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Customizable triangular factorizations of matrices

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Abstract

Customizable triangular factorizations of matrices find their applications in computer graphics and lossless transform coding. In this paper, we prove that any $N \times N$ nonsingular matrix $\boldsymbol{A}$ can be factorized into 3 triangular matrices, $\boldsymbol{A} = \boldsymbol{PLUS}$, where $\boldsymbol{P}$ is a permutation matrix, $\boldsymbol{L}$ is a unit lower triangular matrix, $\boldsymbol{U}$ is an upper triangular matrix of which the diagonal entries are customizable and can be given by all means as long as its determinant is equal to that of $\boldsymbol{A}$ up to a possible sign adjustment, and $\boldsymbol{S}$ is a unit lower triangular matrix of which all but $N-1$ off-diagonal elements are set zeros and the positions of those $N-1$ elements are also flexibly customizable, such as a single-row, a single-column, a bidiagonal matrix or other specially patterned matrices. A pseudo-permutation matrix, which is a simple unit upper triangular matrix with off-diagonal elements being 0, 1 or -1, can take the role of the permutation matrix $\boldsymbol{P}$ as well. In some cases, $\boldsymbol{P}$ may be the identity matrix. Besides PLUS, a customizable factorization also has other alternatives, LUSP, PSUL or SULP for lower $\boldsymbol{S}$, and PULS, ULSP, PSLU, SLUP for upper $\boldsymbol{S}$.

Keywords: Triangular matrix; Triangular factorization; Reversible integer transform; Rotation by shears

1. Introduction

As a classical method in numerical linear algebra, matrix factorization generally serves the purpose of restating some given problem in such a way that can be solved more readily [6]. LU triangular factorization is a mechanism for characterizing what

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occurs in Gaussian elimination for computationally convenience of obtaining a solution to the original linear system [4]. Vaserstein and Wheland [11] proved the theorem that every invertible matrix $\boldsymbol{A}$ over any ring with Bass stable rank 1 has an LUL factorization, $\boldsymbol{A} = \boldsymbol{LUM}$, where $\boldsymbol{L}$ and $\boldsymbol{M}$ are lower triangular and $\boldsymbol{U}$ is upper triangular. In [7], Nagarajan et al. proved the LUL factorization for any square matrix $\boldsymbol{A}$ over a field, and all the diagonal entries in $\boldsymbol{L}$ and $\boldsymbol{U}$ are equal to 1.

In the applications of computer graphics, unit triangular factorizations, such as ULU [10] and LULU [8], are used to convert a rotation into a few one-dimensional shears, which can be carried out very efficiently by "string copy with offset" at the memory controller level. However, single-row matrices and single-column matrices (beam shears and slice shears in [2]) are favorable choices. Therefore, we wish the factor matrices to be customizable so as to implement with hardware most efficiently.

In order to implement linear transforms by reversible integer mapping in the applications of lossless source coding by means of transformation, Hao and Shi [5] found some elementary reversible matrices (ERM) for integer transformation, in which all the diagonal entries are customizable as 1, -1, i or $-$i, and they suggested a PLUS factorization for any matrix $\boldsymbol{A}$ with $\det \boldsymbol{A} = \pm 1$ or $\det \boldsymbol{A} = \pm \mathrm{i}$, $\boldsymbol{A} = \boldsymbol{PLUS}$, where $\boldsymbol{P}$ is a permutation matrix, $\boldsymbol{L}$, $\boldsymbol{U}$ and $\boldsymbol{S}$ are triangular ERMs (TERM), $\boldsymbol{L}$ is a lower TERM, $\boldsymbol{U}$ is an upper TERM, and $\boldsymbol{S}$ is also a single-row ERM (SERM). Their technique is practised with a ladder structure, which was proposed by Bruekers and van den Enden in 1992 [1], and widely accepted as the lifting scheme in wavelet theory [3,9]. Different ERMs have different properties. For instance, single-row ERMs make less rounding error, single-column ERMs are efficient for parallel computing, and a bidiagonal ERM transformation can be implemented with a pipeline structure. In specific applications, we wish the factorization to be customizable for specific customized purposes.

In this paper, we propose a new matrix factorization framework and generalize the PLUS factorization to all nonsingular matrices and make customizable the diagonal entries of the upper triangular matrix $\boldsymbol{U}$ and the positions of some off-diagonal elements in the special unit lower triangular matrix $\boldsymbol{S}$.

2. Notations and definitions

We use a lowercase letter to stand for a scalar number. A bold lowercase letter is for a vector and the bracketed indices are used for its elements. Bold uppercase letters represent matrices, and their elements are also bracket indexed. For example, the kth element of vector $\boldsymbol{a}$ is $\boldsymbol{a}(k)$, and the mth row and nth column element of matrix $\boldsymbol{A}$ is $\boldsymbol{A}(m, n)$.

$\boldsymbol{I}$ stands for the identity matrix, whose dimensions are conformable with other matrices in a formula, and $\boldsymbol{e}_k$ is for the kth standard basis vector formed as the kth column of the identity matrix. $\boldsymbol{E}_{m,n}$ denotes a matrix that all the elements are zeros except the entry-(m, n) being one.

We also use a vector notation for a matrix, such as for an $N \times N$ matrix $\boldsymbol{A}$, $\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_N]$. The kth leading principal submatrices of matrix $\boldsymbol{A}$ are denoted by $\boldsymbol{A}_k$, and $\boldsymbol{A}_k = [\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_N]_k = [\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_k]_k$ for $k = 1, 2, 3, \dots, N$.

A *customized matrix* $\boldsymbol{S}$ of size $N \times N$ is a special unit triangular matrix that the positions of $N - 1$ off-diagonal elements are specially patterned as required and all other off-diagonal elements are set zeros. In the process of factorization, those $N - 1$ elements can also be considered as $N - 1$ unknowns in the customized positions. A customized matrix is also called a special matrix in this paper. Scalar numbers $s_1, s_2, \dots, s_{N-1}$ are used to identify those $N - 1$ elements in the special matrix $\boldsymbol{S}$. For descriptive convenience, we also use $\boldsymbol{s}_k = [s_1, s_2, \dots, s_{N-1}, s_N]^{\mathrm{T}}$ for a vector of those numbers whose kth element is 0 ($s_k = 0$) and the other $N - 1$ elements are those scalar numbers but not subscripted in that order.

A *customization matrix* $\boldsymbol{B}$ is a triangular Boolean matrix that indicates the customized positions of the special matrix $\boldsymbol{S}$, in which an element is set zero, $\boldsymbol{B}(m, n) = 0$, if the corresponding element in the same position of the specially customized matrix is fixed as zero, $\boldsymbol{S}(m, n) = 0$, and otherwise we set $\boldsymbol{B}(m, n) = 1$.

A *customized diagonal* of a triangular matrix is that all the entries on the main diagonal of the matrix are customized by all means as long as its determinant is equal to that of the given matrix up to a possible sign adjustment. The customized diagonal entries of upper triangular matrix $\boldsymbol{U}$ are denoted as $\operatorname{diag}(\boldsymbol{U}) = [d_1, d_2, \dots, d_N]^{\mathrm{T}}$.

A *pseudo-permutation* matrix is a unit triangular matrix that all the off-diagonal elements are 0, 1 or -1, which plays the role of the permutation matrix to transform some submatrices into necessarily nonsingular ones as what a permutation matrix does in factorization.

A *customized triangular factorization* of a matrix is a factorization of a possible permutation or pseudo-permutation matrix and three triangular matrices, among which one triangular matrix is unit, one has a customized diagonal and the other is a customized matrix corresponding to a customization matrix. For instance, $\boldsymbol{A} = \boldsymbol{PLUS}$, where $\boldsymbol{P}$ is a permutation or an upper pseudo-permutation matrix, $\boldsymbol{L}$ is a unit lower triangular matrix, $\boldsymbol{U}$ is an upper triangular matrix with a customized diagonal, and $\boldsymbol{S}$ is a customized special matrix. Although there are other alternatives, we also refer to a customizable triangular factorization as a PLUS factorization in this paper.

