On Model Theory of Bi-approximation Semantics

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joint work with Tomoyuki Suzuki and Jiří Velebil

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- Bi-approximation semantics (T. Suzuki) provides a relational semantics to lattice-based logics, as e.g. substructural logics.
- Relates to work by M. Gehrke, N. Galatos, P. Jipsen,... motivated by similar goals
- What has been done by now includes a natural definition of validity-preserving morphisms, dual relation to algebraic semantics, first-order correspondence, canonicity results and Sahlqvist theorem (series of papers by T.Suzuki 2010-2013)

We would like to do

- ofer a more general categorial view on the polarity-based frames
- to prove a definability theorem in the spirit of Goldblatt and Thomason abstractly
- to prove the definability theorem using first-order model theory

Main references for this talk are:

- Tomoyuki Suzuki, Morphisms on bi-approximation semantics , Advances in Modal Logic 2012, vol.9, 2012, pp.494-515. College Publications.
- Unpublished notes on the category of frames seen as modules by Jiří Velebil.

Polarity frames

- A **polarity** (X, Y, N) is a binary relation N on two nonempty sets X and Y.
- N generates a preorder on X and Y:

$$
\begin{aligned}
x \leq x' &\equiv \forall y(x'Ny \longrightarrow xNy) \\
y' \leq y &\equiv \forall x(xNy' \longrightarrow xNy)
\end{aligned}
$$

- A pair (L, U) of subsets of X and Y is called a **cut**, iff L are the lowerbounds of U, and U are the upperbounds of L w.r.t. N.

Doppelgänger valuation

A valuation is a map V assigning to each atom p a cut $V(p) = (V^{\downarrow}(p), V_{\uparrow}(p))$ of states where p is **assumed** and states where p is **concluded**.

Lattice fragment of the language

Any valuation on $F = (X, Y, N)$ generates semantics relations $\Vdash^x$ and $\Vdash_y$ as follows:

- $\Vdash^x \varphi \wedge \psi \;\Leftrightarrow \Vdash^x \varphi$ and $\Vdash^x \psi$
- $\Vdash^x \varphi \vee \psi \;\Leftrightarrow\; \forall y(\Vdash_y \varphi \vee \psi \Rightarrow xNy)$
- $\Vdash_y \varphi \vee \psi \;\Leftrightarrow \Vdash_y \varphi$ and $\Vdash_y \psi$
- $\Vdash_y \varphi \wedge \psi \;\Leftrightarrow\; \forall x(\Vdash^x \varphi \wedge \psi \Rightarrow xNy)$

Residuated polarity frame

A polarity frame $F = (X, Y, N, R, O)$ where $R : Y \longrightarrow\!\!\!\!\!\!\!\!\!\!/\;\;\;\; X \times X$ is a ternary monotone relation:

$$x_1' \leq x_1,\ x_2' \leq x_2,\ y \leq y' \text{ and } R(x_1, x_2, y) \Rightarrow R(x_1', x_2', y')$$

and $O = (O_X, O_Y)$ is a cut.

additional properties of R and O

1. $x' \leq x \quad \Leftrightarrow \quad (\exists o \in O_X)(\forall y)(R(x, o, y) \Rightarrow x' \leq y)$
   $x' \leq x \quad \Leftrightarrow \quad (\exists o \in O_X)(\forall y)(R(o, x, y) \Rightarrow x' \leq y)$
2. tightness of R...
3. associativity, commutativity of R if needed ...

