}$ (resp. $\Omega^{\circ}_{\mathrm{S}}\mathcal{M}$). We prove by induction that for all η, ${f_*}^{-1}(U) \in M_\eta$ for $U \in L_\eta$.

For $\eta = 0$ we have ${f_*}^{-1}(\varphi(u)) = \varphi(f(u))$.

Assume that the condition holds at η and let $U \in L_{\eta+1}$. The case $U \in L_\eta$ is trivial. If U is an intersection of elements of L_η then the condition holds by preservation of intersections under inverse image. If U is the union of a dd subset of L_η then it suffices to remark that the inverse image of this subset is a dd subset of M_η and that unions are preserved under inverse image.

The case where η is a limit ordinal is trivial. ∎

So we can define the functor $\mathbf{Pt} : \mathbf{Sstr}^{\mathrm{op}} \to \mathbf{Ssp}$ defined on morphisms by $\mathbf{Pt}f = f_*$. Conversely we have the functor $\mathbf{\Omega}_{\mathrm{S}} : \mathbf{Ssp} \to \mathbf{Sstr}^{\mathrm{op}}$ defined by $\mathbf{\Omega}_{\mathrm{S}}(X, \Omega_{\mathrm{S}}(X)) = \Omega_{\mathrm{S}}(X)$ and $\mathbf{\Omega}_{\mathrm{S}}f = f^{-1}$.

Lemma 9 *Let X be an S-space. Then $\varphi : \Omega_{\mathrm{S}}(X) \to \Omega_{\mathrm{S}}^{\circ}\Omega_{\mathrm{S}}(X)$ is an isomorphism, and for any $U \in \Omega_{\mathrm{S}}(X)$ we have $\Phi^{-1}(\varphi(U)) = U$, so that Φ is an S-map from X to $\mathbf{Pt\Omega}_{\mathrm{S}}X$.*

Proof: It suffices to remark that, for $x \in X$ and $U \in \Omega_{\mathrm{S}}(X)$, we have

$$x \in U \quad \text{iff} \quad \Phi(x) \in \varphi(U) \ .$$

The lemma follows easily from this fact. ∎

Proposition 2 *The functor $\mathbf{\Omega}_{\mathrm{S}}$ is left adjoint to $\mathbf{Pt}$.*

Proof: It suffices to remark that the family of morphisms $(\varphi_{\mathcal{L}})$ defines the unit of the adjunction and the family (Φ_X) its counit. ∎

Definition 12 *An S-structure $\mathcal{L}$ is* spatial *iff $\varphi_{\mathcal{L}}$ is an isomorphism between $\mathcal{L}$ and $\Omega_{\mathrm{S}}^{\circ}\mathcal{L}$.*

Definition 13 *A S-space X is said* sober *if Φ_X is an isomorphism.*

As a consequence of the previous proposition we get the following:

Corollary 1 *the functors $\mathbf{Pt}$ and $\mathbf{\Omega}_{\mathrm{S}}$ define an equivalence between the dual of the category of spatial S-structures and the category of sober S-spaces.*

2.2 Duality in the algebraic case

From now on, we restrict our interest to the special case where the S-spaces are meet-cpo's and the S-topology is the one induced by stability. Actually:

Proposition 3 *For any meet-cpo X, the set $\Omega_{\mathrm{S}}^{\wedge}X$ is an S-topology on X called the stable S-topology on X.*

Proof: Let $\mathcal{D} \subseteq \Omega_{\mathrm{S}}^{\wedge}X$ be a dd set. We prove that $V = \bigcup \mathcal{D}$ is in $\Omega_{\mathrm{S}}^{\wedge}X$. That V is Scott open is standard. Let $x, y \in V$ be bounded. Let $U, U' \in \mathcal{D}$ be such that $x \in U$ and $y \in U'$. Then since x and y are bounded, $U \cap V$ is non empty and thus has an upper bound $W \in \mathcal{D}$. Since $x \wedge y \in W$, we have $x \wedge y \in V$.

It is obvious that $\Omega_{\mathrm{S}}^{\wedge}X$ has binary glbs (which are intersections), a 0 (the empty set) and a 1 (the whole space). Distributivity is clear too (since glbs are intersections and lubs unions). ∎

Proposition 4 *Let $f : X \to Y$, where X and Y are two algebraic meet-cpo's. Then f is stable iff it is an S-map with respect to the stable S-topologies of X and Y. .*