3. Some lemmas

Lemma 1. *For an $N \times N$ nonsingular matrix $\boldsymbol{A}$ and its factorization of $\boldsymbol{A} = \boldsymbol{LU}$, where $\boldsymbol{L}$ is unit lower triangular and $\boldsymbol{U}$ is upper triangular with diagonal entries $d_1, d_2, \dots, d_N$, the kth leading principal minors $\det \boldsymbol{A}_k = \det \boldsymbol{U}_k = d_1 d_2 \cdots d_k$, where $k = 1, 2, 3, \dots, N$.*

It can be easily proved, since $\boldsymbol{L}$ does not change the leading principal minors of $\boldsymbol{U}$.

Lemma 2. *For $1 \leqslant m < n \leqslant N$, the leading principal minors of matrix $\boldsymbol{A}$ with a simple linear transform are*

$$\det[\boldsymbol{a}_1, \dots, \boldsymbol{a}_{m-1}, \boldsymbol{a}_m + x\boldsymbol{a}_n, \boldsymbol{a}_{m+1}, \dots, \boldsymbol{a}_n, \dots, \boldsymbol{a}_N]_k$$
$$= \begin{cases} \det \boldsymbol{A}_k & (1 \leqslant k < m) \\ \det \boldsymbol{A}_k + x \det[\boldsymbol{a}_1, \dots, \boldsymbol{a}_{m-1}, \boldsymbol{a}_n, \boldsymbol{a}_{m+1}, \dots, \boldsymbol{a}_k]_k & (m \leqslant k < n) \\ \det \boldsymbol{A}_k & (n \leqslant k \leqslant N) \end{cases}.$$

Its proof is straightforward by the properties of matrix determinant.

Lemma 3. *For an $N \times N$ nonsingular matrix $\boldsymbol{A}$, there exists a permutation matrix $\boldsymbol{P}$ such that all the leading principal submatrices of $\boldsymbol{PA}$ are nonsingular.*

Its proof can be found in many books of matrix theory or linear algebra. A proof can also be provided similarly to the following proof of Lemma 4. Note that the inverse of a permutation matrix is its transpose.

Lemma 4. *For an $N \times N$ nonsingular matrix $\boldsymbol{A}$, there exists an upper pseudo-permutation matrix $\boldsymbol{P}$ such that all the leading principal submatrices of $\boldsymbol{PA}$ or $\boldsymbol{P}^{-1}\boldsymbol{A}$ are nonsingular.*

Proof. We prove it with the method of triangular factorization.

Suppose

$$\boldsymbol{A} = \begin{bmatrix} a_{1,1}^{(1)} & a_{1,2}^{(1)} & \cdots & a_{1,N}^{(1)} \\ a_{2,1}^{(1)} & a_{2,2}^{(1)} & \cdots & a_{2,N}^{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ a_{N,1}^{(1)} & a_{N,2}^{(1)} & \cdots & a_{N,N}^{(1)} \end{bmatrix}.$$

First, if $a_{1,1}^{(1)} \neq 0$, we have $\boldsymbol{A}_1 = a_{1,1}^{(1)} \neq 0$. If $a_{1,1}^{(1)} = 0$, there must exist a nonzero among $a_{2,1}^{(1)}$ through $a_{N,1}^{(1)}$ (If all of them are zeros, matrix $\boldsymbol{A}$ should be singular.), say $a_{m,1}^{(1)}$, then we let $\boldsymbol{P}_1 = \boldsymbol{I} + \boldsymbol{E}_{1,m}$ for row operation, such that:

$$\boldsymbol{P}_1\boldsymbol{A} = \begin{bmatrix} p_{1,1}^{(1)} & p_{1,2}^{(1)} & \cdots & p_{1,N}^{(1)} \\ p_{2,1}^{(1)} & p_{2,2}^{(1)} & \cdots & p_{2,N}^{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ p_{N,1}^{(1)} & p_{N,2}^{(1)} & \cdots & p_{N,N}^{(1)} \end{bmatrix},$$

where $(\boldsymbol{P}_1\boldsymbol{A})_1 = p_{1,1}^{(1)} = a_{m,1}^{(1)} \neq 0$, $p_{1,n}^{(1)} = a_{1,n}^{(1)} + a_{m,n}^{(1)}$ for $n = 1, 2, 3, \dots, N$.

Then, the forward elimination of the first column can be achieved by multiplying an elementary Gauss matrix $\boldsymbol{L}_1$:

$$\boldsymbol{L}_1\boldsymbol{P}_1\boldsymbol{A} = \begin{bmatrix} 1 & & \\ -p_{2,1}^{(1)}/p_{1,1}^{(1)} & 1 & \\ \cdots & & \boldsymbol{I} \\ -p_{N,1}^{(1)}/p_{1,1}^{(1)} & & & 1 \end{bmatrix} \boldsymbol{P}_1\boldsymbol{A} = \begin{bmatrix} a_{1,1}^{(2)} & a_{1,2}^{(2)} & \cdots & a_{1,N}^{(2)} \\ 0 & a_{2,2}^{(2)} & \cdots & a_{2,N}^{(2)} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & a_{N,2}^{(2)} & \cdots & a_{N,N}^{(2)} \end{bmatrix},$$

where $a_{1,1}^{(2)} = p_{1,1}^{(1)} \neq 0$.

The process can be continued recursively to obtain the complete factorization. For $k = 2, 3, \ldots, N-1$, if $a_{k,k}^{(k)} \neq 0$, we set $\boldsymbol{P}_k = \boldsymbol{I}$, otherwise, we let $\boldsymbol{P}_k = \boldsymbol{I} + \boldsymbol{E}_{k,k_m}$ $(k < k_m)$ denote an upper pseudo-permutation matrix for the row operations between the kth row and the k_mth row, another row among the $(k+1)$th through the Nth rows, so as to guarantee that the entry-(k, k) in the matrix is not zero, $p_{k,k}^{(k)} = a_{k_m,k}^{(k)} \neq 0$ (If there was no such an element, $\boldsymbol{A}$ should have been singular.), and $\boldsymbol{L}_k$ record the row multipliers used for the Gaussian elimination of column k. Then, we get,

$$\boldsymbol{L}_{N-1}\boldsymbol{P}_{N-1}\cdots\boldsymbol{L}_2\boldsymbol{P}_2\boldsymbol{L}_1\boldsymbol{P}_1\boldsymbol{A} = \begin{bmatrix} a_{1,1}^{(N)} & a_{1,2}^{(N)} & \cdots & a_{1,N}^{(N)} \\ 0 & a_{2,2}^{(N)} & \cdots & a_{2,N}^{(N)} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{N,N}^{(N)} \end{bmatrix} = \boldsymbol{U}.$$

Let $\boldsymbol{P}_{(k+m)}$ denote the lower-right $(N-k)\times(N-k)$ submatrix of $\boldsymbol{P}_{k+m}$. For $m > 0$, we have

$$\begin{aligned} \boldsymbol{P}_{k+m}\boldsymbol{L}_k\boldsymbol{P}_{k+m}^{-1} &= \begin{bmatrix} \boldsymbol{I}_{k-1} & & \\ & 1 & \\ & & \boldsymbol{P}_{(k+m)} \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_{k-1} & & \\ & 1 & \\ & \boldsymbol{l}_k & \boldsymbol{I}_{N-k} \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_{k-1} & & \\ & 1 & \\ & & \boldsymbol{P}_{(k+m)}^{-1} \end{bmatrix}, \\ &= \begin{bmatrix} \boldsymbol{I}_{k-1} & & \\ & 1 & \\ & \boldsymbol{P}_{(k+m)}\boldsymbol{l}_k & \boldsymbol{I}_{N-k} \end{bmatrix}, \\ &= \boldsymbol{L}'_k, \end{aligned}$$

which is still an elementary Gauss matrix.

