Proof Systems for First-order Theories

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Outline

Motivations

- Basic goal: maximal extraction of information from the analysis of proofs in a formal inference system
- Analytic proof systems for pure logic: sequent calculus, natural deduction
- Extension to the analysis of mathematical theories:
  - theories with universal axioms (N and von Plato 1998)
  - geometric theories (N 2003, Simpson 1994 in ND-style)
  - a wide class of non-classical logics, including provability logic (N 2005), intermediate logics (Dyckhoff and N 2011), temporal logics (Boretti and N 2009), epistemic logics (Hakli and N 2011, 2012)
- Proof analysis beyond geometric theories
  - Generalized geometric implications – Sahlqvist fragment (N 2014), knowability logic (Maffezioli, Naibo and N 2013), conditional logics (N and Sbardolini 2014), ...
  - Arbitrary first-order theories/frame conditions?

Design principles and properties

Conversion of axioms into rules of inference is obtained by a *uniform procedure* that has to respect the properties of the logical calculus to which the rules are added.

Natural deduction with *general elimination rules* or better *sequent calculus*.

Sequent calculus as a ground logical calculus abductive tool to find the rules; invertibility key feature

1. Regular extensions

Take the classical Gentzen-Ketonen-Kleene contraction- and cut-free calculus **G3c** and rules that correspond to axioms

$$P_1 \& \dots \& P_m \supset Q_1 \vee \dots \vee Q_n$$

formulation as a **left** rule:

$$\frac{Q_1, \Gamma \rightarrow \Delta \quad \dots \quad Q_n, \Gamma \rightarrow \Delta}{P_1, \dots, P_m, \Gamma, \rightarrow \Delta} R$$

formulation as a **right** rule:

$$\frac{\Gamma \rightarrow \Delta, P_1 \quad \dots \quad \Gamma \rightarrow \Delta, P_m}{\Gamma, \rightarrow \Delta, Q_1, \dots, Q_n} R$$

Cut elimination

The reason why we get *full cut elimination* lies in the form of the rules:

- the rules act on only one side of sequents

- the rules act on atomic formulas

We have, say,

$$\dfrac{\Gamma \Rightarrow \Delta, P \qquad \dfrac{\vdots}{P, \Gamma' \Rightarrow \Delta'}\, R}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, \mathit{Cut}$$

If R is a left rule with P principal, P is not principal in the left premiss of cut and cut can be permuted

Coherent and geometric implications

A formula is Horn iff built from atoms (and $\top$) using only $\wedge$.

A formula is coherent, *aka "positive",* iff built from atoms (and $\top, \bot$) using only $\vee$, $\wedge$ and $\exists$.

A formula is geometric iff built from atoms (and $\top, \bot$) using only $\vee$, $\wedge$, $\exists$ and infinitary disjunctions.

A sentence is a coherent implication iff of the form $\forall \mathbf{x}.C \supset D$, where C, D are coherent [degenerate case with $\top$ as C left unwritten]

A sentence is a geometric implication iff of the form $\forall \mathbf{x}.C \supset D$, where C, D are geometric.

Theorem: Any coherent sentence is equivalent to a finite conjunction of sentences of the form $\forall \mathbf{x}.C \supset D$ where C is Horn and D is a (finite) disjunction of existentially quantified Horn formulae.

Theorem: Any geometric sentence is equivalent to a (possibly infinite) conjunction of sentences of the form $\forall \mathbf{x}.C \supset D$ where C is Horn and D is a (possibly infinite) disjunction of existentially quantified Horn formulae.

Examples

Universal formulae $\forall \mathbf{x}.A$ can be written as finite conjunctions of coherent implications, just by putting A into CNF, distributing $\forall$ past $\wedge$ and rewriting (e.g. $\neg P \vee Q$ as $P \supset Q$). (No $\exists$ is involved. $\top$ and $\bot$ may be useful.)

Theory of *local rings* is axiomatised by coherent implications, including $\forall x \exists y (xy = 1) \vee \exists y ((1 - x)y = 1)$

Theory of *transitive relations* is axiomatised by coherent implication: $\forall xyz.(Rxy \wedge Ryz) \supset Rxz$.

