Indefiniteness of mathematical problems?

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Gödel's Extrinsic Program (1947)

“There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline...that quite irrespective of their intrinsic necessity they would have to be assumed in the same sense as any well-established physical theory."

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Proof. By **Cohen**’s method of forcing.

It is consistent for the continuum to be anything not cofinal with ω. This is necessary as by Julius König’s Theorem $\mathrm{cf}(2^{\aleph_0}) > \aleph_0$.

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A is a **definite totality** iff the logical operation of quantifying over A, $\forall x \in A\, P(x)$, has a determinate truth value for each **definite property** $P(x)$ of elements of A.

The Structure of all Sets

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V, where V is the universe of all sets, **is not a definite totality**, so unbounded quantification over V is not justified on this conception. Indeed, it is essentially indefinite.

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In **Frege: Philosophy of Mathematics**, **Dummett**'s diagnosis of the failure of Frege's logicist project focusses on the adoption of classical quantification. He rejects it in favor of the intuitionistic interpretation of quantification over the relevant domains.

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Dummett argues that classical quantification is illegitimate when the domain is given as the objects which fall under an indefinitely extensible concept.

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Solomon Feferman:
Is the continuum hypothesis a definite mathematical problem?

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- **Classical logic** for bounded (Δ_0) formulas. **Intuitionistic logic** for unbounded quantification.

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- $(*)$ follows from the observation that $\mathbf{AC}_{full}$ implies LEM_{Δ_0} (Diaconescu) and also BOS.
- Note that **T** proves full **Replacement** and **Strong Collection** (considered by Tharp, Beeson, Aczel).
- **T** is quite strong. It proves every theorem of (classical) second order arithmetic. In strength it resides strictly between **second order arithmetic** and **Zermelo set theory**.

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- If $\mathbb{R} \notin L$ then $L \neq L(\mathbb{R})$. However, always $L[\mathbb{R}] = L$.
- Only $L[A]$ is interesting for our purposes.

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- $B = A \cap L[A] \Rightarrow L[A] = L[B] \wedge (V = L[B])^{L[A]}$.

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- There is a Σ_1 formula $\mathrm{wo}(x, y, z)$ such that

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  and if $<_{L[A]}$ denotes the wellordering of $L[A]$ determined by wo, then for any limit $\lambda > \omega$,

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- $(\divideontimes)$ $\lambda > \omega$ limit $\wedge$ $B = A \cap L_\lambda[A] \Rightarrow L_\lambda[A] = L_\lambda[B]$.

Computability over $\langle L[A], \in, A \rangle$

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- If e is such a code and $a_1, \ldots, a_n$ are sets in $L[A]$, we use

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- In this way the structures $\langle L[A], \in, A \rangle$ give rise to **partial combinatory algebras** (**pca's**) or models of **App**.

Realizability over $\langle L[A], \in, A\rangle$

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\begin{array}{rcl}
e \Vdash a \in b & \text{iff} & a \in b \\
e \Vdash a = b & \text{iff} & a = b \\
e \Vdash \varphi \wedge \psi & \text{iff} & (e)_0 \Vdash \varphi \text{ and } (e)_1 \Vdash \psi \\
e \Vdash \varphi \vee \psi & \text{iff} & [(e)_0 = 0 \wedge (e)_1 \Vdash \varphi] \text{ or } [(e)_0 = 1 \wedge (e)_1 \Vdash \psi] \\
e \Vdash \varphi \rightarrow \psi & \text{iff} & \forall d\, [d \Vdash \varphi \Rightarrow [e]^{L[A]}(d) \Vdash \psi] \\
e \Vdash \exists x \theta(x) & \text{iff} & (e)_1 \Vdash \theta((e)_0) \\
e \Vdash \forall x \theta(x) & \text{iff} & \forall a \in L[A]\, [e]^{L[A]}(a) \Vdash \theta(a).
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Above, for a set-theoretic pair $b = \langle u, v\rangle$, we used the notations $(b)_0 = u$ and $(b)_1 = v$. If b is not a pair let $(b)_0 = (b)_1 = 0$.

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Theorem 1. We need a more useful result that exhibits the underlying uniformity. If $\mathcal{D}$ is a **T**-derivation of a formula $\psi(x_1, \ldots, x_n)$, one explicitly constructs a hereditarily finite set $e_{\mathcal{D}}$ such that for all A and all $a_1, \ldots, a_n \in L[A]$,

$$[e_{\mathcal{D}}]^{L[A]}(a_1, \ldots, a_n, \mathbb{R}^{L[A]}) \Vdash \psi(a_1, \ldots, a_n).$$

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Another way of expressing the uniformity and effectiveness of $e_{\mathcal{D}}$ is obtained by viewing $\langle L[A], \in, A\rangle$ as an applicative structure. According to this view, $e_{\mathcal{D}}$ is given by an applicative term t of the theory **App** such that $t\downarrow$ in $L[A]$, i.e.

$$L[A] \models \exists e\, [t \simeq e \wedge e \Vdash \psi].$$

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  Can be obtained from any universe V' such that $V' \models \textbf{ZFC} + \mathrm{GCH}$ (e.g. L) by forcing with $\mathrm{Fn}(\kappa \times \omega, 2)$ where $\kappa = (\aleph_2)^{V'}$.

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- We now code the set of reals $\mathbb{R}$ via a set A of ordinals in such a way that the set of real numbers of V_0 belong to $L[A]$. We thus have
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- Clearly,

$$L[A] \models \neg CH.$$

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- Assume $\mathbf{T} \vdash CH \vee \neg CH$.
- By Theorem 1 there exists an $e \in \mathrm{HF}$ (which does not depend on A) such that

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- Since $L[A] \models \neg CH$ we must have for $d := [e]^{L[A]}(\mathbb{R}^{L[A]})$ that

$$(d)_0 = 1 \wedge L[A] \models \forall b\, b \not\Vdash CH.$$

- Since the statement "$[e]^{L[A]}(\mathbb{R}^{L[A]}) \simeq d$" is $\Sigma_1^{L[A]}$, there exists a π such that

$$d, A, \mathbb{R}^{L[A]} \in L_\pi[A] \wedge L_\pi[A] \models [e]^{L_\pi[A]}(\mathbb{R}^{L[A]}) \simeq d.$$

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- $L_\pi[A] \models (d)_0 = 1$, thus $L[A \cup B] \models (d)_0 = 1$.
- **CONTRADICTION!** as $L[A \cup B] \models d \Vdash CH \vee \neg CH$, which implies $(d)_0 = 0$ by (a).

The End

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Thank You!