For $m > 0$, we also have

$$\boldsymbol{P}_{k+m}\boldsymbol{P}_k = (\boldsymbol{I} + \boldsymbol{E}_{k+m,k_{k+m}}) \cdot (\boldsymbol{I} + \boldsymbol{E}_{k,k_m}) = \boldsymbol{I} + \boldsymbol{E}_{k,k_m} + \boldsymbol{E}_{k+m,k_{k+m}},$$

which is an upper pseudo-permutation matrix.

Then, we have

$$\begin{aligned} &\boldsymbol{L}_{N-1}\boldsymbol{P}_{N-1}\cdots\boldsymbol{L}_2\boldsymbol{P}_2\boldsymbol{L}_1\boldsymbol{P}_1 \\ &\quad = \boldsymbol{L}_{N-1}(\boldsymbol{P}_{N-1}\boldsymbol{L}_{N-2}\boldsymbol{P}_{N-1}^{-1})\cdots(\boldsymbol{P}_{N-1}\cdots\boldsymbol{P}_2\boldsymbol{L}_1\boldsymbol{P}_2^{-1}\cdots\boldsymbol{P}_{N-1}^{-1}) \\ &\qquad \times (\boldsymbol{P}_{N-1}\cdots\boldsymbol{P}_2\boldsymbol{P}_1) \\ &\quad = (\boldsymbol{L}'_{N-1}\boldsymbol{L}'_{N-2}\cdots\boldsymbol{L}'_1)\cdot(\boldsymbol{P}_{N-1}\cdots\boldsymbol{P}_2\boldsymbol{P}_1). \end{aligned}$$

The product of all the elementary Gauss matrices makes a unit lower triangular matrix, and its inverse is unit lower triangular as well, $\boldsymbol{L} = (\boldsymbol{L}'_{N-1}\boldsymbol{L}'_{N-2}\cdots\boldsymbol{L}'_1)^{-1}$. On the other hand, the product of all the pseudo-permutation matrices makes a pseudo-permutation matrix, $\boldsymbol{P} = \boldsymbol{P}_{N-1}\cdots\boldsymbol{P}_2\boldsymbol{P}_1$. Therefore, there exists an upper pseudo-permutation matrix $\boldsymbol{P}$ such that $\boldsymbol{PA} = \boldsymbol{LU}$, of which all its leading principal submatrices are nonsingular.

As a matter of fact, we also have

$$\begin{aligned}(\boldsymbol{P}_{k+m}\boldsymbol{P}_k)^{-1} &= \boldsymbol{P}_k^{-1}\boldsymbol{P}_{k+m}^{-1}\\ &= (\boldsymbol{I} - \boldsymbol{E}_{k,k_m}) \cdot (\boldsymbol{I} - \boldsymbol{E}_{k+m,k_{k+m}})\\ &= \boldsymbol{I} - \boldsymbol{E}_{k,k_m} - \boldsymbol{E}_{k+m,k_{k+m}} + \boldsymbol{E}_{k,k_m}\boldsymbol{E}_{k+m,k_{k+m}}\\ &= \begin{cases}\boldsymbol{I} - \boldsymbol{E}_{k,k_m} - \boldsymbol{E}_{k+m,k_{k+m}} & (k_m \neq k+m)\\ \boldsymbol{I} - \boldsymbol{E}_{k,k_m} - \boldsymbol{E}_{k+m,k_{k+m}} + \boldsymbol{E}_{k,k_{k+m}} & (k_m = k+m)\end{cases}.\end{aligned}$$

For $k_m = k + m$ and $m > 0$, we have $k_m < k_{k+m}$, which guarantees that all the elements of $(\boldsymbol{P}_{N-1}\cdots\boldsymbol{P}_2\boldsymbol{P}_1)^{-1}$ are 0, 1 or -1. This also makes an upper pseudo-permutation matrix, $\boldsymbol{P} = (\boldsymbol{P}_{N-1}\cdots\boldsymbol{P}_2\boldsymbol{P}_1)^{-1}$. Thus, for any nonsingular matrix $\boldsymbol{A}$, we have a triangular factorization of $\boldsymbol{P}^{-1}\boldsymbol{A} = \boldsymbol{LU}$, where $\boldsymbol{P}$ is an upper pseudo-permutation matrix. It implies that all the leading principal submatrices of $\boldsymbol{P}^{-1}\boldsymbol{A}$ are nonsingular.

Here completes the proof. □

4. Customized with a single-row matrix

Theorem 1. *Given the customized diagonal entries of an upper triangular matrix $\boldsymbol{U}$ as $d_1, d_2, \ldots, d_N$, an $N \times N$ matrix $\boldsymbol{A}$ has a PLUS factorization of $\boldsymbol{A} = \boldsymbol{PLUS}$ if and only if* $\det \boldsymbol{A} = \pm d_1 d_2 \cdots d_N \neq 0$, *where $\boldsymbol{P}$ is a permutation or an upper pseudo-permutation matrix, $\boldsymbol{L}$ is a unit lower triangular matrix, $\boldsymbol{S}$ is a unit single-row matrix with $N-1$ elements in the customized positions of $\boldsymbol{S}(N, k)$ for $k = 1, 2, 3, \ldots, N-1$, or $\boldsymbol{S} = \boldsymbol{I} + \boldsymbol{e}_N\boldsymbol{s}_N^{\mathrm{T}}$.*

Proof. Its necessity is easy to verify.

For a nonsingular matrix $\boldsymbol{A}$, $\boldsymbol{AS}^{-1}$ is also nonsingular. So, there must exist a permutation or pseudo-permutation matrix $\boldsymbol{P}$ such that all the leading principal submatrices of $\boldsymbol{P}^{-1}\boldsymbol{AS}^{-1}$ are nonsingular, and we have $\det \boldsymbol{A} = \det \boldsymbol{P} \det \boldsymbol{U} = \pm d_1 d_2 \cdots d_N \neq 0$.

A proof of its sufficiency is given as follows.

Suppose $\boldsymbol{S} = \boldsymbol{I} + \boldsymbol{e}_N\boldsymbol{s}_N^{\mathrm{T}} = \boldsymbol{I} + \boldsymbol{e}_N \cdot [s_1, s_2, \ldots, s_{N-1}, 0]$, and $\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_N]$. Then, we have $\boldsymbol{AS}^{-1} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_N] \cdot (\boldsymbol{I} - \boldsymbol{e}_N\boldsymbol{s}_N^{\mathrm{T}}) = [\boldsymbol{a}_1 - s_1\boldsymbol{a}_N, \boldsymbol{a}_2 - s_2\boldsymbol{a}_N, \ldots, \boldsymbol{a}_{N-1} - s_{N-1}\boldsymbol{a}_N, \boldsymbol{a}_N]$.

For the given diagonal entries of matrix $\boldsymbol{U}$, by Lemma 1, the following-after problem is that if there exists a unit lower single-row matrix $\boldsymbol{S}$ such that the kth leading principal minors of $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{S}^{-1}$ are $\det(\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{S}^{-1})_k = d_1 d_2 \cdots d_k$, where $k = 1, 2, 3, \ldots, N-1$.