Interpreting substructural language

1. $\Vdash^x 1 \;\Leftrightarrow\; x \in O_X$
2. $\Vdash^x \varphi \otimes \psi \;\Leftrightarrow\; \forall y(\Vdash_y \varphi \otimes \psi \Rightarrow xNy)$
3. $\Vdash^x \varphi \to \psi \;\Leftrightarrow\; \forall x', y(\Vdash^{x'} \varphi \text{ and } \Vdash_y \psi \Rightarrow R(x', x, y))$
4. $\Vdash^x \psi \leftarrow \varphi \;\Leftrightarrow\; \forall x', y(\Vdash^{x'} \varphi \text{ and } \Vdash_y \psi \Rightarrow R(x, x', y))$
5. $\Vdash_y 1 \;\Leftrightarrow\; y \in O_Y$
6. $\Vdash_y \varphi \otimes \psi \;\Leftrightarrow\; \forall x, x'(\Vdash^x \varphi \text{ and } \Vdash^{x'} \psi \Rightarrow R(x, x', y))$
7. $\Vdash_y \varphi \to \psi \;\Leftrightarrow\; \forall x(\Vdash^x \varphi \to \psi \Rightarrow xNy)$
8. $\Vdash_y \psi \leftarrow \varphi \forall x(\Vdash^x \psi \leftarrow \varphi \Rightarrow xNy)$

Interpreting sequents

$$F, V \Vdash (\varphi \Rightarrow \psi) \;\text{ IFF } \forall x, y(\Vdash^x \varphi \text{ and } \Vdash_y \psi \Rightarrow xNy)$$

Morphisms of polarity frames

A **frame morphism** from $F_1 = (X_1, Y_1, N_1)$ to $F_2 = (X_2, Y_2, N_2)$ is a pair of (monotone) maps $p : X_1 \longrightarrow X_1$ and $f : Y_1 \longrightarrow Y_1$ satisfying:

1. $\forall x, y(p(x) N_2 f(y) \Rightarrow x N_1 y)$
2. for all $x_1 \in X_1$ and $y_2 \in Y_2$:
$$\forall y_1 [y_2 \leq f(y_1) \Rightarrow x_1 N_1 y_1] \Rightarrow p(x_1) N_2 y_2$$
3. for all $x_2 \in X_2$ and $y_1 \in Y_1$:
$$\forall x_1 [p(x_1) \leq x_2 \Rightarrow x_1 N_1 y_1] \Rightarrow x_2 N_2 f(y_1)$$

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1. for all $x_1 \in X_1$ and $y_2 \in Y_2$:

$$\forall y_1 [y_2 \leq f(y_1) \Rightarrow x_1 N_1 y_1] \Leftrightarrow p(x_1) N_2 y_2$$

2. for all $x_2 \in X_2$ and $y_1 \in Y_1$:

$$\forall x_1 [p(x_1) \leq x_2 \Rightarrow x_1 N_1 y_1] \Leftrightarrow x_2 N_2 f(y_1)$$

Morphisms of polarity frames

A **frame morphism** from $F_1 = (X_1, Y_1, N_1)$ to $F_2 = (X_2, Y_2, N_2)$ is a pair of (monotone) maps $p : X_1 \longrightarrow X_1$ and $f : Y_1 \longrightarrow Y_1$ **reflecting cuts**:

$$(L, U) \text{ is a cut on } F_2 \; \Rightarrow (p^{-1}[L], f^{-1}[U]) \text{ is a cut on } F_1$$

Morphisms of residuated polarity frames

1. for all x_2, x_2', y

   $$\forall x_1, x_1'[p(x_1) \leq x_2 \text{ and } p(x_1') \leq x_2' \Rightarrow R_1(x_1, x_1', y)] \Leftrightarrow R_2(x_2, x_2', f(y))$$

2. for all x_2, x_1', y_2

   $$\forall x_1, y_1[p(x_1) \leq x_2 \text{ and } y_2 \leq f(y_1) \Rightarrow R_1(x_1, x_1', y_1)] \Leftrightarrow R_2(x_2, p(x_1'), y_2)$$

3. for all x_1, x_2', y_2

   $$\forall x_1', y_1[p(x_1') \leq x_2' \text{ and } y_2 \leq f(y_1) \Rightarrow R_1(x_1, x_1', y_1)] \Leftrightarrow R_2(p(x_1), x_2', y_2)$$

