On Model Theory of Bi-approximation Semantics

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Bi-approximation semantics (T. Suzuki) provides a relational semantics to lattice-based logics, as e.g. substructural logics. Relates to work by M. Gehrke, N. Galatos, P. Jipsen,... motivated by similar goals. What has been done by now includes a natural definition of validity-preserving morphisms, dual relation to algebraic semantics, first-order correspondence, canonicity results and Sahlqvist theorem (series of papers by T. Suzuki 2010-2013).

We would like to do

- offer a more general categorial view on the polarity-based frames
- to prove a definability theorem in the spirit of Goldblatt and Thomason abstractly
- to prove the definability theorem using first-order model theory
Main references for this talk are:


- Unpublished notes on the category of frames seen as modules by Jiří Velebil.
Polarity frames

- A **polarity** \((X, Y, N)\) is a binary relation \(N\) on two nonempty sets \(X\) and \(Y\).
- \(N\) generates a preorder on \(X\) and \(Y\):

  \[
  x \leq x' \equiv \forall y (x' Ny \rightarrow x Ny) \\
  y' \leq y \equiv \forall x (x Ny' \rightarrow x Ny)
  \]

- A pair \((L, U)\) of subsets of \(X\) and \(Y\) is called a **cut**, iff \(L\) are the lowerbounds of \(U\), and \(U\) are the upperbounds of \(L\) w.r.t. \(N\).

Doppelgänger valuation

A valuation is a map \(V\) assigning to each atom \(p\) a cut \(V(p) = (V\downarrow(p), V\uparrow(p))\) of states where \(p\) is **assumed** and states where \(p\) is **concluded**.
Lattice fragment of the language

Any valuation on \( F = (X, Y, N) \) generates semantics relations \( \vdash^x \) and \( \vdash^y \) as follows:

- \( \vdash^x \varphi \land \psi \iff \vdash^x \varphi \text{ and } \vdash^x \psi \)
- \( \vdash^x \varphi \lor \psi \iff \forall y(\vdash^y \varphi \lor \psi \Rightarrow xNy) \)
- \( \vdash^y \varphi \lor \psi \iff \vdash^y \varphi \text{ and } \vdash^y \psi \)
- \( \vdash^y \varphi \land \psi \iff \forall x(\vdash^x \varphi \land \psi \Rightarrow xNy) \)
Residuated polarity frame

A polarity frame $F = (X, Y, N, R, O)$ where $R : Y \rightrightarrows X \times X$ is a ternary monotone relation:

$$x_1' \leq x_1, \ x_2' \leq x_2, \ y \leq y' \text{ and } R(x_1, x_2, y) \Rightarrow R(x_1', x_2', y')$$

and $O = (O_X, O_Y)$ is a cut.

<table>
<thead>
<tr>
<th>additional properties of $R$ and $O$</th>
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<tbody>
<tr>
<td>1. $x' \leq x$ $\iff$ $(\exists o \in O_X)(\forall y)(R(x, o, y) \Rightarrow x' \leq y)$</td>
</tr>
<tr>
<td>2. tightness of $R$...</td>
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<td>3. associativity, commutativity of $R$ if needed ...</td>
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Interpreting substructural language

1. $\vdash^x 1 \iff x \in O_X$
2. $\vdash^x \varphi \otimes \psi \iff \forall y (\vdash_y \varphi \otimes \psi \Rightarrow xNy)$
3. $\vdash^x \varphi \to \psi \iff \forall x', y (\vdash^x \varphi \text{ and } \vdash_y \psi \Rightarrow R(x', x, y))$
4. $\vdash^x \psi \leftarrow \varphi \iff \forall x', y (\vdash^x \varphi \text{ and } \vdash_y \psi \Rightarrow R(x, x', y))$
5. $\vdash_y 1 \iff y \in O_Y$
6. $\vdash_y \varphi \otimes \psi \iff \forall x, x' (\vdash^x \varphi \text{ and } \vdash^{x'} \psi \Rightarrow R(x, x', y))$
7. $\vdash_y \varphi \to \psi \iff \forall x (\vdash^x \varphi \to \psi \Rightarrow xNy)$
8. $\vdash_y \psi \leftarrow \varphi \forall x (\vdash^x \psi \leftarrow \varphi \Rightarrow xNy)$

Interpreting sequents

$F, V \vdash (\varphi \Rightarrow \psi) \iff \forall x, y (\vdash^x \varphi \text{ and } \vdash_y \psi \Rightarrow xNy)$
Morphisms of polarity frames

A **frame morphism** from $F_1 = (X_1, Y_1, N_1)$ to $F_2 = (X_2, Y_2, N_2)$ is a pair of (monotone) maps $p : X_1 \rightarrow X_1$ and $f : Y_1 \rightarrow Y_1$ satisfying:

1. $\forall x, y \ (p(x) N_2 f(y) \Rightarrow x N_1 y)$
2. for all $x_1 \in X_1$ and $y_2 \in Y_2$:
   \[
   \forall y_1 \ [y_2 \leq f(y_1) \Rightarrow x_1 N_1 y_1] \Rightarrow p(x_1) N_2 y_2
   \]
3. for all $x_2 \in X_2$ and $y_1 \in Y_1$:
   \[
   \forall x_1 \ [p(x_1) \leq x_2 \Rightarrow x_1 N_1 y_1] \Rightarrow x_2 N_2 f(y_1)
   \]
Morphisms of polarity frames

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   \forall y_1[ y_2 \leq f(y_1) \Rightarrow x_1 N_1 y_1] \Leftrightarrow p(x_1) N_2 y_2
   \]

2. for all $x_2 \in X_2$ and $y_1 \in Y_1$:
   \[
   \forall x_1[ p(x_1) \leq x_2 \Rightarrow x_1 N_1 y_1] \Leftrightarrow x_2 N_2 f(y_1)
   \]
Morphisms of polarity frames