By Lemma 2, we have

$$\begin{aligned}\det(\boldsymbol{A}\boldsymbol{S}^{-1})_k &= \det[\boldsymbol{a}_1 - s_1\boldsymbol{a}_N, \boldsymbol{a}_2 - s_2\boldsymbol{a}_N, \ldots, \boldsymbol{a}_k - s_k\boldsymbol{a}_N]_k \\ &= \det[\boldsymbol{a}_1 - s_1\boldsymbol{a}_N, \boldsymbol{a}_2 - s_2\boldsymbol{a}_N, \ldots, \boldsymbol{a}_k]_k \\ &\quad - s_k \det[\boldsymbol{a}_1 - s_1\boldsymbol{a}_N, \boldsymbol{a}_2 - s_2\boldsymbol{a}_N, \ldots, \boldsymbol{a}_{k-1} - s_{k-1}\boldsymbol{a}_N, \boldsymbol{a}_N]_k \\ &= \det[\boldsymbol{a}_1 - s_1\boldsymbol{a}_N, \boldsymbol{a}_2 - s_2\boldsymbol{a}_N, \ldots, \boldsymbol{a}_k]_k \\ &\quad - s_k \det[\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_{k-1,}\boldsymbol{a}_N]_k.\end{aligned}$$

By Lemma 3 and Lemma 4, it can be seen that, for the nonsingular matrix $\boldsymbol{A}$, there must exist a permutation or an upper pseudo-permutation matrix $\boldsymbol{P}$ to interchange or add rows of $\boldsymbol{A}$ such that all the leading principal submatrices of $\boldsymbol{P}^{-1}[\boldsymbol{a}_N, \boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_{N-1}]$ are nonsingular, which can guarantee that

$$\det(\boldsymbol{P}^{-1} \cdot [\boldsymbol{a}_N, \boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_{k-1}])_k \neq 0$$

and

$$\det(\boldsymbol{P}^{-1} \cdot [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_{k-1}, \boldsymbol{a}_N])_k \neq 0 \quad \text{for } k = 1, 2, 3, \ldots, N-1.$$

Then, let

$$\begin{aligned}\det(\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{S}^{-1})_k &= \det(\boldsymbol{P}^{-1} \cdot [\boldsymbol{a}_1 - s_1\boldsymbol{a}_N, \boldsymbol{a}_2 - s_2\boldsymbol{a}_N, \ldots, \boldsymbol{a}_k])_k \\ &\quad - s_k \det(\boldsymbol{P}^{-1} \cdot [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_{k-1,}\boldsymbol{a}_N])_k \\ &= d_1 d_2 \cdots d_k.\end{aligned}$$

For the unknown s_k, the equation definitely has a solution as

$$s_k = \frac{\det(\boldsymbol{P}^{-1} \cdot [\boldsymbol{a}_1 - s_1\boldsymbol{a}_N, \boldsymbol{a}_2 - s_2\boldsymbol{a}_N, \ldots, \boldsymbol{a}_k])_k - d_1 d_2 \cdots d_k}{\det(\boldsymbol{P}^{-1} \cdot [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_{k-1}, \boldsymbol{a}_N])_k},$$

where $k = 1, 2, 3, \ldots, N-1$.

Therefore, under the condition of $\det \boldsymbol{P}^{-1}\boldsymbol{A} = \det \boldsymbol{U} = d_1 d_2 \cdots d_N \neq 0$ or $\det \boldsymbol{A} = \det \boldsymbol{P} \det \boldsymbol{U} = \pm d_1 d_2 \cdots d_N \neq 0$, all s_k exist. Then, we obtain a single-row matrix $\boldsymbol{S}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{S}^{-1} = \boldsymbol{L}\boldsymbol{U}$, and $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{L}\boldsymbol{U}\boldsymbol{S}$ exists.

The proof is completed. □

Corollary 1. *A sufficient condition for the customized factorization of* $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}\boldsymbol{S}$ (*or* $\boldsymbol{P} = \boldsymbol{I}$) *is that all the leading principal submatrices of* $[\boldsymbol{a}_N, \boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_{N-1}]$ *are nonsingular, where* $\boldsymbol{S} = \boldsymbol{I} + \boldsymbol{e}_N \boldsymbol{s}_N^{\mathrm{T}}$.

From the proof of Theorem 1, we can see its proof.

5. Customized with a single-column matrix

Theorem 2. *Given the customized diagonal entries of an upper triangular matrix* $\boldsymbol{U}$ *as* $d_1, d_2, \ldots, d_N$, *an* $N \times N$ *matrix* $\boldsymbol{A}$ *has a PLUS factorization of* $\boldsymbol{A} = \boldsymbol{PLUS}$ *if and only if* $\det \boldsymbol{A} = \pm d_1 d_2 \cdots d_N \neq 0$, *where* $\boldsymbol{P}$ *is a permutation or an upper pseudo-permutation matrix,* $\boldsymbol{L}$ *is a unit lower triangular matrix,* $\boldsymbol{S}$ *is a unit single-column matrix with* $N-1$ *elements in the customized positions of* $\boldsymbol{S}(k+1, 1)$ *for* $k = 1, 2, 3, \ldots, N-1$, *or* $\boldsymbol{S} = \boldsymbol{I} + \boldsymbol{s}_1 \boldsymbol{e}_1^{\mathrm{T}}$.

Proof. Its necessity is clear.

A proof of its sufficiency is provided here.

Suppose $\boldsymbol{S} = \boldsymbol{I} + \boldsymbol{s}_1 \boldsymbol{e}_1^{\mathrm{T}} = \boldsymbol{I} + [0, s_1, s_2, \ldots, s_{N-1}]^{\mathrm{T}} \cdot \boldsymbol{e}_1^{\mathrm{T}}$, we have $\boldsymbol{S}^{-1} = \boldsymbol{I} - \boldsymbol{s}_1 \boldsymbol{e}_1^{\mathrm{T}}$ and

$$\begin{aligned} \boldsymbol{AS}^{-1} &= [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_N] \cdot (\boldsymbol{I} - \boldsymbol{s}_1 \boldsymbol{e}_1^{\mathrm{T}}) \\ &= [\boldsymbol{a}_1 - s_1 \boldsymbol{a}_2 - s_2 \boldsymbol{a}_3 - \cdots - s_{N-1} \boldsymbol{a}_N, \boldsymbol{a}_2, \boldsymbol{a}_3, \ldots, \boldsymbol{a}_{N-1}, \boldsymbol{a}_N]. \end{aligned}$$

For $k = N-1, N-2, \ldots, 3, 2, 1$, by Lemma 2, the kth leading principal minors of $\boldsymbol{AS}^{-1}$ are

$$\begin{aligned} \det(\boldsymbol{AS}^{-1})_k &= \det[\boldsymbol{a}_1 - s_1 \boldsymbol{a}_2 - s_2 \boldsymbol{a}_3 - \cdots - s_{N-1} \boldsymbol{a}_N, \boldsymbol{a}_2, \boldsymbol{a}_3, \ldots, \boldsymbol{a}_{N-1}, \boldsymbol{a}_N]_k \\ &= \det[\boldsymbol{a}_1 - s_1 \boldsymbol{a}_2 - s_2 \boldsymbol{a}_3 - \cdots - s_{N-1} \boldsymbol{a}_N, \boldsymbol{a}_2, \boldsymbol{a}_3, \ldots, \boldsymbol{a}_k]_k \\ &= \det[\boldsymbol{a}_1 - s_k \boldsymbol{a}_{k+1} - s_{k+1} \boldsymbol{a}_{k+2} - \cdots - s_{N-1} \boldsymbol{a}_N, \boldsymbol{a}_2, \boldsymbol{a}_3, \ldots, \boldsymbol{a}_k]_k \\ &= \det[\boldsymbol{a}_1 - s_{k+1} \boldsymbol{a}_{k+2} - \cdots - s_{N-1} \boldsymbol{a}_N, \boldsymbol{a}_2, \boldsymbol{a}_3, \ldots, \boldsymbol{a}_k]_k \\ &\quad - s_k \det[\boldsymbol{a}_{k+1}, \boldsymbol{a}_2, \boldsymbol{a}_3, \ldots, \boldsymbol{a}_k]_k. \end{aligned}$$

