Computational Semantics for Dependent Type Theories

Samson Abramsky

Department of Computer Science, University of Oxford
Motivation

HoTT is hot!!

The HoTT book, IAS year, Awodey $7M grant, . . .

HoTT is essentially (intensional) Martin-Löf type theory with one new axiom. Much of the current energy and enthusiasm stems from the discovery of a new semantics in homotopy types.

Question 1:
Can we find a natural intensional computational semantics for HoTT/DTT?

Question 2:
Can we combine linear and dependent types?
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Sketch of a programme

- Attempt to find a game semantics for HoTT (or DTT generally).
- Joint work with Radha Jagadeesan, Kohei Kishida, Matthijs Vákár, Norihiro Yamada.
- Still in a preliminary stage.

What we have so far:
- A coherence space semantics (SA and MV, paper soon)
- Progress towards a game semantics (RJ)
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The Plan

Give a relaxed talk, emphasise ideas rather than technical details.

Firstly:

Why Games? Why Coherence Spaces?

Give a "native" intensional semantics coming from the computational side. Naturally linear, so points to the way to linear forms of dependent type theory.

Computational/informatic content:

Information, time vs. Geometry, space

Can exploit techniques for building universal domains, solving "domain equations" (i.e. reflexive type equations).
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Give a *relaxed* talk 😊
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The ideas **behind** semantics

Examples:

- Homotopy semantics for DTT give rise to HoTT
- Coherence space semantics gave rise to Linear Logic

Splitting the atom of computation, twice: a voyage through semantics

A basic setting: higher-order (partial) functions

- Type 0: $N$
- Type 1: $N \to N$
- Type 2: $[N \to N] \to N$

Functionals are not so unfamiliar: e.g. the quantifiers!

- $\forall, \exists$: $[N \to B] \to B$
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Not just a tool for metamathematics . . .
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\forall, \exists : [\mathbb{N} \rightarrow \mathbb{B}] \rightarrow \mathbb{B}
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Splitting the Atom I

Set-theoretically, higher-order functions are monsters. E.g. $F : \mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{N}$ is defined by its graph:

$\text{graph}(F) = \{ (f, n) \mid f \in \text{dom}(F) \}$

Each ordered pair in the graph may contain an infinite amount of information.

The first splitting: look at finite pieces of information.

Information tokens:

$t := (\{ (n_1, m_1), \ldots, (n_k, m_k) \}, m)$

$F | = t \equiv \forall f. [ \bigwedge_{i=1}^{k} f(n_i) = m_i ] \rightarrow F(f) = m$

$F$ should be determined by the finite pieces of information it satisfies.
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The coherence space revolution

Gave birth to Linear Logic (Girard 1987).

A radically simple form of token-based finite information semantics.

So simple that:

- It makes visible the linear decompositions of intuitionistic types
- It allows classical dualities at the linear level

On the other hand, not closed under lifting!

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Review of Coherence Spaces

The underlying structure of a coherence space is just an undirected reflexive graph. The coherence space is then the set of cliques of the graph, ordered by set inclusion.

The novelty of coherence spaces lies in the constructions on reflexive graphs, which lead to a categorical model of full Classical Linear Logic — indeed, this model gave rise to Linear Logic.

Notation: $|X|$, $\vdash \dashv X$. $|X|$ is the set of tokens.

We write $\dashv$ for the irreflexive part of $\vdash \dashv$, $\vdash$ for the complement of $\vdash \dashv$ (which is irreflexive), and $\dashv \vdash$ for the complement of $\dashv$. 
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Notation: $(|X|, \subseteq_X)$. $|X|$ is the set of tokens.

We write $\bowtie$ for the irreflexive part of $\subseteq$, $\bar{\subseteq}$ for the complement of $\subseteq$ (which is irreflexive), and $\bar{\bowtie}$ for the complement of $\bowtie$. 
Linear Type Constructions on Coherence Spaces

The linear type constructions combine set-theoretic operations on the token sets with logical operations on the coherence relations.

Linear negation

Given a coherence space $X$, we define $X^\perp$ with $|X^\perp| = |X|$, and $⌣⌢X^\perp = ⌢⌣X$.