Special morphisms

- a frame morphism $(p, f) : F_1 \longrightarrow F_2$ is N-**embedding** if

$$\forall x, y(xN_1y \Rightarrow p(x)N_2f(y))$$

- a frame morphism $(p, f) : F_1 \longrightarrow F_2$ is N-**separating** if for all $x_2 \in X_2$ and $y_2 \in Y_2$,

$$\forall x_1, y_1[p(x_1) \leq x_2 \text{ and } y_2 \leq f(y_1) \Rightarrow x_1N_1y_1] \Rightarrow p(x_1)N_2f(y_1)$$

Morphisms of residuated polarity frames

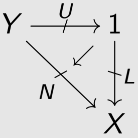
1. generalise to model morphisms by requirement of atomic harmony
2. model morphisms preserve assuming and concluding of every formula
3. N-**embeddings** of frames **reflect** validity of sequents
4. N-**separating** morphisms of frames **preserve** validity of sequents

Frames as modules

Consider 2-category of **preorders** and **monotone relations** (modules).
A frame F is a monotone relation $N : Y \longrightarrow\!\!\!\!\!\!\!\!\!\!\!\!\mapsto X$

Cuts

A **cut** on F is a diagram



that is simultaneously a right Kan extension and a right Kan lifting:

1. $L = [\![U, N]\!]$, i.e. $L(x) = \bigwedge_y (U(y) \longrightarrow N(x, y))$
2. $U = \{\![L, N]\!\}$, i.e. $U(y) = \bigwedge_x (L(x) \longrightarrow N(x, y))$

Reflecting cuts morphisms

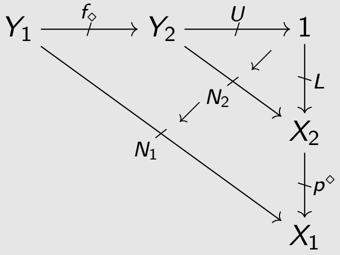
A **morphism** from $N_1 : Y_1 \nrightarrow X_1$ to $N_2 : Y_2 \nrightarrow X_2$ consists of a pair $f : Y_1 \longrightarrow Y_2$, $p : X_1 \longrightarrow X_2$ with:

$$\begin{array}{ccc} Y_1 & \xrightarrow{f_\diamond} & Y_2 \\ {\scriptstyle N_1}\big\downarrow & \leftarrow & \big\downarrow{\scriptstyle N_2} \\ X_1 & \xleftarrow[p^\diamond]{} & X_2 \end{array}$$

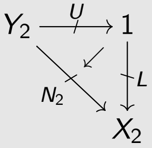
and such that ...

Cut reflection

... when pasted as follows:



yields a cut, for every cut



Polarity frames as separated modules

A frame $N : Y \longrightarrow\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\:\:\:\:\:\:\:\: / \;\;\; X$ is a polarity frame (separated frame), if Y (seen as a module) is the right Kan lift of N through N, and X (as a module) is the right Kan extension of N along N:

1. $y' \leq y \;=\; \bigwedge_x [N(x, y') \Rightarrow N(x, y)]$
2. $x' \leq x \;=\; \bigwedge_y [N(x, y) \Rightarrow N(x', y)]$, meaning that

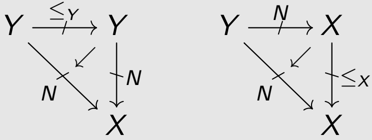


exhibit Y as $\{[N, N]\}$ and X as $[\![N, N]\!]$.

The 2-category of polarity frames

- objects - separated frames
- 1-cells - cut-reflecting morphisms
- 2-cells

$$(p_1, f_1) \sqsubseteq (p_2, f_2) \;\Leftrightarrow\; f_1 \leq f_2 \text{ and } p_2 \leq p_1$$

The 2-category of polarity frames

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Examples

- $\leq_X \colon X \longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;\; X$
- $2_\wedge : 2 \longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;\; 2^{op}$ where $2_\wedge(u, v) = u \wedge v$
- a morphism from a frame N to $2_\wedge$ is precisely a cut on N.