Proof: Let f be an S-map and let us prove that it is stable, the converse being trivial. Indeed we do not really need algebraicity of domains, but only a T_0 property for the S-topolgy of Y, insured for instance by spatiality. But there is no place here to go into these developments. The fact that f is Scott-continuous is plain, since any S-open is Scott open, and since the S-open sets of the shape $\uparrow k$ where k is compact form a basis of the Scott-topology. In particular, f is monotonous. Now let $x, y \in X$ be bounded. We have $f(x \wedge y) \leq f(x) \wedge f(y)$. Take any compact k in Y such that $f(x) \wedge f(y) \in \uparrow k$. We have $f(x), f(y) \in \uparrow k$, hence $x, y \in f^{-1}(\uparrow k)$, and thus since $f^{-1}(\uparrow k) \in \Omega_{\mathrm{S}}^{\wedge}X$, we have $x \wedge y \in f^{-1}(\uparrow k)$, that is $f(x \wedge y) \geq k$ and therefore $f(x \wedge y) \geq f(x) \wedge f(y)$. ∎

Let now $\mathcal{L}$ be any S-structure.

Proposition 5 *The poset* $(\mathrm{pt}(\mathcal{L}), \subseteq)$ *is a meet-cpo.*

Proof: The proof that $\mathrm{pt}(\mathcal{L})$ is a cpo is straightforward. Let $p, q \in \mathrm{pt}(\mathcal{L})$ be two bounded points. Let $r \in \mathrm{pt}(\mathcal{L})$ be an upper bound of p and q. We prove that $p \cap q$ is a point. That it is upper closed and stable under binary meets is clear. Let $D \subseteq \mathcal{L}$ be a dd set such that $\bigvee D \in p \cap q$. Let $u \in p \cap D$ and $v \in q \cap D$. Then $u, v \in r$ and thus $u \wedge v \in r$, therefore $u \wedge v \neq 0$. Since D is dd, there exists a $w \in D$ greater than u and v. We have thus $w \in p \cap q$ and we conclude.

Furthermore, $\mathrm{pt}(\mathcal{L})$ has a bottom, namely the empty set. ∎

Proposition 6 *If* $\mathcal{L}$ *is algebraic then it is spatial and* $\mathrm{pt}(\mathcal{L})$ *is an algebraic cpo. Futhermore* $\varphi(\mathcal{L}) \simeq \Omega_{\mathrm{S}}^{\wedge}\mathrm{pt}(\mathcal{L})$ *and* $\mathrm{Kpt}(\mathcal{L}) \simeq \mathrm{Pr}\mathcal{L}^{\mathrm{op}}$.

Therefore, the functor $\mathbf{Pt} : \mathbf{Sstr} \to \mathbf{Ssp}$ *restricts to a functor* $\mathbf{ASL} \to \mathbf{AIC}$ *that we still note* $\mathbf{Pt}$.

Proof: We assume that $\mathcal{L}$ is algebraic.

We prove first that $\mathrm{pt}(\mathcal{L})$ is algebraic and that the compact elements of this cpo are the points of the shape $\uparrow \alpha$. For this we just have to prove that, for any $p \in \mathrm{pt}(\mathcal{L})$, p is the union of the set $D = \{\uparrow \alpha \mid \alpha \in p\}$ and that this set is directed. The fact that $\bigcup D \subseteq p$ is clear. Conversely, let $u \in p$, then by algebraicity $u = \bigvee \pi(u)$ and thus, since $\pi(u)$ is dd, $\pi(u) \cap p \neq \emptyset$. Let α be in this intersection, we have $u \in \uparrow \alpha \in D$. Let us prove now that D is directed. Let $\uparrow \alpha, \uparrow \beta \in D$, we have $\alpha, \beta \in p$, and thus by algebraicity again, there is a $\gamma \leq \alpha, \beta$ such that $\gamma \in p$, so we have $\uparrow \alpha, \uparrow \beta \subseteq \uparrow \gamma \in p$.

Let $U \in \Omega_{\mathrm{S}}(\mathrm{pt}(\mathcal{L}))$. We prove that the set $B_U = \{\alpha \mid \uparrow \alpha \in U\}$ is dd. Let $\alpha, \beta \in B_U$ be such that $\alpha \wedge \beta \neq 0$. Then there exists a γ such that $\gamma \leq \alpha, \beta$ (by algebraicity). Thus $\uparrow \alpha \cap \uparrow \beta \in U$. By algebraicity of $\mathrm{pt}(\mathcal{L})$, we know that $\uparrow \alpha \cap \uparrow \beta$ is the directed union of all cones $\uparrow \gamma$ such that $\gamma \geq \alpha, \beta$, thus since U is Scott open there is such a gamma satisfying $\uparrow \gamma \in U$. Therefore we can define a function $\psi : \Omega_{\mathrm{S}}(\mathrm{pt}(\mathcal{L})) \to \mathcal{L}$ by $\psi(U) = \bigvee B_U$. Clearly, ψ is monotonous. We conclude by proving that $\varphi \circ \psi = \mathrm{Id}$ and $\psi \circ \varphi = \mathrm{Id}$.

