

Cuts and completions

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(Joint questions with Sam van Gool)

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The proof systems of Gentzen



In 1934/1935 Gentzen introduced two proof systems: natural deduction and sequent calculi.

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Features:

- few axioms and less freedom in the choice of rules;
- the inference rules describe (explain) the meaning of the language;
- meta-mathematical properties (consistency) follow naturally;
- both classical and intuitionistic logic have such proof systems.

Sequent calculi



Dfn A *sequent* is an ordered pair $\Gamma \Rightarrow \Delta$, where Γ, Δ are multisets of (propositional) formulas. Its *interpretation* is $I(\Gamma \Rightarrow \Delta) = \bigwedge \Gamma \rightarrow \bigvee \Delta$.

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Axioms and rules for implication and disjunction of G3i (for IPC): $|\Delta| \leq 1$

$$\Gamma, p \Rightarrow p \text{ Ax}$$

$$\Gamma, \perp \Rightarrow \Delta \text{ L}\perp$$

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \text{ L}\vee$$

$$\frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2} (i = 1, 2) \text{ R}\vee$$

$$\frac{\Gamma, A \rightarrow B \Rightarrow A \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta} \text{ L}\rightarrow \quad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \text{ R}\rightarrow$$

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Thm (Gentzen)

Cut is admissible in G3i: G3i + Cut is conservative over G3i.

$$\frac{\Gamma \Rightarrow A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \text{ Cut}$$

Proof of cut-elimination



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- Local transformation steps that move the cuts in a proof upwards.
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Advantages

- Modular, elementary, and constructive.
- Adaptable to almost all sequent calculi that have cut-elimination.
- Meta-mathematical properties follow easily.

Algebraic proof



Belardinelli, Jipsen, Ono (2004): an algebraic proof of cut-elimination.

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Proof (for G3i)

$(G, \preceq, \cdot, \wedge, \vee, \rightarrow, 0, 1)$ is a *Gentzen structure* if $0, 1 \in G$, \preceq is a subset of $G^* \times (G \cup \{\varepsilon\})$, and the binary operations satisfy (ommitting \wedge):

$$xa \preceq a \qquad 0 \preceq a \qquad x \preceq 1 \qquad \frac{x \preceq a_i}{x \preceq a_1 \vee a_2} \quad (i = 1, 2)$$

$$\frac{xa \preceq c \quad xb \preceq c}{x(a \vee b) \preceq c} \quad \frac{x \preceq a \quad yb \preceq c}{xy(a \rightarrow b) \preceq c} \quad \frac{xa \preceq b}{x \preceq a \rightarrow b} \quad \frac{xab \preceq c}{x(a \cdot b) \preceq c} \quad \frac{x \preceq a \quad y \preceq b}{x \preceq a \cdot b}$$

G^* consists of the finite multisets which elements are in G .

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Rmk Gentzen structures need not be *strongly transitive*:

$$\frac{x \preceq a \quad ay \preceq c}{xy \preceq c}$$

The *free Gentzen structure* is (because of cut-elimination).

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$(\Gamma \Rightarrow \Delta)^a \equiv \Gamma$ and $(\Gamma \Rightarrow \Delta)^c \equiv \Delta$. Suppose $\mathcal{G} \not\models S$ ($S^a \not\equiv S^c$ in \mathcal{G}).

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Construct a strongly transitive Gentzen structure \mathcal{G}_c that refutes S .

- \mathcal{G}_c is a commutative residuated lattice ($ab \preceq c$ iff $a \preceq b \rightarrow c$).

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Ciabattoni, Galatos, Terui (2011) use similar methods to characterize, for structural rules in \mathcal{N}_2 , the ones that preserve analyticity when added to the Lambek calculus FL.

Question



Are there other proofs of cut-elimination that use algebras?

The Schütte method



Emerged as a method to prove the completeness of sequent calculi.

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- Given that $\not\vdash_{G3i} S$, construct a Kripke model \mathcal{K} that refutes S .
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This proves cut-elimination for $G3i$ as well.

In this setting it is more convenient to work with multi-conclusion sequents and the calculus LJ' instead of $G3i$.

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If $\not\vdash_{\text{LJ}'}$ S , generate all possible “derivations” bottom-up, the *tableaux* of S . For a node a , $\text{sq}(a)$ is the sequent at a .

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Choose in every tableau an open branch. For two nodes a, b on a branch:

$$a \sim b \equiv \text{no application of } R \rightarrow \text{ in } [a, b]\text{-segment}$$

Defining $\bar{a} \Vdash p \equiv \exists b \in \bar{a} (p \in \text{sq}(b)^a)$ gives a Kripke model that refutes S .



Observations

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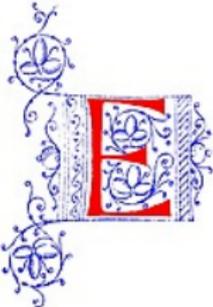


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- What is the relation between \mathcal{A}_G and \mathcal{G}_c ?
- The Schütte method is easily extendable to predicate logic. And the method using completions?
- Can the method using completions be applied to Gentzen structures corresponding to multi-conclusion sequents?

The  nd