By Lemmas 3 and 4, for the nonsingular matrix $\boldsymbol{A}$, there exists a permutation or an upper pseudo-permutation matrix $\boldsymbol{P}$ such that all the leading principal submatrices of $\boldsymbol{P}^{-1}[\boldsymbol{a}_2, \boldsymbol{a}_3, \ldots, \boldsymbol{a}_{N-1}, \boldsymbol{a}_N]$ are nonsingular, which can guarantee that

$$\det(\boldsymbol{P}^{-1} \cdot [\boldsymbol{a}_2, \boldsymbol{a}_3, \ldots, \boldsymbol{a}_{N-1}, \boldsymbol{a}_N])_k \neq 0$$

and

$$\det(\boldsymbol{P}^{-1} \cdot [\boldsymbol{a}_{k+1}, \boldsymbol{a}_2, \boldsymbol{a}_3, \ldots, \boldsymbol{a}_k])_k \neq 0 \quad \text{for } k = 1, 2, 3, \ldots, N-1.$$

Then, let

$$\begin{aligned} \det(\boldsymbol{P}^{-1}\boldsymbol{AS}^{-1})_k &= \det(\boldsymbol{P}^{-1} \cdot [\boldsymbol{a}_1 - s_{k+1} \boldsymbol{a}_{k+2} - \cdots - s_{N-1} \boldsymbol{a}_N, \boldsymbol{a}_2, \boldsymbol{a}_3, \ldots, \boldsymbol{a}_k])_k \\ &\quad - s_k \det(\boldsymbol{P}^{-1} \cdot [\boldsymbol{a}_{k+1}, \boldsymbol{a}_2, \boldsymbol{a}_3, \ldots, \boldsymbol{a}_k])_k \\ &= d_1 d_2 \cdots d_k. \end{aligned}$$

For the unknown s_k, the equation definitely has a solution as

$$s_k = \frac{\det(\boldsymbol{P}^{-1} \cdot [\boldsymbol{a}_1 - s_{k+1}\boldsymbol{a}_{k+2} - \cdots - s_{N-1}\boldsymbol{a}_N, \boldsymbol{a}_2, \boldsymbol{a}_3, \ldots, \boldsymbol{a}_k])_k - d_1 d_2 \cdots d_k}{\det(\boldsymbol{P}^{-1} \cdot [\boldsymbol{a}_{k+1}, \boldsymbol{a}_2, \boldsymbol{a}_3, \ldots, \boldsymbol{a}_k])_k},$$

where $k = N-1, N-2, \ldots, 3, 2, 1$.

Therefore, under the condition of $\det \boldsymbol{P}^{-1}\boldsymbol{A} = \det \boldsymbol{U} = d_1 d_2 \cdots d_N \neq 0$ or $\det \boldsymbol{A} = \det \boldsymbol{P} \det \boldsymbol{U} = \pm d_1 d_2 \cdots d_N \neq 0$, all s_k exist. Then, we find a single-column matrix $\boldsymbol{S}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{S}^{-1} = \boldsymbol{L}\boldsymbol{U}$, and $\boldsymbol{A} = \boldsymbol{PLUS}$ exists.

Thus, the proof is complete. □

Corollary 2. *A sufficient condition for the customized factorization of* $\boldsymbol{A} = \boldsymbol{LUS}$ *(or* $\boldsymbol{P} = \boldsymbol{I}$*), is that all the leading principal submatrices of* $[\boldsymbol{a}_2, \boldsymbol{a}_3, \ldots, \boldsymbol{a}_N]$ *are nonsingular, where* $\boldsymbol{S} = \boldsymbol{I} + \boldsymbol{s}_1\boldsymbol{e}_1^{\mathrm{T}}$.

Its proof is given in the proof to Theorem 2.

6. Customized with a bidiagonal matrix

Theorem 3. *Given the customized diagonal entries of an upper triangular matrix* $\boldsymbol{U}$ *as* $d_1, d_2, \ldots, d_N$, *an* $N \times N$ *matrix* $\boldsymbol{A}$ *has a PLUS factorization of* $\boldsymbol{A} = \boldsymbol{PLUS}$ *if and only if* $\det \boldsymbol{A} = \pm d_1 d_2 \cdots d_N \neq 0$, *where* $\boldsymbol{P}$ *is a permutation or an upper pseudo-permutation matrix,* $\boldsymbol{L}$ *is a unit lower triangular matrix,* $\boldsymbol{S}$ *is a unit lower bidiagonal matrix with* $N-1$ *elements in the customized positions of* $\boldsymbol{S}(k+1, k)$ *for* $k = 1, 2, 3, \ldots, N-1$.

Proof. Its necessity is obvious.

A proof of its sufficiency is given below.

Suppose the bidiagonal matrix is formed as

$$\begin{aligned}
\boldsymbol{S} &= \begin{bmatrix} 1 & & & & \\ s_1 & 1 & & & \\ & s_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & s_{N-1} & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & & & & \\ s_1 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ & s_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & s_{N-1} & 1 \end{bmatrix} \\
&= \boldsymbol{S}_1 \cdot \boldsymbol{S}_2 \cdots \boldsymbol{S}_{N-1},
\end{aligned}$$

we have

$$\begin{aligned}
S^{-1} &= S_{N-1}^{-1}\cdots S_2^{-1}\cdot S_1^{-1} \\
&= \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & -s_{N-1} & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ & -s_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \end{bmatrix} \\
&\quad \times \begin{bmatrix} 1 & & & & \\ -s_1 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \end{bmatrix}.
\end{aligned}$$

By Lemma 2, it follows that for all $k = N-1, N-2, \ldots, 3, 2, 1$, $S_{k-1}^{-1}\cdots S_2^{-1}\cdot S_1^{-1}$ do not change the kth leading principal minors of $AS_{N-1}^{-1}\cdots S_k^{-1}$. Take a notation of $[a_1^{(k+1)}, a_2^{(k+1)}, \ldots, a_{N-1}^{(k+1)}, a_N^{(k+1)}] = A\cdot S_{N-1}^{-1}\cdot S_{N-2}^{-1}\cdots S_{k+1}^{-1}$, we have

$$\begin{aligned}
\det(AS^{-1})_k &= \det\left(A\cdot S_{N-1}^{-1}\cdot S_{N-2}^{-1}\cdots S_2^{-1}\cdot S_1^{-1}\right)_k \\
&= \det\left(A\cdot S_{N-1}^{-1}\cdot S_{N-2}^{-1}\cdots S_k^{-1}\right)_k \\
&= \det\left(\left[a_1^{(k+1)}, a_2^{(k+1)}, \ldots, a_{N-1}^{(k+1)}, a_N^{(k+1)}\right]\cdot S_k^{-1}\right)_k \\
&= \det\left[a_1^{(k+1)}, \ldots, a_{k-1}^{(k+1)}, a_k^{(k+1)} - s_k a_{k+1}^{(k+1)}, a_{k+1}^{(k+1)}, \ldots, a_N^{(k+1)}\right]_k \\
&= \det\left[a_1^{(k+1)}, \ldots, a_{k-1}^{(k+1)}, a_k^{(k+1)} - s_k a_{k+1}^{(k+1)}\right]_k \\
&= \det\left[a_1^{(k+1)}, \ldots, a_{k-1}^{(k+1)}, a_k^{(k+1)}\right]_k \\
&\quad - s_k \det\left[a_1^{(k+1)}, \ldots, a_{k-1}^{(k+1)}, a_{k+1}^{(k+1)}\right]_k
\end{aligned}$$

and for the nonsingular matrix A, by Lemmas 3 and 4, a permutation or an upper pseudo-permutation matrix P can be found such that $\det(P^{-1}[a_1^{(k+1)}, \ldots, a_{k-1}^{(k+1)}, a_{k+1}^{(k+1)}])_k \neq 0$ and then the equation $\det(P^{-1}AS^{-1})_k = d_1 d_2 \cdots d_k$ has a solution for the unknown s_k.

Therefore, under the condition of $\det A = \pm d_1 d_2 \cdots d_N \neq 0$, all s_k exist for $k = N-1, N-2, \ldots, 3, 2, 1$. Then, we get a bidiagonal matrix S such that $P^{-1}AS^{-1} = LU$, and $A = PLUS$ exists.