Note that $X^{\perp\perp} = X$.

Multiplicatives

Given coherence spaces $X$, $Y$:

$|X \otimes Y| = |X \O Y| = |X \Rightarrow Y| := |X| \times |Y|$.

$(a, b) \⌣⌢(c, d) \equiv a \⌣⌢c \mod X \land b \⌣⌢d \mod Y$.

$(a, b) \⌢(c, d) \equiv a \⌣⌢c \mod X \lor b \⌣⌢d \mod Y$.

$(a, b) \Rightarrow(c, d) \equiv a \⌣⌢c \mod X \Rightarrow b \⌣⌢d \mod Y$.

Note that $X \O Y = (X^\perp \otimes Y^\perp)^\perp$, $X \Rightarrow Y = X^\perp \O Y$.

N.B. Interdefinabilities follow from propositional calculus!
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$$(a, b) \odot (c, d) \text{ mod } X \otimes Y \equiv a \odot c \text{ mod } X \land b \odot d \text{ mod } Y$$  

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Linear Type Constructions ctd

Additives

Given coherence spaces $X, Y$:

$$|X \oplus Y| = |X| \cap |Y| := |X| + |Y|.$$  

We represent the disjoint union $|X| \cap |Y|$ concretely as $X \times \{0\} \cup Y \times \{1\}$.

$$(a, i) \dashv \vdash (b, j) \mod |X \oplus Y| \equiv i = j \land a \dashv \vdash b \mod Z_i (a, i) \dashv \vdash (b, j) \mod |X \cap Y| \equiv i = j \Rightarrow a \dashv \vdash b \mod Z_i$$

where $Z_0 = X$ and $Z_1 = Y$.

Exponentials

Given a coherence space $X$, we define $|!X| = X^{\text{fin}}$ with $x \dashv \vdash y \mod !X \equiv x \cup y \in X$. 

Note that $X \cap Y = (X \perp \perp Y \perp \perp)$.
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**Additives** Given coherence spaces $X$, $Y$:

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**Exponentials** Given a coherence space $X$, we define

$$|!X| = X_{\text{fin}}$$

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Stable functions and traces

We recall that a function \( f : X \to Y \) is stable if it is monotone, and preserves directed unions and pull-backs.

Proposition

A monotone and continuous function \( f : X \to Y \) is stable if and only if, whenever \( b \in f(x) \) for \( x \in X \), there exists a finite clique \( s \subseteq x \) such that \( b \in f(s) \), and moreover for all \( s' \subseteq x \) such that \( b \in f(s') \), \( s \subseteq s' \).

We define the trace of a stable function:

\[
\text{Tr}(f) := \left\{ (s, b) \in X_{\text{fin}} \times \mathcal{P}(Y) \mid b \in f(s) \land \forall s' \subseteq s, b \in f(s') \Rightarrow s' = s \right\}.
\]

Proposition

A stable function can be uniquely recovered from its trace:

\[
f(x) = \left\{ b \mid \exists (s, b) \in \text{Tr}(f), s \subseteq x \right\}.
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The trace defines a bijection between stable functions \( f : X \to Y \) and the points of the coherence space \( X \Rightarrow_Y := !X \to Y \).
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Splitting the Atom II: Games

Consider a token such as $\{(3,2), (4,3)\}$. Many possible "schedules" for realising such a token as a process. E.g. $(N \Rightarrow N) \Rightarrow N^q_3$. $q_2^q_4^3^1$. 

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Consider a token such as \(((3, 2), (4, 3)), 1\).
Splitting the Atom II: Games
Consider a token such as $\{(3, 2), (4, 3)\}, 1$.

Many possible “schedules” for realising such a token as a process.
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Consider a token such as \( \{(3, 2), (4, 3)\}, 1 \).