Theory of *partial order* is axiomatised by coherent implications, including: $\forall xy.\ (x \leq y \wedge y \leq x) \supset x = y$

Theory of *strongly directed relations* is axiomatised by coherent implication: $\forall xyz.(Rxy \wedge Rxz) \supset \exists u.Ryu \wedge Rzu$

(Infinitary) theory of *torsion abelian groups* is axiomatised by geometric implications, including $\forall x.\ \bigvee_{n>1} (nx = 0)$

Other examples

Obs.: Reformulation of the axioms and/or choice of basic concepts may be crucial for obtaining a geometric axiomatization.

Fields: $\neg\, a = 0 \supset \exists y\; a \cdot y = 1$ is not geometric, but the equivalent $a = 0 \vee \exists y\; a \cdot y = 1$ is.

Robinson arithmetic: $\neg\, a = 0 \supset \exists y\; a = s(y)$ is not geometric, but the equivalent $a = 0 \vee \exists y\; a = s(y)$ is

Real-closed fields:
$\neg\, a_{2n+1} = 0 \supset \exists y\; a_{2n+1} \cdot y^{2n+1} + a_{2n} \cdot y^{2n} + \ldots a_1 \cdot y + a_0 = 0$ is not geometric, but the equivalent
$a_{2n+1} = 0 \vee \exists y\; a_{2n+1} \cdot y^{2n+1} + a_{2n} \cdot y^{2n} + \ldots a_1 \cdot y + a_0 = 0$ is.

Classical projective geometry: Not a geometric theory!
Axiom of existence of three non-collinear points.
$\exists x \exists y \exists z (\neg\, x = y \,\&\, \neg\, z \in ln(x, y))$
if the basic notions are replaced by the constructive notions of apartness between points and lines and “outsideness” of a point from a line, a geometric axiomatization is found:
$\exists x \exists y \exists z (x \neq y \,\&\, z \notin ln(x, y))$

Properties of coherent theories

For the time being we ignore infinitary theories: so we will just discuss coherence. “Geometric” is often used synonymously with “coherent”, and *vice versa*.

“Barr’s Theorem”: Coherent implications form a “Glivenko Class”, i.e. if a sequent $I_1, \ldots, I_n \Rightarrow I_0$ is provable classically, then it can be proved intuitionistically, provided each I_i is a coherent implication.

Coherent theories (i.e. those axiomatised by coherent implications) are preserved by pullback along geometric morphisms between topoi.

Coherent theories are those whose class of models is closed under directed limits.

Coherent theories are “exactly the theories expressible by natural deduction rules in a certain simple form in which only atomic formulas play a critical part” (Simpson 1994).

Coherent implications can be converted directly to inference rules in such a fashion that admissibility of the structural rules of the underlying calculus is unaffected (Negri 2003).

Conversion of coherent implications to rules: Example

- The coherent implication $\forall xyz.(x \leq y \wedge y \leq z) \supset (y \leq x \vee z \leq y)$ is converted to the inference rule

$$\frac{y \leq x, x \leq y, y \leq z, \Gamma \Rightarrow \Delta \qquad z \leq y, x \leq y, y \leq z, \Gamma \Rightarrow \Delta}{x \leq y, y \leq z, \Gamma \Rightarrow \Delta}$$

(If $\leq$ is a partial order, this says that the depth is at most 2.)

- The coherent implication $\forall xyz.(x \leq y \wedge x \leq z) \supset \exists w(y \leq w \wedge z \leq w)$ is converted to the inference rule (in which w is fresh, i.e. not in the conclusion):

$$\frac{y \leq w, z \leq w, x \leq y, x \leq z, \Gamma \Rightarrow \Delta}{x \leq y, x \leq z, \Gamma \Rightarrow \Delta}$$

(If $\leq$ is a partial order, this says that it is “strongly directed”.)