$$\begin{aligned} \psi\varphi(u) &= \bigvee\{\alpha \mid \uparrow \alpha \in \varphi(u)\} \\ &= \bigvee\{\alpha \mid u \in \uparrow \alpha\} \\ &= u \end{aligned}$$

by algebraicity of $\mathcal{L}$. On the other hand:

$$\begin{aligned} \varphi\psi(U) &= \{p \mid \psi(U) \in p\} \\ &= \{p \mid B_U \cap p \neq \emptyset\} \\ &= \{p \mid \exists \alpha \in p \ \uparrow \alpha \in U\} \\ &= U \end{aligned}$$

by algebraicity of $\mathrm{pt}(\mathcal{L})$.

The fact that $\mathbf{Pt} : \mathbf{Sstr} \to \mathbf{Ssp}$ restricts to a functor $\mathbf{ASL} \to \mathbf{AIC}$ is a consequence of proposition 4. ∎

Proposition 7 *Let* X *be an algebraic meet-cpo. Then* $(X, \Omega_{\mathrm{S}}^{\wedge}X)$ *is a sober S-space and* $\Omega_{\mathrm{S}}^{\wedge}X$ *is an algebraic S-structure. Furthermore,* $\mathrm{Pr}\Omega_{\mathrm{S}}^{\wedge}X \simeq \mathrm{K}X^{\mathrm{op}}$.

Therefore, the functor $\mathbf{\Omega}_{\mathrm{S}} : \mathbf{Ssp} \to \mathbf{Sstr}$ *restricts to a functor* $\mathbf{AIC} \to \mathbf{ASL}$ *that we still note* $\mathbf{\Omega}_{\mathrm{S}}$.

Proof: We first prove that $\Omega_{\mathrm{S}}^{\wedge}X$ is algebraic and that its dd-prime elements are the stable opens of the shape $\uparrow k$ where $k \in X$ is compact. This amounts to proving that for any $U \in \Omega_{\mathrm{S}}^{\wedge}X$, the set $P_U = \{\uparrow k \mid k \in U \cap \mathrm{K}X\}$ is dd. Let $\uparrow k, \uparrow l \in P_U$ having a non empty intersection. This means that k and l are bounded. Thus $k \wedge l \in U$, and since U is Scott open and X is algebraic, this implies that k and l have a compact lower bound m in U. Then $\uparrow k, \uparrow l \subseteq \uparrow m \in P_U$.

Now we define a map $\Psi : \mathrm{pt}(\Omega_{\mathrm{S}}^{\wedge}X) \to X$ by $\Psi(p) = \bigvee B_p$ where

$$B_p = \{k \in \mathrm{K}X \mid \uparrow k \in p\} \ .$$

Actually, let us prove that this set is directed. Let $k, l \in B_p$. Then $\uparrow k \cap \uparrow l \in p$ since p is a filter. Since $\Omega_{\mathrm{S}}^{\wedge}X$ is algebraic, $\uparrow k \cap \uparrow l$ is the dd lub of its prime lower bounds in $\Omega_{\mathrm{S}}^{\wedge}X$, which are of the shape $\uparrow m$ with $m \geq k, l$. Since p is a dd-prime filter, there is a compact $m \geq k, l$ such that $\uparrow m \in p$, that is $m \in B_p$. So Ψ is well defined by completeness of X.

We finish the proof by a simple calculation:

$$\begin{aligned} \Phi\Psi(p) &= \{U \mid \exists k \in \mathrm{K}X \ \uparrow k \in p \text{ and } k \in U\} \\ &= \{U \mid U \in p\} = p \end{aligned}$$

and

$$\begin{aligned} \Psi\Phi(x) &= \bigvee\{k \in \mathrm{K}X \mid x \in \uparrow k\} \\ &= \bigvee\{k \in \mathrm{K}X \mid x \geq k\} \\ &= x \end{aligned}$$

by algebraicity of X.

The fact that $\mathbf{\Omega}_{\mathrm{S}}$ restricts to a functor **AIC** $\to$ **ASL** results, for morphisms, from proposition 4. ∎

To summarize:

Proposition 8 *The functors* **Pt** *and* $\mathbf{\Omega}_{\mathrm{S}}$ *define an equivalence of categories between the category* **ASL** *of algebraic S-structures with S-morphisms and the dual of the category* **AIC** *of algebraic meet-cpos with stable maps.*

3 Intrinsic stable ordering

In the following, we assume $\mathcal{L}$ to be a fixed algebraic S-structure.