Many possible “schedules” for realising such a token as a process. E.g.

\[
\begin{array}{ccc}
( & N & \Rightarrow & N & ) & \Rightarrow & N \\
 & q & & q & & q & & q \\
 & 3 & & 2 & & \cdot & & \cdot \\
 & 4 & & \cdot & & \cdot & & \cdot \\
 & 1 & & \cdot & & \cdot & & \cdot \\
\end{array}
\]
Coherence spaces as a collapse of game semantics

Forgetful map from schedules — i.e. plays in a game semantics — to tokens. Game semantics decomposes information tokens into finer structures. New subtleties arise: Partial tokens Sequential definability. E.g. consider composition. New possibilities also arise: Full abstraction and full completeness Compositional model-checking So we view the coherence space semantics as a stepping stone to a game semantics.
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So we view the coherence space semantics as a stepping stone to a game semantics.
Dependent Type Theory

Key ideas (over and above simple propositional type theories):
- Types depending on values of other types (syntactically, on terms).
- Thus no clean separation between syntax of types and terms; and order in
contexts is important!
- Identity types.
- Universes.

Judgement forms:
- $\Gamma \vdash \Gamma$ is a valid context
- $\Gamma \vdash \sigma$ type
- $\Gamma \vdash M : \sigma$ $M$ is a term of type $\sigma$ in context $\Gamma$

Plus equality judgements for contexts, types and terms.

Too many rules!
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\[ \vdash \Gamma \text{ ctxt} \quad \Gamma \text{ is a valid context} \]

\[ \Gamma \vdash \sigma \text{ type} \quad \sigma \text{ is a type in context } \Gamma \]

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Context Formation

Rules for context formation:

\[ \diamond \quad \text{ctxt} \quad \frac{\Gamma \vdash \sigma \text{ type}}{\Gamma, x : \sigma \text{ ctxt}} \]
Context Formation

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Note the mutual recursion between type and context formation.
Context Formation

Rules for context formation:

\[ \Gamma \vdash \sigma \text{ type} \]
\[ \emptyset \text{ ctxt} \]
\[ \Gamma, x : \sigma \text{ ctxt} \]

Note the mutual recursion between type and context formation.

A useful intuition:

In interpreting the simply-typed \( \lambda \)-calculus in a cartesian closed category, we interpret the comma in contexts as a product, and the turnstile in typing judgements as function space. Thus a type judgement

\[ A_1, \ldots, A_n \vdash t : A \]

is interpreted as a morphism

\[ f : A_1 \times \cdots \times A_n \to A. \]
Context Formation

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In interpreting dependent types, we should interpret the comma in contexts as a dependent sum, and the turnstile as a dependent product. This is the appropriate way to think of dependent type families.
Dependent Sums and Products
Dependent Sums and Products

Same type formation rules:

\[\begin{array}{c}
\Gamma \vdash \sigma \text{ type} \\
\Gamma, x : \sigma \vdash \tau \text{ type} \\
\hline
\Gamma \vdash \Pi x : \sigma.\tau \text{ type}
\end{array}\]

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\]

The introduction rules:

\[
\Gamma, \, x : \sigma \vdash M : \tau \\
\quad \quad \Gamma \vdash \lambda x : \sigma.\! M : \Pi x : \sigma.\tau \\
\quad \quad \Gamma \vdash \text{Pair} (M, N) : \Sigma x : \sigma.\tau
\]

The "canonical elements" of these types are those given by the introduction rules. The elimination rules give application and projections respectively. The equality rules give the usual \(\beta\)-conversions.
Dependent Sums and Products

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\end{align*}
\]

\[
\begin{align*}
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Identity Types

Formation: $\Gamma \vdash M : \sigma$

$\Gamma \vdash N : \sigma$

$\Gamma \vdash \text{Id}_\sigma(M, N)$ type

Introduction: $\Gamma \vdash M : \sigma$

$\Gamma \vdash \text{Refl}_\sigma(M) : \text{Id}_\sigma(M, M)$

The reflexivity terms are the canonical elements.

But in the intensional theory, there can be many other elements.

The homotopy interpretation: paths!

i.e. witnesses giving ways in which one object can be transformed into another.