Conversion, in general

Use a canonical form for geometric implications: they are equivalent to conjunctions of formulas

$$\forall \overline{x}(P_1 \& \dots \& P_m \supset \exists \overline{y}_1 M_1 \vee \cdots \vee \exists \overline{y}_n M_n)$$

P_i atomic formula
$M_i \equiv \bigwedge_j Q_{i_j}$ conjunction of atomic formulas
none of the variables in $\overline{y}_j$ are free in P_i.
each is converted to a rule of the form

$$\frac{\overline{Q}_1(\overline{z}_1/\overline{y}_1), \overline{P}, \Gamma \Rightarrow \Delta \quad \dots \quad \overline{Q}_n(\overline{z}_n/\overline{y}_n), \overline{P}, \Gamma \Rightarrow \Delta}{\overline{P}, \Gamma \Rightarrow \Delta}\, GRS$$

- The eigenvariables z_i must not be free in $\overline{P}, \Gamma, \Delta$.

Coherent/geometric theories are not enough

- *Axiom of non-collinearity* of classical projective and affine geometry

$$\exists x \exists y \exists z (\neg x = y \,\&\, \neg z \in ln(x, y))$$

- *Axiom of existence of a least upper bound*

$$\forall xy \exists z ((x \leqslant z \,\&\, y \leqslant z) \,\&\, \forall w (x \leqslant w \,\&\, y \leqslant w \supset z \leqslant w))$$

are not geometric implications.

Co-geometric theories

- a formula is *co-geometric* if it does not contain $\supset$ or $\exists$.

- a *co-geometric implication* has the form, with A and B co-geometric formulas,

$$\forall x \ldots \forall z(A \supset B)$$

- canonical form: conjunctions of
$\forall \overline{x}(\forall y_1 M_1 \& \ldots \& \forall y_n M_n \supset P_1 \vee \cdots \vee P_m)$
with the M_i disjunctions of atoms

- classical projective and affine geometries with the axiom of non-collinearity are co-geometric: write non-collinearity as

$$\neg \forall x \forall y \forall z(x = y \vee z \in ln(x, y))$$

$$\frac{\Gamma \Rightarrow \Delta, x = y, z \in ln(x, y)}{\Gamma \Rightarrow \Delta}\, \text{\textit{Non-coll}}$$

(x, y, z not free in the conclusion)

Co-geometric theories

Formulate **right** rules as mirror images of geometric rules:

$$\frac{\Gamma \to \Delta, P_{1_1}, \ldots, P_{1_k} \quad \ldots \quad \Gamma \to \Delta, P_{m_1}, \ldots, P_{m_l}}{\Gamma, \to \Delta, Q_1, \ldots, Q_n,}\, R$$

The P_i can contain eigenvariables

Basic proof-theoretical results go through as for geometric rules

The duality between geometric and co-geometric theories used for changing the primitive notions in the sequent formulation of a theory.

Meta-theoretical results imported from one theory to its dual by exploiting the symmetry of their associated sequent calculi.

Herbrand's theorem for geometric and co-geometric theories (N and von Plato 2011).

A motivating problem: Knowability logic

Epistemic conceptions of truth justify the *knowability principle*:

If A is true, then it is possible to know that A $A \supset \Diamond\mathcal{K}A$ (**KP**)

The Church-Fitch paradox (1945–1962): formal derivation from the knowability principle to *(collective) omniscience*:

All truths are actually known $A \supset \mathcal{K}A$ (**OP**)

The main goal has been to show that the paradox does not affect an *intuitionistic* conception of truth.

The derivation of the paradox is indeed done in classical logic. Intuitionistic logic proves its negative version, but to prove intuitionistic *underivability* of the positive version, a careful proof analysis is needed.

Knowability logic

Intuitionistic logic blocks the derivation of **OP** from the Moore-instance ($A \& \neg \mathcal{K} A$) of **KP**. Is this enough? No!

So the goal has been to develop a proof theory for *knowability logic*: a cut-free sequent system for bimodal logic extended by the knowability principle.

The knowability principle does not reduce to atomic instances, so it cannot be translated into rules through the methodology of “axioms as rules”.

Something similar can be done...

Labelled Calculi for Modal and Intuitionistic Logics

Labelled sequent calculi obtained by the internalization of Kripke semantics (Negri 2005)

Something similar (Dyckhoff & Negri 2012) can be done for intuitionistic logic, with special rules for implication using a partial order $\leq$ (including reflexivity and transitivity) as the accessibility relation.