Definition 14 *Let $u, v \in \mathcal{L}$. We say that u is stably less than v and we write $u \sqsubseteq v$ iff $u \leq v$ and*

$$\forall \alpha \ \alpha \leq v \text{ and } u \wedge \alpha \neq 0 \Rightarrow \alpha \leq u \ .$$

Proposition 9 *i) $\sqsubseteq$ is an order included in $\leq$.*

ii) If $u, v \in \mathcal{L}$ are stably bounded then $u \wedge v$ is the glb of u and v wrt $\sqsubseteq$.

iii) If $D \subseteq \mathcal{L}$ is directed wrt $\sqsubseteq$, then $\bigvee D$ is the lub of D wrt $\sqsubseteq$.

Proof:

i) We have just to check transitivity: $u \sqsubseteq v \sqsubseteq w$ and $\alpha \leq w$ and $\alpha \wedge u \neq 0$ then $\alpha \wedge v \neq 0$ so $\alpha \leq v$ and $\alpha \wedge u \neq 0$, hence $\alpha \leq u$.

ii) Let $u, v \in \mathcal{L}$ be stably bounded; we have to show that $u \wedge v \sqsubseteq u$; so let $\alpha \leq u$ such that $\alpha \wedge u \wedge v \neq 0$; now there exists w such that $u \sqsubseteq w \sqsupseteq v$, because u, v are bounded, so we can state that $\alpha \leq w$ because $u \leq w$ and $\alpha \wedge u \neq 0$ because $\alpha \wedge u \geq \alpha \wedge u \wedge v$; by the same argument $\alpha \wedge v \neq 0$, so we conclude that $\alpha \leq u \wedge v$.

iii) Let $u \in D$; it is easily verified that $u \sqsubseteq \sup D$ iff $\forall \alpha, \forall v \in D\ \alpha \leq v, \alpha \wedge u \neq 0 \Rightarrow \alpha \leq u$; but for $w \in D, u, v \sqsubseteq w$ we can deduce that if $\alpha \leq v, u \wedge \alpha \neq 0$ then $\alpha \leq w \sqsupseteq u$ hence $\alpha \leq u$, which proves that $u \sqsubseteq \bigvee D$.
Let suppose now that exists a w which is another stable upper bound for D and $w \sqsubseteq \bigvee D$; then we can conclude that $w = \bigvee D$ because $\sqsubseteq$ is included in $\leq$.

∎

Proposition 10 *Let $a, b, u \in \mathcal{L}$.*

i) If $a \sqsubseteq b$ then $a = b$.

ii) If $a, b \sqsubseteq u$ then either $a = b$ or $a \wedge b = 0$.

Proof:

i) If $\alpha \leq b$ then we can find a $\gamma \leq a$ such that $\alpha \leq \gamma$; this γ exists because by lemma 6 $\pi(a), \pi(b)$ are directed set and $\pi(a) \subseteq \pi(b)$; so for $\beta \leq a$ there exists γ, such that $\alpha, \beta \geq \gamma \in \pi(b)$; it follows that the set $H = \{\gamma \leq b \mid \gamma \wedge a \neq 0\}$ is a directed set included in $\pi(b)$ and moreover $b = \bigvee H$; but $a \sqsubseteq b$ implies that $H \subseteq \pi(a)$ so $b = \bigvee H \leq \bigvee \pi(a) = a$.

ii) Suppose $a \wedge b \neq 0$; then $a \wedge b \sqsubseteq a$; let $\alpha \leq a$, then there exists $\gamma \leq a$ such that $\alpha \leq \gamma$ and $a \wedge b \wedge \gamma \neq 0$ (because $\pi(a)$ is directed); now $a \wedge b \sqsubseteq a$ implies $\gamma \leq a \wedge b$; it follows that $H = \{\gamma \leq a \mid a \wedge b \wedge \alpha \neq 0\}$ is directed and $a = \bigvee H = a \wedge b$; the same holds for b giving the desired equality $a = a \wedge b = b$.

∎

Definition 15 *Let $u \in \mathcal{L}$. We call* trace *of u and we note* $\mathrm{tr}(u)$ *the set of connected stable lower bounds of u.*

Remark: A connected stable lower bound of u is just a connected maximal lower bound of u so in the previous definition the reference to definition of stable ordering is not necessary.

Proposition 11 *i) $u \sqsubseteq v$ iff $\mathrm{tr}(u) \subseteq \mathrm{tr}(v)$.*

ii) $u \leq v$ iff for all $a \in \mathrm{tr}(u)$ there exists a $b \in \mathrm{tr}(v)$ such that $a \leq b$ (and then this b is unique).