The proof is complete. $\square$

Corollary 3. *If* $\det[\boldsymbol{a}_1^{(k+1)}, \ldots, \boldsymbol{a}_{k-1}^{(k+1)}, \boldsymbol{a}_{k+1}^{(k+1)}]_k \neq 0$ (*see the proof of Theorem* 3) *for all* $k = N-1, N-2, \ldots, 3, 2, 1$, *permutations or pseudo-permutations are unnecessary* (*or* $\boldsymbol{P} = \boldsymbol{I}$), *then* $\boldsymbol{A}$ *has a customized factorization of* $\boldsymbol{A} = \boldsymbol{LUS}$, *where* $\boldsymbol{S}$ *is a unit lower bidiagonal matrix.*

7. Generic customizable triangular factorizations

Theorem 4. *For an arbitrary* $N \times N$ *nonsingular matrix* $\boldsymbol{A}$, *a customizable triangular factorization* $\boldsymbol{A} = \boldsymbol{PLUS}$ *exists only if* $\sum_{n=1}^{k} \sum_{m=n+1}^{N} \boldsymbol{B}(m,n) \geqslant k$ *and* $\sum_{m=1}^{k} \sum_{n=1}^{N-m} \boldsymbol{B}(N-m+1,n) \geqslant k$ *hold for* $k = 1, 2, \ldots, N-1$, *where* $\boldsymbol{B}$ *is the customization matrix of the customized matrix* $\boldsymbol{S}$.

Proof. First, we investigate the dependence of the elements in the customized matrix.

Suppose the elements of the special matrix $\boldsymbol{S}$ are $\boldsymbol{S}(m,n) = s_{m,n}$ for $m > n$, $\boldsymbol{S}(m,n) = 1$ for $m = n$, and $\boldsymbol{S}(m,n) = 0$ for $m < n$, then the matrix can be written as a product of $N-1$ single-column matrices or a product of $N-1$ single-row matrices:

$$\boldsymbol{S} = \begin{bmatrix} 1 & & & & \\ s_{2,1} & 1 & & & \\ s_{3,1} & & 1 & & \\ \cdots & & & \ddots & \\ s_{N,1} & & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & \\ & 1 & & \\ & s_{3,2} & 1 & \\ & \cdots & & \ddots & \\ & s_{N,2} & & & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & s_{N,N-1} & 1 \end{bmatrix}$$

$$= \boldsymbol{C}_1 \cdot \boldsymbol{C}_2 \cdots \boldsymbol{C}_{N-1}.$$

$$\boldsymbol{S} = \begin{bmatrix} 1 & & & & \\ s_{2,1} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ s_{3,1} & s_{3,2} & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ s_{N,1} & s_{N,2} & \cdots & s_{N,N-1} & 1 \end{bmatrix}$$

$$= \boldsymbol{R}_2 \cdot \boldsymbol{R}_3 \cdots \boldsymbol{R}_N,$$

where $\boldsymbol{S}(m,n) = s_{m,n} = 0$ if $\boldsymbol{B}(m,n) = 0$, and $\boldsymbol{S}(m,n) = s_{m,n}$ are in the customized positions if $\boldsymbol{B}(m,n) = 1$.

The inverse of the customized matrix can be written as

$$
\begin{aligned}
\boldsymbol{S}^{-1} &= \boldsymbol{C}_{N-1}^{-1} \cdots \boldsymbol{C}_2^{-1} \cdot \boldsymbol{C}_1^{-1} \\
&= \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & -s_{N,N-1} & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -s_{3,2} & 1 & \\ & \cdots & & \ddots & \\ & -s_{N,2} & & & 1 \end{bmatrix} \\
&\quad \times \begin{bmatrix} 1 & & & & \\ -s_{2,1} & 1 & & & \\ -s_{3,1} & & 1 & & \\ \cdots & & & \ddots & \\ -s_{N,1} & & & & 1 \end{bmatrix}
\end{aligned}
$$

or

$$
\begin{aligned}
\boldsymbol{S}^{-1} &= \boldsymbol{R}_N^{-1} \cdots \boldsymbol{R}_3^{-1} \cdot \boldsymbol{R}_2^{-1} \\
&= \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ -s_{N,1} & -s_{N,2} & \cdots & -s_{N,N-1} & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ -s_{3,1} & -s_{3,2} & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \\
&\quad \times \begin{bmatrix} 1 & & & & \\ -s_{2,1} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}.
\end{aligned}
$$

By Lemmas 3 and 4, for the nonsingular matrix $\boldsymbol{A}$, there exists a permutation or an upper pseudo-permutation matrix $\boldsymbol{P}$ such that all the $N-1$ submatrices of $\boldsymbol{P}^{-1}\boldsymbol{A}$ are nonsingular as needed.

By Lemma 1, for customizable triangular factorizations, all the first $N-1$ leading principal minors of $\boldsymbol{P}^{-1}\boldsymbol{A}$ should be modifiable to the customized diagonal numbers through right-multiplication by $\boldsymbol{S}^{-1}$. From Lemma 2, it follows that a nonzero element $s_{m,n} = \boldsymbol{S}(m,n)$ $(m > n)$ in $\boldsymbol{C}_k^{-1}$ or $\boldsymbol{R}_k^{-1}$ or $\boldsymbol{S}$ can be used to modify the leading principal minors from the nth to the $(m-1)$th, where $1 \leqslant k \leqslant N-1$. Therefore, right-multiplication by $\boldsymbol{C}_k^{-1} \cdots \boldsymbol{C}_2^{-1} \cdot \boldsymbol{C}_1^{-1}$ or $\boldsymbol{R}_N^{-1} \cdots \boldsymbol{R}_{k+2}^{-1} \cdot \boldsymbol{R}_{k+1}^{-1}$ can modify all the first $N-1$ leading principal minors of $\boldsymbol{P}^{-1}\boldsymbol{A}$. Nevertheless, $\boldsymbol{C}_{N-1}^{-1} \cdots \boldsymbol{C}_{k+2}^{-1} \cdot \boldsymbol{C}_{k+1}^{-1}$ cannot change the first k leading principal minors, and $\boldsymbol{R}_k^{-1} \cdots \boldsymbol{R}_3^{-1} \cdot \boldsymbol{R}_2^{-1}$ can do nothing to the last $N-k$ (from the kth to the $(N-1)$th) leading principal minors.

Consequently, the $N-1$ customized element locations in $\boldsymbol{S}$ should be distributed from the first column and to the last row.

Let those elements in the customized positions be the unknowns. Based on the above discussion, we see that, for the first k column customized numbers $\sum_{n=1}^{k}\sum_{m=n+1}^{N}\boldsymbol{B}(m,n)$ and the first k leading principal minors of $\boldsymbol{P}^{-1}\boldsymbol{A}$, or for the last k row customized numbers $\sum_{m=1}^{k}\sum_{n=1}^{N-m}\boldsymbol{B}(N-m+1,n)$ and the last k leading principal minors of $\boldsymbol{P}^{-1}\boldsymbol{A}$, there are k equations induced and must be satisfied. As a result, for an arbitrary nonsingular matrix $\boldsymbol{A}$, it is necessary that $\sum_{n=1}^{k}\sum_{m=n+1}^{N}\boldsymbol{B}(m,n)\geqslant k$ and $\sum_{m=1}^{k}\sum_{n=1}^{N-m}\boldsymbol{B}(N-m+1,n)\geqslant k$ such that those equations are solvable and then $\boldsymbol{S}$ exists, which completes the proof. □

Corollary 4. *If the corresponding submatrices are sequentially nonsingular as required, then permutations are not necessary (or $\boldsymbol{P}=\boldsymbol{I}$), and consequently the nonsingular matrix has a customized factorization of $\boldsymbol{A}=\boldsymbol{LUS}$, where $\boldsymbol{S}$ is a customized unit lower triangular matrix.*

Its proof is easy to see as stated in previous corollaries and the proofs to the Theorem 1–3.