Rules for implication are then:

$$\frac{x \leq y, x : A \to B, \Gamma \Rightarrow y : A, \Delta \quad x \leq y, x : A \to B, y : B, \Gamma \Rightarrow \Delta}{x \leq y, x : A \to B, \Gamma \Rightarrow \Delta} \; L\to$$

and, with y fresh,

$$\frac{x \leq y, y : A, \Gamma \Rightarrow y : B, \Delta}{\Gamma \Rightarrow x : A \to B, \Delta} \; R\to$$

A labelled calculus for knowability logic

Extend the intuitionistic labelled calculus by rules for $\mathcal{K}$ and $\Diamond$:

- $x \Vdash \mathcal{K}A$ iff for all y, $xR_{\mathcal{K}}y$ implies $y \Vdash A$
- $x \Vdash \Diamond A$ iff for some y, $xR_{\Diamond}y$ and $y \Vdash A$

The clauses are converted into rules:

$$\frac{y : A, x : \mathcal{K}A, xR_{\mathcal{K}}y, \Gamma \Rightarrow \Delta}{x : \mathcal{K}A, xR_{\mathcal{K}}y, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\mathcal{K}} \qquad \frac{xR_{\mathcal{K}}y, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \mathcal{K}A}\ {\scriptstyle R\mathcal{K}}$$

$$\frac{xR_{\Diamond}y, y : A, \Gamma \Rightarrow \Delta}{x : \Diamond A, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\Diamond} \qquad \frac{xR_{\Diamond}y, \Gamma \Rightarrow \Delta, x : \Diamond A, y : A}{xR_{\Diamond}y, \Gamma \Rightarrow \Delta, x : \Diamond A}\ {\scriptstyle R\Diamond}$$

In $R\mathcal{K}$ and $L\Diamond$, y does not appear in Γ and Δ

From (modal) axioms to (frame) rules

Recall that various extensions are obtained by adding the frame properties that correspond to the added axioms, for example

Logic	Axiom	Frame property	Rule
T	$\Box A \supset A$	$\forall x\, xRx$ reflexivity	$\dfrac{xRx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$
4	$\Box A \supset \Box\Box A$	$\forall xyz(xRy \,\&\, yRz \supset xRz)$ trans.	$\dfrac{xRz, \Gamma \Rightarrow \Delta}{xRy, yRz, \Gamma \Rightarrow \Delta}$
E	$\Diamond A \supset \Box\Diamond A$	$\forall xyz(xRy \,\&\, xRz \supset yRz)$ euclid.	$\dfrac{yRz, \Gamma \Rightarrow \Delta}{xRy, xRz, \Gamma \Rightarrow \Delta}$
B	$A \supset \Box\Diamond A$	$\forall xy(xRy \supset yRx)$ symmetry	$\dfrac{yRx, \Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta}$
D	$\Box A \supset \Diamond A$	$\forall x \exists y\, xRy$ seriality	$\dfrac{xRy, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, y$
2	$\Diamond\Box A \supset \Box\Diamond A$	$\forall xyz(xRy \,\&\, xRz \supset \exists w(yRw \,\&\, zRw))$	$\dfrac{yRw, zRw, \Gamma \Rightarrow \Delta}{xRy, xRz, \Gamma \Rightarrow \Delta}\, w$
W	$\Box(\Box A \supset A) \supset \Box A$	trans., irref., and no infinite R-chains	modified $\Box$ rules

but *knowability* is different from all such cases...

Finding the right rules for knowability logic

The calculus itself is used to find the frame condition and the rules needed, by root-first proof search:

$$\frac{}{\Rightarrow x : A \supset \Diamond \mathcal{K} A}$$

Finding the right rules for knowability logic

The calculus itself is used to find the frame condition and the rules needed, by root-first proof search:

$$\dfrac{\overline{x \leqslant y, y : A \Rightarrow y : \Diamond \mathcal{K} A}}{\Rightarrow x : A \supset \Diamond \mathcal{K} A}\ R\supset$$

Finding the right rules for knowability logic

The calculus itself is used to find the frame condition and the rules needed, by root-first proof search:

$$\dfrac{\dfrac{\overline{x \leqslant y, yR_\Diamond z, y : A \Rightarrow y : \Diamond\mathcal{K}A}}{x \leqslant y, y : A \Rightarrow y : \Diamond\mathcal{K}A}\; Ser_\Diamond}{\Rightarrow x : A \supset \Diamond\mathcal{K}A}\; R\supset$$