Proof:

i) If $u \sqsubseteq v$ and $a \in \mathrm{tr}(u)$ then $a \sqsubseteq u \sqsubseteq v$ so $a \sqsubseteq v$ and hence $a \in \mathrm{tr}(v)$. Conversely let $\mathrm{tr}(u) \subseteq \mathrm{tr}(v)$, let $\alpha \leq v$ be such that $\alpha \wedge u \neq 0$. Then there exists an unique $a \in \mathrm{tr}(v)$ such that $\alpha \leq a$ (because α is connected); now $\alpha \wedge u \neq 0$ implies $a \in \mathrm{tr}(u)$ (by disjointness of traces) so we get $\alpha \leq a$ which proves $u \sqsubseteq v$.

ii) If $u \leq v$ and $a \in \mathrm{tr}(u)$ then $a \leq v = \dot{\bigvee} \mathrm{tr}(v)$ so there exists an unique $b \in \mathrm{tr}(v)$ which is greater than a. The converse follows from monotonicity of sup.

■

Proposition 12 *The cpo $(\mathcal{L}, \sqsubseteq)$ is isomorphic to the coherence space $\mathcal{C}(\mathcal{L})$ the web of which is the set of connected elements of $\mathcal{L}$ endowed with the coherence relation defined by: $a \frown b$ iff $a \wedge b = 0$. This isomorphism $\Theta : \mathcal{L} \to \mathcal{C}(\mathcal{L})$ is given by: $\Theta(u) = \mathrm{tr}(u)$ and $\Theta^{-1}(A) = \bigvee A$.*

Proof: We have already shown that $u = \dot{\bigvee}_{i \in I} (\bigvee \Delta_i)$ where for each i the set Δ_i is directed and $\bigcup_{i \in I}(\Delta_i) = \pi(u)$, so $u = \dot{\bigvee}_{i \in I} a_i$ where $\{a_i \mid i \in I\}$ is the set of (disjont) connected elements of $\mathcal{L}$ such that $\bigvee \Delta_i = a_i \sqsubseteq u$ for any i; this shows that Θ is a bijection. The fact that is an isomorphism follows from proposistion 11.i) . ■

Corollary 2 *The cpo $(\mathcal{L}, \sqsubseteq)$ is algebraic. But in general, the cardinality of its basis is strictly larger than the cardinality of the basis of $(\mathcal{L}, \leq)$.*

Proposition 13 *The evaluation map $\mathrm{Ev} : \mathcal{C}(\mathcal{L}) \times \mathrm{pt}(\mathcal{L}) \to \mathbf{O}$ defined by $\mathrm{Ev}(u, p) = \top$ iff $u \in p$ is stable.*

Proof: Let $u, v \sqsubseteq w$ and $p, p' \subseteq q$; we have to show that $\mathrm{Ev}(u \wedge v, p \cap p') = \mathrm{Ev}(u, p) \wedge \mathrm{Ev}(v, p')$, i.e. $u \in p$ and $v \in p' \Rightarrow u \wedge v \in p \cap p'$ the other implication being trivial; now $u \in p$ and $v \in p'$ iff there exists $a \sqsubseteq u$, $b \sqsubseteq v$ such that $a \in p$, $b \in p'$ (because p, p' are dd-prime filters), and $u, v \sqsubseteq w$ forces $a = b$ or $a \wedge b = 0$; but $a \wedge b = 0$ implies $0 \in q$ contradicting the fact that q is a dd-prime filter; hence $u \wedge v \in p \cap p'$. ■

4 Function spaces

In this section, we assume that $\mathcal{L}$ and $\mathcal{M}$ are two algebraic S-structures.

4.1 Stable ordering of S-morphism

In this paragraph we introduce the notions of stable ordering between S-morphism and trace of an S-morphism and then we show that these definitions are the natural ones, that is stable ordering between S-morphism corresponds (by duality) to the Berry' order on stable maps.

Definition 16 *Let $f, g : \mathcal{L} \to \mathcal{M}$ be two S-morphisms.*

- *We say that f is stably less than g and we write $f \sqsubseteq g$ iff for all $\alpha \in \mathcal{L}$ we have $f(\alpha) \sqsubseteq g(\alpha)$.*
- *The trace of f is the set $\mathrm{tr}(f) = \{(\alpha, b) \in \mathrm{Pr}\mathcal{L} \times \mathrm{Conn}(\mathcal{M}) \mid b \sqsubseteq f(\alpha)\}$.*

Proposition 14 *For any α, $f(\alpha) \sqsubseteq g(\alpha)$ iff for any u, $f(u) \sqsubseteq g(u)$.*