The 2-category of polarity frames

- objects - separated frames
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$$(p_1, f_1) \sqsubseteq (p_2, f_2) \;\Leftrightarrow\; f_1 \leq f_2 \text{ and } p_2 \leq p_1$$

Factorisation

- *N*-**embeddings** are **order-mono**
- *N*-**separating** morphisms are **order-epi**
- Every frame morphism has an *N*-separating-*N*-embedding factorisation

Lattices and polarity frames

The dual picture:

$$\mathsf{Fr}^{op} \underset{\mathit{Pred}}{\overset{\mathit{Stone}}{\underset{\longrightarrow}{\overset{\longleftarrow}{?}}}} \mathsf{Lat}$$

Explanation:

1. $\mathit{Pred} : \mathbb{F} \mapsto [\mathbb{F}, 2_\wedge]$.
   The **predicates** on $\mathbb{F}$ are the **cuts** of $\mathbb{F}$ with

$$\begin{aligned} (L, U) \wedge (L', U') &= (L \cap L', U'') \\ (L, U) \vee (L', U') &= (L'', U \cap U') \end{aligned}$$

   This is a lattice.

2. $\mathit{Stone} : \mathbb{A} \mapsto (\mathscr{F}, \mathscr{I}, N)$.
   The **Stone polarity frame** of $\mathbb{A}$ is based on **filters** on $\mathbb{A}$, **ideals** on $\mathbb{A}$, related by

$$F \; N \; I \equiv (F \cap I \neq \emptyset).$$

   This is a separated frame.

Lattices and polarity frames

The dual picture:

$$\mathsf{Fr}^{op} \underset{Pred}{\overset{Stone}{\underset{?}{\rightleftarrows}}} \mathsf{Lat}$$

On morphisms:

1. For $(p, f) : \mathbb{F}_2 \longrightarrow \mathbb{F}_1$ define $Pred(p, f) : Pred(\mathbb{F}_1) \longrightarrow Pred(\mathbb{F}_2)$ as

$$(L_2, U_2) \mapsto (p^{-1}[L_2], f^{-1}[L_1])$$

2. For $h : \mathbb{A} \longrightarrow \mathbb{B}$ define $Stone(h) : (\mathscr{F}_B, \mathscr{I}_B, N_B) \longrightarrow (\mathscr{F}_A, \mathscr{I}_A, N_A)$ as

$$\begin{array}{rcl} p(F_B) & \mapsto & h^{-1}[F_B] \\ f(I_B) & \mapsto & h^{-1}[I_B] \end{array}$$

Residuated lattices and residuated frames

The lifted dual picture:

$$\mathrm{RFr}^{op} \underset{Pred^{\#}}{\overset{Stone^{\#}}{\rightleftarrows}} \mathrm{RL} \quad (?)$$

Structure of $Pred^{\#}(\mathbb{F})$:

$$\begin{aligned}
(L, U) \otimes (L', U') &= (L'', \{y \mid \forall x \in L, x' \in L'.R(x, x', y)\}) \\
(L, U) \to (L', U') &= (\{x' \mid \forall x \in L, y \in U'.R(x, x', y)\}, U'') \\
(L', U') \leftarrow (L, U) &= (\{x' \mid \forall x \in L, y \in U'.R(x', x, y)\}, U'') \\
1 &= (O_X, O_Y)
\end{aligned}$$

This is a residuated lattice.