Finding the right rules for knowability logic

The calculus itself is used to find the frame condition and the rules needed, by root-first proof search:

$$\dfrac{\dfrac{\dfrac{\overline{x \leqslant y, yR_{\Diamond}z, y : A \Rightarrow y : \Diamond\mathcal{K}A, z : KA}}{x \leqslant y, yR_{\Diamond}z, y : A \Rightarrow y : \Diamond\mathcal{K}A}\, R\Diamond}{x \leqslant y, y : A \Rightarrow y : \Diamond\mathcal{K}A}\, Ser_{\Diamond}}{\Rightarrow x : A \supset \Diamond\mathcal{K}A}\, R{\supset}$$

Finding the right rules for knowability logic

The calculus itself is used to find the frame condition and the rules needed, by root-first proof search:

$$\cfrac{\cfrac{\cfrac{\cfrac{\overline{x \leqslant y, yR_\Diamond z, zR_\mathcal{K} w, y : A \Rightarrow y : \Diamond\mathcal{K}A, w : A}}{x \leqslant y, yR_\Diamond z, y : A \Rightarrow y : \Diamond\mathcal{K}A, z : KA}\ R\mathcal{K}}{x \leqslant y, yR_\Diamond z, y : A \Rightarrow y : \Diamond\mathcal{K}A}\ R\Diamond}{x \leqslant y, y : A \Rightarrow y : \Diamond\mathcal{K}A}\ Ser_\Diamond}{\Rightarrow x : A \supset \Diamond\mathcal{K}A}\ R\supset$$

Finding the right rules for knowability logic

The calculus itself is used to find the frame conditions and the rules needed, by root-first proof search:

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{x \leqslant y, \overline{y \leqslant w}, yR_{\Diamond} z, zR_{\mathcal{K}} w, \overline{y : A} \Rightarrow y : \Diamond\mathcal{K}A, \overline{w : A}}{x \leqslant y, yR_{\Diamond} z, zR_{\mathcal{K}} w, y : A \Rightarrow y : \Diamond\mathcal{K}A, w : A}\ \Diamond\mathcal{K}\text{-}Tr}{x \leqslant y, yR_{\Diamond} z, y : A \Rightarrow y : \Diamond\mathcal{K}A, z : KA}\ R\mathcal{K}}{x \leqslant y, yR_{\Diamond} z, y : A \Rightarrow y : \Diamond\mathcal{K}A}\ R\Diamond}{x \leqslant y, y : A \Rightarrow y : \Diamond\mathcal{K}A}\ Ser_{\Diamond}}{\Rightarrow x : A \supset \Diamond\mathcal{K}A}\ R\supset
$$

the uppermost sequent is derivable by *monotonicity*.

Finding the right rules for knowability logic (cont.)

The two extra-logical rules used are:

$$\frac{xR_\Diamond y, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Ser}_\Diamond \qquad \frac{x \leqslant z, xR_\Diamond y, yR_\mathcal{K} z, \Gamma \Rightarrow \Delta}{xR_\Diamond y, yR_\mathcal{K} z, \Gamma \Rightarrow \Delta}\ \Diamond\mathcal{K}\text{-}\mathit{Tr}$$

Ser$_\Diamond$ has the condition $y \notin \Gamma, \Delta$. The rules correspond to the frame properties

$$\forall x \exists y . xR_\Diamond y \qquad \textbf{Ser}_\Diamond$$

$$\forall x \forall y \forall z (xR_\Diamond y \,\&\, yR_\mathcal{K} z \supset x \leqslant z) \qquad \Diamond\mathcal{K}\textbf{-Tr}$$

The universal frame property $\Diamond\mathcal{K}$**-Tr** is, however, **too strong**: The instance of rule $\Diamond\mathcal{K}$-*Tr* used in the derivation of **KP** is not applied (root first) to an arbitrary sequent, but to one in which the middle term is the eigenvariable introduced by *Ser*$_\Diamond$.

So we have the requirements:

- *$\Diamond\mathcal{K}$-Tr has to be applied above Ser$_\Diamond$*
- The middle term of $\Diamond\mathcal{K}$-*Tr* is the eigenvariable of *Ser*$_\Diamond$.