Proof: We have just to prove the "left to right" implication. Let $\alpha \leq f(u)$ and $\alpha \wedge g(u) \neq 0$. Then there exist $\beta, \beta' \leq u$ such that $\alpha \leq f(\beta)$ and $\alpha \wedge g(\beta') \neq 0$ (because f and g preserve the dd lubs). It follows that $\beta \wedge \beta' \neq 0$ (because otherwise $0 = g(\beta \wedge \beta') = g(\beta) \wedge g(\beta') \geq \alpha \wedge g(\beta)$) and therefore, by using the fact that $\pi(u)$ is dd, there exists $\gamma \leq u$, such that $\beta, \beta' \leq \gamma$, $\alpha \leq f(\gamma)$, $\alpha \wedge g(\gamma) \neq 0$, so we conclude that $\alpha \leq g(u)$. ■

Remember that we have already defined a notion of stable ordering between stable maps. The following connects the two notions.

Proposition 15 *Let $f, g : \mathcal{L} \to \mathcal{M}$ be two S-morphisms. Then $f \sqsubseteq g$ iff $f_* \sqsubseteq g_*$.*

Proof: Suppose $f_* \sqsubseteq g_*$; this means that $\forall x' \geq x(f_*(x) = g_*(x) \wedge f_*(x'))$; we have to show that for any prime $\beta \in \mathcal{L} \simeq \Omega_S(\mathrm{pt}(\mathcal{L}))$ ${f_*}^{-1}(\beta) \sqsubseteq {g_*}^{-1}(\beta)$, i.e. $\forall \alpha(\alpha \subseteq {g_*}^{-1}(\beta)$ and $\alpha \cap {f_*}^{-1}(\beta) \neq \emptyset \Rightarrow \alpha \subseteq {f_*}^{-1}(\beta))$ which is the same (by using the isomorphism between $\mathrm{Pr}\mathcal{L}$ and $\mathrm{Kpt}(\mathcal{L})^{\mathrm{op}}$) that for $y \in \mathrm{Kpt}(\mathcal{M})$ and for any $x \in \mathrm{Kpt}(\mathcal{M})$ $(\uparrow x \subseteq {g_*}^{-1}(\uparrow y)$ and $\uparrow x \cap {f_*}^{-1}(\uparrow y) \neq \emptyset \Rightarrow \uparrow x \subseteq {f_*}^{-1}(\uparrow y))$ which is true whenever, if $g_*(x) \geq y$ and there exists $z \geq x$ such that $f_*(z) \geq y$ then $f_*(x) \geq y$. Now, by hypothesis, we have $f_*(x) = g_*(x) \wedge f_*(z)$ hence the last condition is true.

By definition $f_*(p) = f^{-1}(p)$, so let $p \subseteq p'$ and what we have to show is that

$$f^{-1}(p) = g^{-1}(p) \cap f^{-1}(p')$$

which is true iff for any u $f(u) \in p$ iff $g(u) \in p$ and $f(u) \in p'$. This last equivalence is true because if $f(u) \in p$ then $g(u) \in p$ (because $f(u) \leq g(u)$) and $f(u) \in p'$ (because $p \subseteq p'$). On the other side, let $f(u) \in p'$ and $g(u) \in p$; then there exist $\alpha \leq f(u), \alpha \in p'$ and $\alpha' \leq g(u), \alpha' \in p$; now $\alpha' \leq g(u)$ and $\alpha' \wedge f(u) \neq 0$ because $\alpha \wedge \alpha' \in p'$ so $\alpha' \leq f(u)$ which allows to us to conclude that $f(u) \in p$. ∎

Proposition 16 *For any $u \in \mathcal{L}$ we have $f(u) = \bigvee\{b \mid \exists \alpha \leq u\ (\alpha, b) \in \mathrm{tr}(f)\}$ and $f \sqsubseteq g$ iff $\mathrm{tr}(f) \subseteq \mathrm{tr}(g)$.*

Proof: In fact $f(\alpha) = \dot{\bigvee} \{b \mid b \sqsubseteq f(\alpha)\}$ so

$$\begin{aligned} f(u) &= f(\bigvee \pi(u)) \\ &= \bigvee f(\pi(u)) \\ &= \bigvee\{b \mid \exists \alpha \leq u\ (\alpha, b) \in \mathrm{tr}(f)\} \end{aligned}$$

and

$$\begin{aligned} f \sqsubseteq g \quad & \text{iff} \quad \forall \alpha\ f(\alpha) \sqsubseteq g(\alpha) \\ & \text{iff} \quad \forall \alpha\ \mathrm{tr}(f(\alpha)) \subseteq \mathrm{tr}(g(\alpha)) \\ & \text{iff} \quad \mathrm{tr}(f) \subseteq \mathrm{tr}(g) \end{aligned}$$