Residuated lattices and residuated frames

The lifted dual picture:

$$\mathrm{RFr}^{op} \underset{Pred^{\#}}{\overset{Stone^{\#}}{\underset{?}{\leftrightarrows}}} \mathrm{RL}$$

Structure of $Stone^{\#}(\mathbb{A})$:

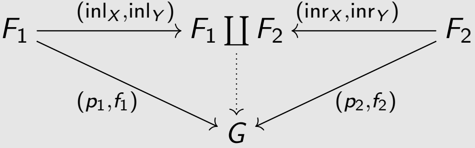
$$\begin{aligned} O_F &= \{F \mid 1 \in F\} \\ O_F &= \{I \mid 1 \in I\} \\ R(F, F', I) &= F * F' \subseteq I \end{aligned}$$

where

$$F * F' = \{a \mid \exists b \in F, b' \in F'. b \otimes b' \leq a\}.$$

is a residuated polarity frame.

Coproducts



Coproduct of polarity frames:

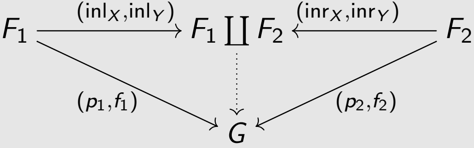
1. $F_1 \coprod F_2$ is defined on the **disjoint union** of the **underlying sets** as $(X_1 \uplus X_2, Y_1 \uplus Y_2, N)$ with

$$\neg xNy \equiv \exists i(x \in X_i, y \in Y_i, \neg xN_i y)$$

2. this affects the preorder N generates:

$$x \leq x' \equiv \begin{cases} \exists i(x \in X_i, x' \in X_i, x \leq_i x') \text{ or} \\ x \text{ is a bottom element in its component} \end{cases}$$

Coproducts



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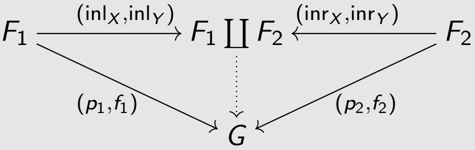
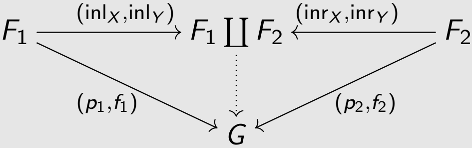
Coproducts



Coproducts of residuated polarity frames is $(X_1 \uplus X_2, Y_1 \uplus Y_2, N, R, O_X, O_Y)$ with

$$
\begin{aligned}
\neg R(x, x', y) &\equiv \exists i(x \in X_i, x' \in X_i, y \in Y_i, \neg R_i(x, x', y)) \\
O_X &= \biguplus O_{X_i} \\
O_Y &= \biguplus O_{Y_i}
\end{aligned}
$$

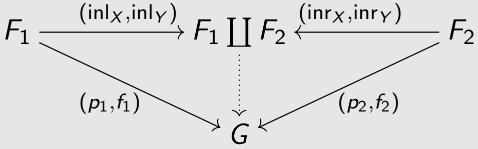
Coproducts



Notice:

$$Pred(\coprod_{i \in I} F_i) \cong \prod_{i \in I} (Pred\, F_i)$$

Coproducts



Notice:

$$Pred^{\#}(\coprod_{i \in I} F_i) \cong \prod_{i \in I}(Pred^{\#} F_i)$$

Subframes

We say that F_1 is (isomorphic to) a subframe of F_2

$$F_1 \overset{(p,f)}{\rightarrowtail} F_2$$

if (p, f) is an N-embedding.

Example - pair generated polarity subframes

For F and a pair (x, y) with $\neg xNy$ we define the pair generated subframe $F_{(x,y)}$ as the smallest subframe containing (x, y) and closed under finite iterations of $\neg N$.

Notice:

1. (p, f) need not be injective. It is an **order-mono**.
2. preserves validity of sequents from F_2 to F_1.
3. Each polarity frame is a morphic image of its pair-generated subframes.

Images of frames

We say that F_2 is a N-separating image of F_1

$$F_1 \xrightarrow{(p,f)} \!\!\!\!\rightarrow F_2$$

if (p, f) is an N-separating morphism.