Finding the right rules for knowability logic (cont.)

The system of rules is equivalent to the frame property

$$\forall x \exists y (x R_{\diamond} y \,\&\, \forall z (y R_{\mathcal{K}} z \supset x \leqslant z)) \quad \textbf{KP-Fr}$$

The system with rules $\diamond\mathcal{K}$-*Tr* and $Ser_{\diamond}$ that respect the side condition is a cut-free equivalent of the system that employs ***KP-Fr*** as an axiomatic sequent in addition to the structural rules.

The rules that correspond to **KP-Fr do not** follow the geometric rule scheme. However, all the structural rules are still admissible in the presence of such rules.

The system obtained by the addition of suitable combinations of these two rules provides a complete contraction- and cut-free system for the *knowability logic* **G3KP**, that is, intuitionistic bimodal logic extended with **KP**.

Intuitionistic solution to Fitch's paradox through an exhaustive proof analysis in **KP**: **OP** is **not** derivable in **G3KP**.
cf. Maffezioli, Naibo, and N (2012).

3. Generalizing geometric implications

(Negri 2013)

Normal form for geometric implications:

$$\forall \overline{x}(\& P_i \supset \exists y_1 M_1 \vee \cdots \vee \exists y_n M_n) \qquad GA$$

P_i atomic formulas, M_j conjunctions of atomic formulas $Q_{j_1} \; \& \; \ldots \; \& \; Q_{j_{k_j}}$, y_j not free in the P_i.

Geometric implications are taken as the base case in the inductive definition of generalized geometric implications.

$GA_0 \equiv GA \qquad GRS_0 \equiv GRS$

$GA_1 \equiv \forall \overline{x}(\& P_i \supset \exists y_1 \& GA_0 \vee \cdots \vee \exists y_m \& GA_0)$

$GA_{n+1} \equiv \forall \overline{x}(\& P_i \supset \exists y_1 \& GA_n \vee \cdots \vee \exists y_m \& GA_n)$

here $\& GA_i$ denotes a conjunction of GA_i-axioms.

Examples

- *Frame condition for the* knowability principle, $A \supset \Diamond\mathcal{K}A$ *is* $\forall x \exists y (xR_\Diamond y \,\&\, \forall z(yR_\mathcal{K} z \supset x \leqslant z))$ *and corresponds to the system of rules*

$$\begin{cases} \dfrac{xR_\Diamond y, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Ser_\Diamond\ (y\ fresh) \\[2em] \dfrac{x \leqslant z, xR_\Diamond y, yR_\mathcal{K} z, \Gamma \Rightarrow \Delta}{xR_\Diamond y, yR_\mathcal{K} z, \Gamma \Rightarrow \Delta}\ \Diamond\mathcal{K}\text{-}Tr \end{cases}$$

- *Continuity axiom* $\forall \epsilon \exists \delta \forall x (x \in B(\delta) \supset f(x) \in B(\epsilon))$ *is in* GA_1.
- *The class* GA_1 *does not require the presence of quantifiers:* $(P \supset Q) \vee (Q \supset P)$ *is in* GA_1 *(a degenerate case without variables).*

  The system of rules has the form

$$\dfrac{\begin{array}{c} \dfrac{Q, P, \Gamma' \Rightarrow \Delta'}{P, \Gamma' \Rightarrow \Delta'} \\ \vdots \\ \Gamma \Rightarrow \Delta \end{array} \qquad \begin{array}{c} \dfrac{P, Q, \Gamma'' \Rightarrow \Delta''}{Q, \Gamma'' \Rightarrow \Delta''} \\ \vdots \\ \Gamma \Rightarrow \Delta \end{array}}{\Gamma \Rightarrow \Delta}$$

The Sahlqvist fragment

- Conversion into systems of rules works in full generality for the class of generalized geometric implications
- gives operative characterisation in terms of Glivenko classes: generalised geometric implication $\equiv$ no negative $\supset$, $\forall$
- Kracht formulas (frame correspondents of Sahlqvist formulas) belong to the same class
- thus conversion of g.g.i. into rules gives proof systems for all the Sahlqvist fragment

Other motivations, and another solution

(Dyckhoff and Negri 2014)

Not all theories are axiomatised by geometric implications, e.g.

