Cuts and completions

Rosalie Iemhoff
Utrecht University

ALCOP, May 15, 2014

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(Joint questions with Sam van Gool)

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The proof systems of Gentzen

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Features:

- few axioms and less freedom in the choice of rules;
- the inference rules describe (explain) the meaning of the language;
- meta-mathematical properties (consistency) follow naturally;
- both classical and intuitionistic logic have such proof systems.

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Axioms and rules for implication and disjunction of G3i (for IPC): $|\Delta| \leq 1$

$$\Gamma, p \Rightarrow p \ \ \text{Ax} \qquad\qquad \Gamma, \bot \Rightarrow \Delta \ \ \text{L}\bot$$

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}\ \text{L}\vee \qquad\qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\ (i = 1, 2)\ \text{R}\vee$$

$$\frac{\Gamma, A \to B \Rightarrow A \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \to B \Rightarrow \Delta}\ \text{L}\to \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}\ \text{R}\to$$

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Thm (Gentzen)
Cut is admissible in G3i: G3i + Cut is conservative over G3i.

$$\frac{\Gamma \Rightarrow A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{Cut}$$

Proof of cut-elimination

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- Local transformation steps that move the cuts in a proof upwards.
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Advantages

- Modular, elementary, and constructive.
- Adaptable to almost all sequent calculi that have cut-elimination.
- Meta-mathematical properties follow easily.

Algebraic proof

Belardinelli, Jipsen, Ono (2004): an algebraic proof of cut-elimination.

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Proof (for G3i)
$(G, \preccurlyeq, \cdot, \wedge, \vee, \rightarrow, 0, 1)$ is a *Gentzen structure* if $0, 1 \in G$, $\preccurlyeq$ is a subset of $G^* \times (G \cup \{\varepsilon\})$, and the binary operations satisfy (ommitting $\wedge$):

$$xa \preccurlyeq a \qquad 0 \preccurlyeq a \qquad x \preccurlyeq 1 \qquad \frac{x \preccurlyeq a_i}{x \preccurlyeq a_1 \vee a_2}\ (i = 1, 2)$$

$$\frac{xa \preccurlyeq c \quad xb \preccurlyeq c}{x(a \vee b) \preccurlyeq c} \quad \frac{x \preccurlyeq a \quad yb \preccurlyeq c}{xy(a \rightarrow b) \preccurlyeq c} \quad \frac{xa \preccurlyeq b}{x \preccurlyeq a \rightarrow b} \quad \frac{xab \preccurlyeq c}{x(a \cdot b) \preccurlyeq c} \quad \frac{x \preccurlyeq a \quad y \preccurlyeq b}{x \preccurlyeq a \cdot b}$$

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Rmk Gentzen structures need not be *strongly transitive*:

$$\frac{x \preccurlyeq a \quad ay \preccurlyeq c}{xy \preccurlyeq c}$$

The *free Gentzen structure* is (because of cut-elimination).

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Construct a strongly transitive Gentzen structure $\mathcal{G}_c$ that refutes S.

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Ciabattoni, Galatos, Terui (2011) use similar methods to characterize, for structural rules in $\mathcal{N}_2$, the ones that preserve analycity when added to the Lambek calculus FL.

Question

Are there other proofs of cut-elimination that use algebras?

The Schütte method

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- Given that $\nvdash_{\mathsf{G3i}} S$, construct a Kripke model $\mathcal{K}$ that refutes S.
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In this setting it is more convenient to work with multi-conclusion sequents and the calculus LJ$'$ instead of G3i.

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If $\nvdash_{\mathrm{LJ}'} S$, generate all possible "derivations" bottom-up, the *tableaux* of S. For a node a, sq(a) is the sequent at a.

Choose in every tableau an open branch. For two nodes a, b on a branch:

$$a \sim b \equiv \text{no application of } R \to \text{ in } [a, b]\text{-segment}$$

Defining $\overline{a} \Vdash p \equiv \exists b \in \overline{a}(p \in \mathrm{sq(b)}^{\mathrm{a}})$ gives a Kripke model that refutes S.

Observations

- For every Gentzen structure $\mathcal{G} \not\models S$ there exists a tableau $T_{\mathcal{G}}$ for S such that all sequents in $T_{\mathcal{G}}$ are refuted in $\mathcal{G}$.

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- What is the relation between $\mathcal{A}_{\mathcal{G}}$ and $\mathcal{G}_c$?
- The Schütte method is easily extendable to predicate logic. And the method using completions?
- Can the method using completions be applied to Gentzen structures corresponding to multi-conclusion sequents?

The End