Notice:

1. (p, f) need not be surjective. It is an **order-epi**.
2. preserves validity of sequents from F_1 to F_2.

From the dual picture:

1. If $F_1 \xrightarrow{(p,f)} \!\!\!\!\!\!\rightarrow F_2$ then $Pred\, F_2 \rightarrowtail{}^{Pred(p,f)} Pred\, F_1$, i.e. $F_1 \overset{(p,f)}{\twoheadrightarrow} F_2$ then $Pred\, F_2 \overset{Pred(p,f)}{\rightarrowtail} Pred\, F_1$
2. If $F_1 \overset{(p,f)}{\rightarrowtail} F_2$ then $Pred\, F_2 \overset{Pred(p,f)}{\twoheadrightarrow} Pred\, F_1$
3. If $A_1 \overset{h}{\twoheadrightarrow} A_2$ then $Stone A_2 \overset{Stone(h)}{\rightarrowtail} Stone A_1$
4. If $A_1 \overset{h}{\rightarrowtail} A_2$ then $Stone A_2 \overset{Stone(h)}{\twoheadrightarrow} Stone A_1$

holds for both polarity and residuated frames (both *Stone*,*Pred* and $Stone^{\#}$,$Pred^{\#}$).

Goldblatt-Thomason Theorem for classes of residuated polarity frames

Suppose $\mathbf{C}$ is a class of frames closed under the canonical extensions ($F \in \mathbf{C}$ implies that $Stone^{\#} Pred^{\#} F \in \mathbf{C}$). Then the following are equivalent:

1. $\mathbf{C}$ is modally definable (by a set of sequents).
2. $\mathbf{C}$ has the following closure properties:
    1. If F_1 is in $\mathbf{C}$, $(p, f) : F_1 \longrightarrow F_2$ is N-separating, then F_2 is in $\mathbf{C}$.
    2. If F_2 is in $\mathbf{C}$, $(p, f) : F_1 \longrightarrow F_2$ is N-embedding, then F_1 is in $\mathbf{C}$.
    3. If F_i for all $i \in I$ are in $\mathbf{C}$, then $\coprod_{i \in I} F_i$ is in $\mathbf{C}$.
    4. If $Stone^{\sharp} Pred^{\sharp}(F)$ is in $\mathbf{C}$, then F is in $\mathbf{C}$.

A proof of the theorem

1. Assume F satisfies the logic of $\mathbf{C}$. Then $Pred(F)$ satisfies the corresponding equational theory of the variety generated by the complex algebras of $\mathbf{C}$.
2. Therefore $Pred F$ is in $HSP(Cm(\mathbf{C}))$
3. $Pred(F) \twoheadleftarrow B \rightarrowtail \prod(Pred F_i) \cong Pred \coprod F_i$ with all $F_i \in \mathbf{C}$
4. $Stone Pred(F) \rightarrowtail Stone B \twoheadleftarrow Stone Pred \coprod F_i$

A model-theoretic proof of the theorem

- Assume **C** is closed under ultraproducts. Assume F validates the logic of the class. Assume w.l.o.g. that F is generated by $\neg xNy$.
- Put $At_F = \{p_{(L,U)} \mid (L,U) \in Pred F\}$ and generate language $\mathscr{L}(At)_F$. Consider F with the obvious valuation as the model $\mathscr{M}$.
- Define $\Delta = \{\alpha \Rightarrow \beta \mid \mathscr{M} \Vdash^x \alpha, \mathscr{M} \Vdash_y \beta\}$.
- Each $\Delta' \subseteq_\omega \Delta$ is refutable in **C**, w.l.o.g. in a pair-generated frame (model).
- Therefore Δ is refutable in **C**, w.l.o.g. in a pair-generated frame (model). Consider a countably saturated ultrapower $\mathscr{N}$ of this model, on a frame G in **C**.
- Show that $G \longrightarrow\!\!\!\!\!\rightarrow Stone Pred F$