- The "McKinsey condition" (a frame condition for modal logic, related to the McKinsey axiom $\Box\Diamond A \supset \Diamond\Box A$)

  $$\forall x \exists y.\ xRy \wedge (\forall z.\ yRz \supset y = z)$$

  is not a coherent implication.
- The "strict seriality condition"

  $$\forall x.\ \exists y.\ xRy \wedge \neg(yRx)$$

are not geometric implications; they are are generalised geometric implications; such theories can be treated as systems of rules BUT:

- Systems of rules are non-local, require bookkeeping of the dependence of eigenvariables
- Not all first-order axioms are generalised geometric implications, e.g. $\forall xyx(xRy \& yRz \supset xRz) \supset \forall v(\neg vRv)$

Solutions

The technique we adopt is called “atomisation”, and was introduced by Skolem (1920) (in his proofs of Löwenheim’s theorems). Also called “Morleyization”.

- Introduce a new unary predicate symbol M (for Maximal), and replace the McKinsey condition $\forall x \exists y.\ xRy \wedge (\forall z. yRz \supset y = z)$ by two geometric implications:

$$\forall x(\exists y.\ xRy \wedge M(y))$$

$$\forall yz(M(y) \wedge yRz \supset y = z)$$

- Introduce a new binary predicate symbol S, and replace the strict seriality condition $\forall x. \exists y.\ xRy \wedge \neg(yRx)$ by two geometric implications:

$$\forall x \exists y. xRy \wedge ySx$$

$$\forall xy(xRy \wedge xSy \supset \bot)$$

What is going on here ?

Conservative and Skolem Extensions

Let $\mathcal{T}$ be a theory in a language $\mathcal{L}$ and $\mathcal{L}'$ be a language extending $\mathcal{L}$. A theory $\mathcal{T}'$ in $\mathcal{L}'$ is a conservative extension of $\mathcal{T}$ iff (i) every theorem of $\mathcal{T}$ is a theorem of $\mathcal{T}'$ and (ii) every theorem of $\mathcal{T}'$ expressed in $\mathcal{L}$ is a theorem of $\mathcal{T}$.

So, rather than proving theorems (expressed in $\mathcal{L}$) in $\mathcal{T}$ it suffices to prove them in $\mathcal{T}'$, where it may be easier.

A related condition is that of being a “Skolem extension”. $\mathcal{T}'$ is a Skolem extension of $\mathcal{T}$ iff (i) every theorem of $\mathcal{T}$ is a theorem of $\mathcal{T}'$ and (ii) for some substitution of $\mathcal{L}$-formulae for predicate symbols not in $\mathcal{L}$, every theorem of $\mathcal{T}'$ becomes a theorem of $\mathcal{T}$.

Results

Theorem If $\mathcal{T}'$ is a Skolem extension of $\mathcal{T}$, then (i) it is a conservative extension and (ii) they are satisfiable in the same domains.

Proof Routine.

Theorem Every first-order axiomatic theory has a Skolem extension axiomatised by coherent implications.

Proof Generalisation of the two tricks given above: one can replace each axiom by its prenex normal form, with the body in DNF, and use one technique to strip off pairs $\forall\mathbf{x}\exists\mathbf{y}$ of quantifiers and the other technique to get rid of negated atoms. New predicate symbols are introduced, along with coherent implications that (partially) express their meanings.

Corollary Every first-order axiomatic theory has a conservative extension axiomatised by coherent implications.

Corollary Every first-order axiomatic theory has an $\forall\exists$-extension with models in the same domains.
Remark This corollary is the result proved by Skolem 1920, as a means to simplify the proofs of Löwenheim's theorems about cardinality of models.

A view through rules

Translation into rules

- Gives an alternative (simpler) proof of the conservativity result
- Defines the rule system equivalent to the given first-order theory
- Extends the labelled approach to arbitrary first-order frame conditions

Labelled Calculi for Intermediate Logics

Intermediate (aka "superintuitionistic") logics are those, e.g. Gödel-Dummett logic and Jankov-De Morgan logic, between intuitionistic and classical logic.

They can usually be presented using some "frame conditions", e.g. $\forall xy.\ (x \leq y) \vee (y \leq x)$ for G-D logic; so we need to incorporate such conditions into the rules of the sequent calculus.