Gives a completely new, positive way of thinking about intensional identity types.
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Towards a coherence space semantics

It is useful to define the extended trace, which gives minimum data points for finite outputs:

\[
\text{Tr}^* (f) := \{(s, t) \in X_{\text{fin}} \times Y_{\text{fin}} | t \subseteq f(s) \land \forall s' \subseteq s. t \subseteq f(s') \Rightarrow s' = s\}.
\]

Note that this extended trace can be derived from the basic version:

\[
(s, \{b_1, \ldots, b_n\}) \in \text{Tr}^* (f) \iff s = \bigcup_{i=1}^{n} s_i \land (s_i, b_i) \in \text{Tr}(f), i = 1, \ldots, n.
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We shall show how to construct a category with families, a standard notion of semantics for DTT, based on coherence spaces.
Parameterizations

We define an order relation on coherence spaces: $X \equiv |X| \subseteq |Y| \land \downarrow \uparrow X = \downarrow \uparrow Y \cap |X|^2$. This yields a large cpo $(\text{Coh}, \cdot)$ on the class of coherence spaces. This cpo is algebraic, since every graph is the directed union of its finite sub-graphs, and has pull-backs.

A parameterization on a coherence space $X$ is a stable, continuous function $F : X \to \text{Coh}$. We write $\text{Par}(X)$ for the set of parameterizations on $X$. 
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A parameterization on a coherence space \(X\) is a stable, continuous function \(F : X \to \text{Coh}\). We write \(\text{Par}(X)\) for the set of parameterizations on \(X\).
Dependent sums and products of coherence spaces

We now define constructions on coherence spaces corresponding to dependent sums and products.

Given $F \in \text{Par}(X)$, we define:

$$|\Sigma(X, F)| := \{ (s, u) | s \in X_{\text{fin}}, u \in F(s)_{\text{fin}} \},$$

$$(s, u) \sim (t, v) \text{ mod } \Sigma(X, F) \equiv s \cup t \in X \land u \cup v \in F(s \cup t).$$

Given $F \in \text{Par}(X)$, we define the extended trace of $F$:

$$\text{Tr}^*(F) := \{ (s, t) | s \in X_{\text{fin}}, b \in F(s)_{\text{fin}}, \forall s' \subseteq s, t \in F(s')_{\text{fin}} \Rightarrow s' = s \}.$$

We define

$$|\Pi(X, F)| := \text{Tr}^*(F),$$

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Suppose that $F$ is constant. Define $Y := F(\emptyset)$. Then

$$\Pi(X, F) = X \Rightarrow Y, \Sigma(X, F) = !X \otimes !Y \sim ! (X \times Y).$$
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Suppose that $F$ is constant. Define $Y := F(\emptyset)$. Then

$$\Pi(X, F) = X \Rightarrow Y, \quad \Sigma(X, F) = !X \otimes !Y \cong !(X \& Y).$$
We take the category \( C \) to be the category of coherence spaces and stable maps. For each coherence space \( X \), we set \( \text{Ty}(X) := \text{Par}(X) \). Given \( F \in \text{Par}(X) \), we set \( \text{Tm}(X, F) := \Pi(A, F) \).

For the functorial action on types, given a stable map \( f: X \to Y \), and \( F \in \text{Par}(Y) \), we define \( F \{ f \} \in \text{Par}(X) \) by:

\[
F \{ f \} = F \circ f.
\]

For the functorial action on terms, given \( f: X \to Y \), and \( t \in \Pi(Y, F) \), we define \( t \{ f \} \in \Pi(X, F \{ f \}) \) by:

\[
t \{ f \} = t \circ f.
\]

The empty context is the terminal object \( \top \) in \( C \). Context extension is defined using the dependent sum construction. If \( F \in \text{Par}(X) \), then we define \( X.F := \Sigma(X, F) \).