Remark 1. In Theorem 4, we just proved the necessary condition for generic customizable triangular factorizations, the necessary and sufficient condition is still an unsolved problem.

Remark 2. In above theorems, we can also use column operations to find the necessarily nonsingular submatrices so as to satisfy the necessary conditions. It follows that a column permutation or a lower pseudo-permutation matrix (right-multiplied) can be an alternative. Thus, the customizable triangular factorization can also be a LUSP factorization.

Remark 3. If a customizable triangular factorization is for the transpose of a matrix, we can easily obtain its factorizations of SLUP and PSLU.

Remark 4. Use the anti-identity matrix $\boldsymbol{Q}$, and factorize $\boldsymbol{QAQ}$ first. If $\boldsymbol{QAQ}=\boldsymbol{PLUS}$, then $\boldsymbol{A}=\boldsymbol{QPLUSQ}=(\boldsymbol{QPQ})\cdot(\boldsymbol{QLQ})\cdot(\boldsymbol{QUQ})\cdot(\boldsymbol{QSQ})$, where $\boldsymbol{QPQ}$ is still a permutation or pseudo-permutation matrix, $\boldsymbol{QLQ}$ is a unit upper triangular matrix, $\boldsymbol{QUQ}$ is a lower triangular matrix with the given diagonal entries, and $\boldsymbol{QSQ}$ is a customized unit upper triangular matrix. Therefore, matrix $\boldsymbol{A}$ has a PULS factorization. If $\boldsymbol{QAQ}$ is factorized into LUSP, SLUP, or PSLU, we can also obtain other customizable triangular factorizations of $\boldsymbol{A}$, ULSP, SULP, and PSUL.

Remark 5. If a pseudo-permutation matrix instead of a permutation matrix is used in a customizable triangular factorization, we have $\det\boldsymbol{A}=\det\boldsymbol{U}=d_1d_2\cdots d_N$.

Remark 6. TERM factorization is a special case of customizable triangular factorizations. It is proved by Hao and Shi in [5] that a TERM factorization can be further factorized into a series of SERMs. Analogously, a customizable triangular factorization also provides possibilities for further factorizations into a series of special elementary matrices, which facilitate more potential applications.

8. A matrix factorization algorithm

Generally, customizable triangular matrix factorizations can be implemented by finding the unknowns in the equations of the leading principal minors. An executable algorithm of the customizable triangular factorization PLUS is given herein below. The customized special triangular matrix is a unit lower single-row matrix.

Suppose

$$\boldsymbol{A} = \begin{bmatrix} a_{1,1}^{(1)} & a_{1,2}^{(1)} & \cdots & a_{1,N}^{(1)} \\ a_{2,1}^{(1)} & a_{2,2}^{(1)} & \cdots & a_{2,N}^{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ a_{N,1}^{(1)} & a_{N,2}^{(1)} & \cdots & a_{N,N}^{(1)} \end{bmatrix}$$

and the given diagonal entries of $\boldsymbol{U}$ is $d_1, d_2, \ldots, d_N$, where $d_1 d_2 \cdots d_N = \pm \det \boldsymbol{A}$.

Then, there must exist a permutation or an upper pseudo-permutation matrix $\boldsymbol{P}_1$ for row interchanges or adds, such that:

$$\boldsymbol{P}_1\boldsymbol{A} = \begin{bmatrix} p_{1,1}^{(1)} & p_{1,2}^{(1)} & \cdots & p_{1,N}^{(1)} \\ p_{2,1}^{(1)} & p_{2,2}^{(1)} & \cdots & p_{2,N}^{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ p_{N,1}^{(1)} & p_{N,2}^{(1)} & \cdots & p_{N,N}^{(1)} \end{bmatrix}$$

and $p_{1,N}^{(1)} \neq 0$, and hence there must be a number s_1, such that $p_{1,1}^{(1)} - s_1 \cdot p_{1,N}^{(1)} = d_1$. Then, we can get $s_1 = (p_{1,1}^{(1)} - d_1)/p_{1,N}^{(1)}$, and obtain a product as

$$\boldsymbol{P}_1\boldsymbol{A}\boldsymbol{S}_1 = \boldsymbol{P}_1\boldsymbol{A} \begin{bmatrix} 1 & & \\ & \boldsymbol{I} & \\ -s_1 & \boldsymbol{0} & 1 \end{bmatrix} = \begin{bmatrix} d_1 & p_{1,2}^{(1)} & \cdots & p_{1,N}^{(1)} \\ p_{2,1}^{(1)} - s_1 p_{2,N}^{(1)} & p_{2,2}^{(1)} & \cdots & p_{2,N}^{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ p_{N,1}^{(1)} - s_1 p_{N,N}^{(1)} & p_{N,2}^{(1)} & \cdots & p_{N,N}^{(1)} \end{bmatrix}.$$

Afterwards, the forward elimination of the first column can be achieved by multiplying an elementary Gauss matrix $\boldsymbol{L}_1$:

$$\boldsymbol{L}_1\boldsymbol{P}_1\boldsymbol{A}\boldsymbol{S}_1 = \begin{bmatrix} 1 & & \\ (s_1 p_{2,N}^{(1)} - p_{2,1}^{(1)})/d_1 & 1 & \\ \cdots & & \boldsymbol{I} \\ (s_1 p_{N,N}^{(1)} - p_{N,1}^{(1)})/d_1 & & & 1 \end{bmatrix} \boldsymbol{P}_1\boldsymbol{A}\boldsymbol{S}_1$$

$$= \begin{bmatrix} d_1 & a_{1,2}^{(2)} & \cdots & a_{1,N}^{(2)} \\ 0 & a_{2,2}^{(2)} & \cdots & a_{2,N}^{(2)} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & a_{N,2}^{(2)} & \cdots & a_{N,N}^{(2)} \end{bmatrix}.$$

The process can be continued recursively to obtain the complete factorization. For $k = 2, 3, \ldots, N-1$, $\boldsymbol{P}_k$ denotes a permutation or an upper pseudo-permutation matrix for the row interchange or row addition between the kth row and another row from the $(k+1)$th through the Nth rows so as to guarantee that the kth element in the Nth column is not zero, $p_{k,N}^{(k)} \neq 0$ (If there were no such element, $\boldsymbol{A}$ would have been singular.); $\boldsymbol{S}_k$ coverts $a_{k,k}^{(k)}$ into d_k, where $s_k = (p_{k,k}^{(k)} - d_k)/p_{k,N}^{(k)}$; and $\boldsymbol{L}_k$ record the row multipliers used for the Gaussian elimination of column k. Then, we get,

$$\boldsymbol{L}_{N-1}\boldsymbol{P}_{N-1}\cdots\boldsymbol{L}_2\boldsymbol{P}_2\boldsymbol{L}_1\boldsymbol{P}_1\boldsymbol{A}\boldsymbol{S}_1\boldsymbol{S}_2\cdots\boldsymbol{S}_{N-1} = \begin{bmatrix} d_1 & a_{1,2}^{(N)} & \cdots & a_{1,N}^{(N)} \\ 0 & d_2 & \cdots & a_{2,N}^{(N)} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{N,N}^{(N)} \end{bmatrix} = \boldsymbol{U},$$

where $a_{N,N}^{(N)} = d_N$.