Where (as here) the condition is coherent, this is easy (and we can restrict to cases where x and y are in the conclusion):

$$\frac{x \leq y, \Gamma \Rightarrow \Delta \quad y \leq x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

and we also need reflexivity (for x in the conclusion) and transitivity:

$$\frac{x \leq x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Refl} \qquad \frac{x \leq z, x \leq y, y \leq z, \Gamma \Rightarrow \Delta}{x \leq y, y \leq z, \Gamma \Rightarrow \Delta}\ \mathit{Trans}.$$

Labelled Calculi for Intermediate Logics 2

Not all frame conditions are coherent. E.g., (i) the McKinsey condition (in modal logic) above and (ii) that for the Kreisel-Putnam (intermediate) logic, axiomatised by $(\neg A \to (B \vee C)) \to ((\neg A \to B) \vee (\neg A \to C))$.

The condition is

$$\forall xyz.\ (x \leq y\ \wedge\ x \leq z)\ \supset\ (y \leq z\ \vee\ z \leq y\ \vee\ \exists u.(x \leq u\ \wedge\ u \leq y\ \wedge\ u \leq z\ \wedge\ F(u, y, z)))$$

where $F(u, y, z)$ abbreviates
$\forall v.\ u \leq v \supset \exists w.\ (v \leq w\ \wedge\ (y \leq w \vee z \leq w))$;

By changing F from an abbreviation to a new predicate symbol with an associated coherent implication

$$\forall uvyz.\ (F(u, y, z) \wedge u \leq v) \supset (\exists w(v \leq w \wedge y \leq w) \vee \exists w(v \leq w \wedge z \leq w))$$

we achieve our goal of making the condition coherent. The theory developed above about conservative extensions formalises this idea.

So, the conversion of non-coherent frame conditions to two (or more) coherent ones allows the application of automated coherent reasoning to be applied to a wider range of modal and intermediate logics.

To sum up...

Logical constant	**Left rule**	**Right rule**
&	,	branching
$\vee$	branching	,
$\supset$	split in succ./ant.	split in ant./succ.
positive $\exists$ (geometric axiom)	variable condition	—
negative $\forall$ (cogeometric axiom)	—	variable condition
quantifier alternations beyond $\forall\exists$ generalised geometric implications	systems of rules (same language intuitionistically equivalent)	–
quantifier alternations beyond $\forall\exists$ arbitrary first-order axioms	geometric rules (classical conversion and conservative extension)	–

Prior and Related Work

Was such a simple result really not known before ?

- Skolem (1920) used atomisation to replace a f.-o. formula by a single $\forall\exists$-formula "satisfiable in the same domains". Our technique is a modification of his.
- Antonius (1975) used a similar technique to replace a f.-o. formula by a positive formula (plus lots of coherent implications), with a theorem apparently weaker than what we need. But nevertheless a conservativity result can be proved using her translation.
- Johnstone (2002) [Lemma D.1.5.13, p 858] expresses a similar result in terms of models in Boolean coherent categories. Privately he says he learnt the idea from Sacks in Chicago in 1975–76, but knows no publication prior to his Compendium (2002).
- Bezem & Coquand (2005) use another similar technique: their result is just about satisfiability (the original formula is unsatisfiable iff its coherentisation—lots of coherent implications—is unsatisfiable). Not quite as strong as what we need. Mints (2012) is similar. But nevertheless a conservativity result can be proved using their translation.

Opportunities and Challenges

Better automation: Polonsky's thesis (2010) (building on B & C 2005) is the state of the art. Relevant work also by Giese et al in Oslo.

Better conversion of first-order formulae: again, Polonsky. Lots of opportunities for optimisation and tricks. For example, in $\forall \mathbf{x}.\ C \supset (P \vee \neg Q)$ one would do better to construct $\forall \mathbf{x}.\ (C \wedge Q) \supset P$ than to introduce a new symbol for $\neg Q$. Likewise, $\forall \mathbf{x}.\ C \supset (P \wedge \neg Q)$ splits into $\forall \mathbf{x}.\ C \supset P$ and $\forall \mathbf{x}.\ C \wedge Q \supset \bot$. Not so easy in presence of $\exists$.

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