The projection morphism \( p_F: X.F \to X \) is the first projection \( \Sigma(X, F) \to X \) whose trace is given by:

\[
\{ (\{ (s, u) \}, a) | (s, u) \in |\Sigma(X, F)|, a \in s \}.
\]
A Category with Families based on Coherence Spaces

- We take the category of contexts $\mathcal{C}$ to be the category of coherence spaces and stable maps.
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- The projection morphism $p_F : X.F \to X$ is the first projection $\Sigma(X, F) \to X$ whose trace is given by:
  $$\{((\{s, u\}, a) \mid (s, u) \in |\Sigma(X, F)|, a \in s\}.\)
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- Given \( f : X \to Y \), \( F \in \text{Par}(Y) \), and \( g \in \Pi(X, F\{f\}) \), we define the context morphism extension
  \[
  \langle f, g \rangle_F : X \to \Sigma(Y, F)
  \]
as the pairing map, with trace:
  \[
  \{(s \cup s', (t, u)) \in X_{\text{fin}} \times |\Sigma(Y, F)| \mid (s, t) \in \text{Tr}^*(f) \land (s', u) \in \text{Tr}^*(g) \}.
  \]
Verification

Proposition

The above construction defines a CwF.

Proof

The proof amounts to verifying a number of equations.
Firstly, for coherence spaces \( X, Y, Z \), stable maps \( f : X \rightarrow Y, g : Y \rightarrow Z \), \( F \in \text{Par}(Z) \), and \( t \in \Pi(Z, F) \):

\[
\begin{align*}
F\{\text{id}_X\} &= F & \in & \text{Ty}(X) & \quad \text{(Ty-Id)} \\
F\{g \circ f\} &= F\{g\}\{f\} & \in & \text{Ty}(X) & \quad \text{(Ty-Comp)} \\
t\{\text{id}_Z\} &= t & \in & \text{Tm}(Z, F) & \quad \text{(Tm-Id)} \\
t\{g \circ f\} &= t\{g\}\{f\} & \in & \text{Tm}(X, F\{g \circ f\})) & \quad \text{(Tm-Comp)}
\end{align*}
\]

Furthermore, for \( f : X \rightarrow Y, g : Z \rightarrow X \), \( F \in \text{Par}(Y) \), and \( t \in \Pi(X, F\{f\}) \):

\[
\begin{align*}
\mathbf{p}_F \circ \langle f, t \rangle_F &= f & : & X \rightarrow Y & \quad \text{(Cons-L)} \\
\mathbf{v}_F\{\langle f, t \rangle_F\} &= t & \in & \text{Tm}(X, F\{f\}) & \quad \text{(Cons-R)} \\
\langle f, t \rangle_F \circ g &= \langle f \circ g, t\{g\} \rangle_F & : & Z \rightarrow \Sigma(Y, F) & \quad \text{(Cons-Nat)} \\
\langle \mathbf{p}_F, \mathbf{v}_F \rangle_F &= \text{id}_{\Sigma(Y, F)} & : & \Sigma(Y, F) \rightarrow \Sigma(Y, F) & \quad \text{(Cons-Id)}
\end{align*}
\]
Identity, Totality and Intensionality

Identity types $\sigma(M, N)$ are interpreted by taking the intersections of the cliques denoted by $M$ and $N$. But this gives a model of partial type theory, as previously studied using Scott domains by Palmgren and Stoltenberg-Hansen. To get an appropriate model of standard type theory, we must extend the model with a notion of totality. If this is done, we get a highly intensional (non-well-pointed) model. However, it is not yet clear which principles are satisfied in the internal sense of propositional equality, e.g. from UIP and Function Extensionality.
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A glimpse at linear dependent types

The key issue is how to handle dependent types \( \sigma(x) \) with \( x \) resource-sensitive. If \( x \) is used in forming \( \sigma(x) \), is it still available to form terms of type \( \sigma(x) \)? This is circumvented in the current approach by considering only linear types depending on intuitionistic types, using a dual-context formulation: 

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\Gamma \mid \Theta \vdash M : \sigma
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It is not clear if this is robust with respect to universes. The underlying semantic structure is strict indexed monoidal categories with comprehension. There are a number of other delicate issues, e.g. multiplicative rather than additive forms of quantifiers.
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Towards Games

There are some subtleties in refining the coherence space semantics to a game semantics. In particular, consider the dependent sum. In type theory, we have first and second projections, so we can access the components independently. However, in game terms, the dependence indicates a causality of information flow: to get information about the second component, we need information about the first. It is not clear how this can be captured in game semantics while preserving the required equations.
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