Relational and final coalgebra semantics for dynamic logics

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joint work with
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Informal presentation of Display Calculi

Display calculi: variation of sequent calculi

Why: they are modular, cut elimination meta theorem

How do they work? 1 property = 1 rule

Some rules

$$\frac{X \vdash A \quad Y \vdash B}{X ; Y \vdash A \wedge B} \qquad \frac{X \vdash A}{\circ X \vdash \Diamond A} \qquad \frac{\circ X \vdash Y}{X \vdash \bullet Y}\text{display}$$

; structural symbols $\longrightarrow$ to manipulate structures
$\wedge$ operational symbols $\longrightarrow$ formulas such as $\Diamond A$ are "frozen".

Main feature: display property

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display property $\Longrightarrow$ cut elimination meta theorem

Example: The case of EAK

EAK: Epistemic Action and Knowledge

- Syntax: $\varphi := p \mid \neg\varphi \mid \varphi \vee \varphi \mid \Diamond_a \varphi \mid \langle\alpha\rangle\varphi$
- Intuitive meaning: $\Diamond_a \varphi$ agent a thinks that φ is possible and $\langle\alpha\rangle\varphi$: α can be executed, and after that, φ holds.

Protocol to design the proof system of a given logic:

1/ Algebraic description of the logic,

2/ Existing blocks: intuitionistic/classical logic, normal modal logic

3/ Design of the derivation rules following some conditions.

Aim: Display rules for each connective

Soundness? We need adjunction

Remark: $\langle\alpha\rangle$ distributes over joins $\rightsquigarrow$ Does it have an adjoint?

EAK: Frame semantic

Kripke model: $a \in Agent$, $\alpha \in Action$, $p \in Prop$

$\mathbb{M} := (W, R_a \subseteq W \times W, V)$ or $\mathbb{M} := (W, \sigma_a : W \longrightarrow \mathcal{P}W, V)$

Epistemic update: public announcement $\alpha \cong p$
$\langle\alpha\rangle\varphi$: after p is announced, φ holds.

$$\mathbb{M} = (W, R \subseteq W \times W, V) \rightsquigarrow_\alpha \mathbb{M}_\alpha = (W_\alpha, R_\alpha \subseteq W_\alpha \times W_\alpha, V_\alpha)$$

$$W_\alpha := V(p) \cap W, \quad R_\alpha := R \cap W_\alpha \times W_\alpha, \quad V_\alpha(p) := V(p) \cap W_\alpha.$$

/!\ We lose information about $\mathbb{M}$, some worlds disappear.

Problem: We cannot talk about the inverse of α!
We don't know how to talk about the adjoint?
We need them to define our proof system!

Why is the Final Coalgebra interesting?

What is the final coalgebra $\mathbb{Z}$?

Idea: "We put all the models together and build a gigantic model."

Advantages:

- dynamic modalities are maps on $\mathbb{Z}$.
- We do not need to compute a new model anymore.
- We can interpret the adjoint.

Final Coalgebra: formal definition

Let $\mathbb{C}$ be a category, F be an endofunctor on $\mathbb{C}$.
A F-coalgebra is a pair $(A, \sigma : A \longrightarrow FA)$ with $A \in \mathbb{C}$.
A coalgebra morphism from (A, α) to (B, β) is a arrow $h : A \longrightarrow B$ in $\mathbb{C}$ such that $\beta \circ h = F(h) \circ \alpha$.

$F-Coalg(\mathbb{C})$: the category of F-coalgebras and their morphisms.

Final object

A *final object* in the category $\mathbb{C}$ is an object Z s.t. for any $Y \in C$, there is a unique arrow $f : Y \longrightarrow Z$.

Final Coalgebra

A *final coalgebra* $\mathbb{Z}$ for the functor F is a final object of the category $F-Coalg(\mathbb{C})$.

When does the final coalgebra exist for a *Set*-endofunctor?

The Final Coalgebra Theorem, Aczel & Mendler

Every set-based functor has a final coalgebra.

Category of classes: objects are classes and arrows are functions between classes.

An endofunctor F is called **set-based** if
for each **class** A and each $a \in FA$,
there is a **set** $A_0 \subseteq A$ and $a_0 \in FA_0$
such that $a = F(i_{A_0,A})(a_0)$,
where $i_{A_0,A}$ is the inclusion map $A_0 \hookrightarrow A$.

Idea: F must be entirely defined by what it does on sets.

Final coalgebra semantics for powerset functor

A. Baltag, A Coalgebraic Semantics for Epistemic Programs, 2003.

Kripke model: $a \in Agent$, $\alpha \in Action$, $p \in Prop$
$\mathbb{M} := (W, \sigma_a : W \longrightarrow \mathcal{P}W, V)$

Updated model: $\mathbb{M}_\alpha := (W_\alpha, \sigma'_a : W_\alpha \longrightarrow \mathcal{P}W_\alpha, V_\alpha)$

Final Coalgebra: $\mathbb{Z} := (Z, \zeta_a : Z \longrightarrow \mathcal{P}Z, V)$

$$\begin{array}{ccc} \mathcal{P}Z & \xrightarrow{\langle\alpha\rangle} & \mathcal{P}Z \\ {\scriptstyle \exists! f}\uparrow & & {\scriptstyle \exists! g}\uparrow\downarrow{\scriptstyle g^{-1}} \\ \mathcal{P}W & \xrightarrow[\langle\alpha_{\mathbb{M}}\rangle]{} & \mathcal{P}W_\alpha \end{array}$$

$$[\![\phi]\!]_{\mathbb{M}} = f^{-1}[\![\phi]\!]_{\mathbb{Z}}$$

What about other dynamic logics?

Other logics:

- Monotone modal logic: $\mathbb{M} := (W, \sigma : W \longrightarrow \mathcal{P}\mathcal{P}W)$
- Intuitionistic modal logic: coalgebras over poset
- ...

Questions about monotone modal logic:

- Which action structure should we use?
- How to define the updated model?
- When does the final coalgebra exist?

Conjecture: Every poset-based functor has a final coalgebra.

Final Coalgebra Theorem for *Poset*?

Conjecture: Every poset-based functor has a final coalgebra.

Category of partially ordered classes: objects are classes with a partial order and arrows are order-preserving functions.

An endofunctor F is called **poset-based** if
for each **partially ordered class** $(A, \leq)$ and each $a \in FA$,
there is a **poset** $(A_0, \leq_0)$ and $a_0 \in FA_0$
such that $A_0 \subseteq A$ and $a = F(e_{A_0,A})(a_0)$,
where $e_{A_0,A}$ is an order-embedding $(A_0, \leq_0) \hookrightarrow (A, \leq)$.

Idea: The class $(A, \leq)$ must be entirely defined by the sub-posets $\{(A_i, \leq_i)\}$. That is why we ask for an order-embedding e.

$$\forall a_0, b_0 \in A_0, \quad a_0 \leq b_0 \quad \text{iff} \quad e(a_0) \leq e(b_0)$$

Conclusion

We know:

- Final coalgebra theorem for *Sets*
- Final coalgebra semantics for EAK
- A display type proof system for EAK

We want:

- Final coalgebra theorem for *Posets*
- Epistemic monotone modal logic ($\diamond$ and/or $\langle \alpha \rangle$ monotone). Semantics? Axiomatization?
- A display type proof system for monotone dynamic logics, coalgebraic logics, ...

Giuseppe is going to talk about:

Display calculi for dynamic logics