∎

4.2 Space of function's traces

In the following we define the set of weakly coherent S-structure. For $\mathcal{L}$, $\mathcal{M}$ in this class we can represent the set of S-morphism from $\mathcal{L}$ to $\mathcal{M}$ as a subset of $\mathcal{P}(\mathrm{Pr}\mathcal{L} \times \mathrm{Conn}(\mathcal{M}))$ which we call the set of trace on $\mathcal{L}$, $\mathcal{M}$. It is interesting to remark that, as far we know, when we consider more general domains than dI domains, there is no simple definition of trace of stable map, since there is no more notion of minimal element. Now the definition of connected element idealizes, always by duality, the notion of minimal element, and so allows us to speak about traces in the usual sense. This representation property will be used for proving the cartesian closedness of the category of L-domains with stable maps which is the dual of the category of weakly coherent S-structure with S-morphism.

Definition 17 *An S-structure $\mathcal{L}$ is called* weakly coherent *if for any α, β we have $\mathrm{tr}(\alpha \wedge \beta) \subseteq \mathrm{Pr}\mathcal{L}$.*

Remark: By duality a weakly coherent S-structure is an algebraic meet-cpo such that any finite set A of bounded compact points has a set B of compact upper bounds which is complete, i.e. any two elements of this set are unbounded, and any upper bound of A is greater than an element of B. This is equivalent to say that the set of lower bound of any point is a lattice, and that is the definition of an L-domain.

Proposition 17 *We assume that $\mathcal{L}$ is weakly coherent. Let T be a subset of* $\mathrm{Pr}\mathcal{L} \times \mathrm{Conn}(\mathcal{M})$*. T is the trace of an S-morphism $\mathcal{L} \to \mathcal{M}$ iff it satisfies the three following conditions:*

i) For any α, the set $T_\alpha = \{b \mid (\alpha, b) \in T\}$ is disjoint.

ii) For any $(\alpha, b) \in T$, for any $\alpha' \geq \alpha$, there exists a b' such that $(\alpha', b') \in T$.

iii) For any $(\alpha, b), (\alpha', b') \in T$, for any $c \sqsubseteq b \wedge b'$, there is a $\beta \sqsubseteq \alpha \wedge \alpha'$ such that $(\beta, c) \in T$.

Furthermore, when these conditions hold, the b' of ii) is unique, and the β of iii) too.

Any set satisfying these conditions will be called a trace.

Proof: Let $f : \mathcal{L} \to \mathcal{M}$ be an S-morphism. Let us prove that $T = \mathrm{tr}(f)$ satisfies conditions i), ii) and iii).

i) It simply results from the fact that $T_\alpha = \mathrm{tr}(f(\alpha))$.

ii) It results from proposition 11 ii).

iii) Let $(\alpha, b), (\alpha', b') \in T$. We have

$$\begin{aligned} f(\alpha \wedge \alpha') &= \dot{\bigvee} \{f(\beta) \mid \beta \sqsubseteq \alpha \wedge \alpha'\} \\ &= \dot{\bigvee} \{c \mid \exists \beta \sqsubseteq \alpha \wedge \alpha' \ (\beta, c) \in T\} \end{aligned}$$

and

$$\begin{aligned} f(\alpha) \wedge (\alpha') &= \dot{\bigvee} \{b \wedge b' \mid (\alpha, b), (\alpha', b') \in T\} \\ &= \dot{\bigvee} \{c \mid \exists (\alpha, b), (\alpha', b') \in T \ c \sqsubseteq b \wedge b'\} \ . \end{aligned}$$

Since $f(\alpha) \wedge (\alpha') \leq f(\alpha \wedge \alpha')$ and since the lubs written above are both disjoint lubs of connected elements, we conclude that *iii*) holds for T.

Let now $T \subseteq \mathrm{Pr}\mathcal{L} \times \mathrm{Conn}(\mathcal{M})$ be a set satisfying conditions i), ii) and iii). Remark first that, for any $u \in \mathcal{L}$, the set $F_T(u) = \{b \mid \exists \alpha \leq u \ (\alpha, b) \in T\}$ is dd, so that by setting $f(u) = \bigvee F_T(u)$ we define a function from $\mathcal{L}$ to $\mathcal{M}$. Actually, let $b, b' \in F_T(u)$ be such that $b \wedge b' \neq 0$. Let $\alpha, \alpha' \leq u$ be such that $(\alpha, b), (\alpha', b') \in T$. Then $\alpha \wedge \alpha' \neq 0$ because of condition iii). Since $\pi(u)$ is dd, there exists $\beta \geq \alpha, \alpha'$ such that $\beta \leq u$. By condition ii), we can find $c \geq b$ and $c' \geq b'$ such that $(\beta, c), (\beta, c')$. But $c \wedge c' \geq b \wedge b' \neq 0$, so by condition i), $c = c'$. So we have found in $F_T(u)$ a c such that $c \geq b, b'$, and thus $F_T(u)$ is dd.