Respectively, multiplying all $\boldsymbol{S}_k$ together, all $\boldsymbol{P}_k$ together, and all $\boldsymbol{L}_k$ together, analogous to the proof of Lemma 4, we have one unit single-row triangular matrix $\boldsymbol{S}^{-1}$, one pre-multiplying permutation or pseudo-permutation matrix $\boldsymbol{P}^{-1}$, and one unit lower triangular matrix $\boldsymbol{L}^{-1}$.

$$\boldsymbol{S}_1\boldsymbol{S}_2\cdots\boldsymbol{S}_{N-1} = \begin{bmatrix} 1 & & & \\ & \boldsymbol{I} & & \\ & & 1 & \\ -s_1 & \cdots & -s_{N-1} & 1 \end{bmatrix} = \boldsymbol{S}^{-1},$$

$$\begin{aligned} &\boldsymbol{L}_{N-1}\boldsymbol{P}_{N-1}\cdots\boldsymbol{L}_2\boldsymbol{P}_2\boldsymbol{L}_1\boldsymbol{P}_1 \\ &\quad = \boldsymbol{L}_{N-1}(\boldsymbol{P}_{N-1}\boldsymbol{L}_{N-2}\boldsymbol{P}_{N-1}^{-1})\cdots(\boldsymbol{P}_{N-1}\cdots\boldsymbol{P}_2\boldsymbol{L}_1\boldsymbol{P}_2^{-1}\cdots\boldsymbol{P}_{N-1}^{-1}) \\ &\qquad \times (\boldsymbol{P}_{N-1}\cdots\boldsymbol{P}_2\boldsymbol{P}_1) \\ &\quad = \boldsymbol{L}^{-1}\boldsymbol{P}^{-1}, \end{aligned}$$

where $\boldsymbol{L}^{-1} = \boldsymbol{L}_{N-1}(\boldsymbol{P}_{N-1}\boldsymbol{L}_{N-2}\boldsymbol{P}_{N-1}^{-1})\cdots(\boldsymbol{P}_{N-1}\cdots\boldsymbol{P}_2\boldsymbol{L}_1\boldsymbol{P}_2^{-1}\cdots\boldsymbol{P}_{N-1}^{-1}) = \boldsymbol{L}'_{N-1} \times \boldsymbol{L}'_{N-2}\cdots\boldsymbol{L}'_1$, $\boldsymbol{P}^{-1} = \boldsymbol{P}_{N-1}\cdots\boldsymbol{P}_2\boldsymbol{P}_1$.

The inverse of a unit lower triangular matrix is also a unit lower triangular matrix, the inverse of the pseudo-permutation matrix is also a pseudo-permutation matrix, and the inverse of a permutation matrix is its transpose. Hence, we obtain $\boldsymbol{L}^{-1}\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{S}^{-1} = \boldsymbol{U}$ or $\boldsymbol{A} = \boldsymbol{PLUS}$.

9. Examples

In order to demonstrate the availability of the factorizations presented in this paper, a 4×4 matrix is used as an example for the customizable factorizations. The 4×4 matrix is:

$$\boldsymbol{A} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

Since its determinant is 20, we just customize the elements on the diagonal of $\boldsymbol{U}$ to be 1, 2, 2, 5. There are 20 choices (C_6^3) for the customizable matrix, among which 2 cases of the customized matrix do not satisfy Theorem 4 and are not eligible for customizable triangular factorizations. Then, we get some customizable factorizations with various available customized matrices and concerning appropriate permutations as follows. The customized positions are framed.

$$\begin{aligned}
\boldsymbol{A} &= \boldsymbol{PLUS} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -3 & 2 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & 2 & 5 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 5 \end{bmatrix} \\
&\quad \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \boxed{1} & \boxed{1} & \boxed{1} & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 5/6 & -1/6 & 1 & 0 \\ 5/2 & -7/2 & 5/2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 & 3 & 2 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix} \\
&\quad \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ \boxed{2/3} & 1 & 0 & 0 \\ \boxed{-1/3} & 0 & 1 & 0 \\ \boxed{1/6} & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -2 & 1/2 & 1 & 0 \\ -6 & 3/2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1/5 & 14/5 & 3 \\ 0 & 2 & 10 & 10 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & -5 \end{bmatrix}
\end{aligned}$$

$$\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ \boxed{5} & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & \boxed{2/5} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 15 & -16 & -5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \boxed{6} & 0 & 1 & 0 \\ \boxed{-8} & \boxed{1} & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2/3 & 1 & 0 & 0 \\ 2 & -3/2 & 1 & 0 \\ 4/3 & -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & -1/2 & 3 \\ 0 & 2 & 1/3 & 0 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ \boxed{5/6} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \boxed{-1/2} & 0 & \boxed{3/2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ -7/2 & 1 & 0 & 0 \\ -4 & 1 & 1 & 0 \\ -3/2 & 1 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4/5 & 2 & 1 \\ 0 & 2 & 10 & 15/2 \\ 0 & 0 & 2 & -1/2 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \boxed{3/2} & \boxed{6/5} & 1 & 0 \\ 0 & \boxed{-1/5} & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 1/5 & 2/5 & 1 & 0 \\ -9/10 & 11/5 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 2 & 4 & 9/2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ \boxed{3/2} & 1 & 0 & 0 \\ 0 & \boxed{3/5} & 1 & 0 \\ 0 & \boxed{-1/5} & 0 & 1 \end{bmatrix}.$$

With pseudo-permutation matrices, we get some other customizable factorizations with various customized matrices as follows.

$$
\begin{aligned}
\boldsymbol{A} &= \boldsymbol{PLUS}\\
&= \begin{bmatrix} 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0\\ 2 & 1 & 0 & 0\\ -1 & 0 & 1 & 0\\ -3 & -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 & 2\\ 0 & 2 & -2 & -5\\ 0 & 0 & 2 & 5\\ 0 & 0 & 0 & 5 \end{bmatrix}\\
&\quad \times \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \boxed{1} & \boxed{1} & \boxed{1} & 1 \end{bmatrix}\\
&= \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0\\ -1 & 1 & 0 & 0\\ 1 & 0 & 1 & 0\\ -1 & 5/2 & 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & 2 & 1\\ 0 & 2 & -2 & -4\\ 0 & 0 & 2 & 2\\ 0 & 0 & 0 & 5 \end{bmatrix}\\
&\quad \times \begin{bmatrix} 1 & 0 & 0 & 0\\ \boxed{2} & 1 & 0 & 0\\ \boxed{-2} & 0 & 1 & 0\\ \boxed{1} & 0 & 0 & 1 \end{bmatrix}\\
&= \begin{bmatrix} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0\\ -2 & 1 & 0 & 0\\ 0 & 1/4 & 1 & 0\\ -3 & 5/4 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1/2 & 1/2 & -3\\ 0 & 2 & 2 & -8\\ 0 & 0 & 2 & 5\\ 0 & 0 & 0 & 5 \end{bmatrix}\\
&\quad \times \begin{bmatrix} 1 & 0 & 0 & 0\\ \boxed{4} & 1 & 0 & 0\\ 0 & \boxed{1} & 1 & 0\\ 0 & 0 & \boxed{1/2} & 1 \end{bmatrix}\\
&= \begin{bmatrix} 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0\\ -1 & 1 & 0 & 0\\ -2 & 0 & 1 & 0\\ 5 & -1 & 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 & -1\\ 0 & 2 & 2 & 1\\ 0 & 0 & 2 & 1\\ 0 & 0 & 0 & 5 \end{bmatrix}\\
&\quad \times \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ \boxed{4} & 0 & 1 & 0\\ \boxed{-4} & \boxed{1} & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 5/2 & 1 & 0 \\ 2 & 2 & 1/2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -3/5 & -1 \\ 0 & 2 & 4/5 & -2 \\ 0 & 0 & 2 & 10 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ \boxed{1/2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \boxed{-1/2} & 0 & \boxed{2/5} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 9 & 1 & 0 & 0 \\ 10 & 1 & 1 & 0 \\ 7 & 0 & 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & -1 & -3 \\ 0 & 2 & 12 & 29 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \boxed{-2} & \boxed{-3} & 1 & 0 \\ 0 & \boxed{1} & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 6 & 4 & 1 & 0 \\ 7 & 5 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -1 & -3 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ \boxed{-2} & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & \boxed{-1} & 0 & 1 \end{bmatrix}.$$

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