It is obvious that $f(0) = 0$ and that $f(1) = 1$. Let us prove that f preserves binary glbs. Let $u, v \in \mathcal{L}$. On one hand,

$$f(u \wedge v) = \bigvee \{b \mid \exists \alpha \leq u \wedge v \ (\alpha, b) \in T\} \ .$$

On the other hand,

$$f(u) \wedge f(v) = \bigvee\{b \wedge b' \mid \exists \alpha \leq u, \alpha' \leq v\ (\alpha, b), (\alpha', b') \in T\}$$

and we have to prove that $f(u) \wedge f(v) \leq f(u \wedge v)$ since f is clearly monotonous, by condition ii). So let $(\alpha, b), (\alpha', b') \in T$ be such that $\alpha \leq u$ and $\alpha' \leq v$. Let c be such that $c \sqsubseteq b \wedge b'$. By property iii), there exists a $\beta \sqsubseteq \alpha \wedge \alpha'$ such that $(\beta, c) \in T$. Since $\beta \leq \alpha \wedge \alpha' \leq u \wedge v$, we have $c \leq f(u \wedge v)$, and we conclude.

Last, let $D \subseteq \mathcal{L}$ be dd. We have

$$\begin{aligned} f(\bigvee D) &= \bigvee\{b \mid \exists \alpha \leq \bigvee D\ (\alpha, b) \in T\} \\ &= \bigvee\{b \mid \exists \alpha\ \exists u \in D\ \alpha \leq u \text{ and } (\alpha, b) \in T\} \\ &= \bigvee_{u \in D} f(u) \end{aligned}$$

hence f preserves dd lubs.

To conclude, let us check the second unicity statement, for any T satisfying the three conditions (the first one is easy too and results from i)). Let $(\alpha, b), (\alpha', b') \in T$ and let $c \sqsubseteq b \wedge b'$. Let $\beta, \beta' \sqsubseteq \alpha \wedge \alpha'$ be such that $(\beta, c), (\beta', c) \in T$. There must exist (by iii)) a $\gamma \leq \beta, \beta'$ such that $(\gamma, c) \in T$, so $\beta \wedge \beta' \neq 0$, so $\beta = \beta'$ since they both are in $\mathrm{tr}(\alpha \wedge \alpha')$. ∎

Proposition 18 *Let T be a trace. Let E be any subset of T. Then there exists a trace $[E]_T$ such that $E \subseteq [E]_T \subseteq T$ and which is minimal among the traces having this property.*

Proof: We define $[E]_T = \bigcap A$ where $A = \{T' \mid E \subseteq T' \subseteq T$ and T' is a trace$\}$. We have to show that $[E]_T$ is a trace. Condition *i)* holds for any element of A, so it holds for their intersection. If $(\alpha, b) \in T'$ for any $T' \in A$ then for any $\alpha' \geq \alpha$, there exists a b' such that $(\alpha', b') \in T'$ and since this b' is unique in T, it must be the same for any T', hence condition *ii)* holds. Finally, by using the same unicity argument, condition *iii)* is verified. ∎

As a corollary, we retrieve an already known result:

Proposition 19 *The poset of all traces ordered by inclusion is an L-domain.*

Proof: Checking that the union of a directed family of traces is a trace is straightforward.

Let T be a fixed trace. If S, S' are two traces included in T then by proposition18 $S \cap S'$ is a trace and so it is their glb

We can now prove algebraicity. It simply results from the fact that, for any finite subset E of T, the trace $[E]_T$ is compact. Actually, let $\mathcal{D}$ be a directed family of traces such that $[E]_T \subseteq \bigcup \mathcal{D}$. Since E is finite, there exists a trace $S \in \mathcal{D}$ such that $E \subseteq S$. Now $[E]_T, S \subseteq \bigcup \mathcal{D}$ so $[E]_T \cap S$ is a subtrace of T containing E; hence $[E]_T \subseteq [E]_T \cap S \subseteq S$.

We finally show that the set of lower bounds of a point is a lattice. It suffices to show the existence of finite lubs in $\downarrow(T)$ where T is a trace. The lub of S, S' is $S'' = [S \cup S']_T$; by definition, S'' is a trace greater than S and S' and less than T. Now S'' is the lub of S and S' in $\downarrow(T)$, because of proposition 18.

∎

From this result, it is easy to prove that the category of L-domains and stable functions is cartesian closed. Indeed what remains to be proved is that evaluation and abstraction are stable and this a routine exercise, since stable ordering has been choosen in order to make evaluation stable.

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