A **frame morphism** from $F_1 = (X_1, Y_1, N_1)$ to $F_2 = (X_2, Y_2, N_2)$ is a pair of (monotone) maps $p : X_1 \longrightarrow X_1$ and $f : Y_1 \longrightarrow Y_1$ **reflecting cuts**:

$$(L, U) \text{ is a cut on } F_2 \implies (p^{-1}[L], f^{-1}[U]) \text{ is a cut on } F_1$$
Morphisms of residuated polarity frames

1. for all $x_2, x'_2, y$

$$\forall x_1, x'_1 [p(x_1) \leq x_2 \text{ and } p(x'_1) \leq x'_2 \Rightarrow R_1(x_1, x'_1, y)] \iff R_2(x_2, x'_2, f(y))$$

2. for all $x_2, x'_1, y_2$

$$\forall x_1, y_1 [p(x_1) \leq x_2 \text{ and } y_2 \leq f(y_1) \Rightarrow R_1(x_1, x'_1, y_1)] \iff R_2(x_2, p(x'_1), y_2)$$

3. for all $x_1, x'_2, y_2$

$$\forall x'_1, y_1 [p(x'_1) \leq x'_2 \text{ and } y_2 \leq f(y_1) \Rightarrow R_1(x_1, x'_1, y_1)] \iff R_2(p(x_1), x'_2, y_2)$$
Special morphisms

- a frame morphism \((p, f) : F_1 \rightarrow F_2\) is **\(N\)-embedding** if
  \[
  \forall x, y (x N_1 y \Rightarrow p(x) N_2 f(y))
  \]

- a frame morphism \((p, f) : F_1 \rightarrow F_2\) is **\(N\)-separating** if for all \(x_2 \in X_2\) and \(y_2 \in Y_2\),
  \[
  \forall x_1, y_1 [p(x_1) \leq x_2 \text{ and } y_2 \leq f(y_1) \Rightarrow x_1 N_1 y_1] \Rightarrow p(x_1) N_2 f(y_1)
  \]
Morphisms of residuated polarity frames

1. generalise to model morphisms by requirement of atomic harmony
2. model morphisms preserve assuming and concluding of every formula
3. $N$-embeddings of frames reflect validity of sequents
4. $N$-separating morphisms of frames preserve validity of sequents
Frames as modules
Consider 2-category of preorders and monotone relations (modules). A frame $F$ is a monotone relation $N : Y \rightarrow X$

Cuts
A cut on $F$ is a diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{U} & 1 \\
\downarrow & & \downarrow \\
X & \xleftarrow{L} & Y
\end{array}
\]

that is simultaneously a right Kan extension and a right Kan lifting:

1. $L = \lfloor U, N \rfloor$, i.e. $L(x) = \bigwedge_y (U(y) \rightarrow N(x, y))$
2. $U = \{ [L, N] \}$, i.e. $U(y) = \bigwedge_x (L(x) \rightarrow N(x, y))$
A general categorial picture?

Reflecting cuts morphisms

A morphism from $N_1 : Y_1 \rightarrow X_1$ to $N_2 : Y_2 \rightarrow X_2$ consists of a pair $f : Y_1 \rightarrow Y_2$, $p : X_1 \rightarrow X_2$ with:

\[
\begin{array}{c}
Y_1 \xrightarrow{f} Y_2 \\
\downarrow N_1 \quad \downarrow N_2 \\
X_1 \xleftarrow{p} X_2
\end{array}
\]

and such that ...
Cut reflection

... when pasted as follows:

\[
\begin{array}{c}
Y_1 \xrightarrow{f} Y_2 \xrightarrow{U} 1 \\
\downarrow N_1 \quad \downarrow N_2 \\
X_2 \quad \downarrow L \\
\downarrow p \\
X_1
\end{array}
\]

yields a cut, for every cut

\[
\begin{array}{c}
Y_2 \xrightarrow{U} 1 \\
\downarrow N_2 \quad \downarrow L \\
X_2
\end{array}
\]
Polarity frames as separated modules

A frame $N : Y \longrightarrow X$ is a polarity frame (separated frame), if $Y$ (seen as a module) is the right Kan lift of $N$ through $N$, and $X$ (as a module) is the right Kan extension of $N$ along $N$:

1. $y' \leq y = \bigwedge_x [N(x, y') \Rightarrow N(x, y)]$
2. $x' \leq x = \bigwedge_y [N(x, y) \Rightarrow N(x', y)]$, meaning that

$$
\begin{array}{ccc}
Y & \xrightarrow{\leq_y} & Y \\
\downarrow N & & \downarrow N \\
X & & X
\end{array}
\quad
\begin{array}{ccc}
Y & \xrightarrow{N} & X \\
\downarrow N & & \downarrow \leq_x \\
X & & X
\end{array}
$$

exhibit $Y$ as $\{[N, N]\}$ and $X$ as $\llbracket N, N \rrbracket$. 
The 2-category of polarity frames

- objects - separated frames
- 1-cells - cut-reflecting morphisms
- 2-cells

\[(p_1, f_1) \sqsubseteq (p_2, f_2) \iff f_1 \leq f_2 \text{ and } p_2 \leq p_1\]
The 2-category of polarity frames

- objects - separated frames
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\[(p_1, f_1) \sqsubseteq (p_2, f_2) \iff f_1 \leq f_2 \text{ and } p_2 \leq p_1\]

Examples

- \(\leq_X: X \to X\)
- \(2^\wedge : 2 \to 2^{op}\) where \(2^\wedge(u, v) = u \wedge v\)
- a morphism from a frame \(N\) to \(2^\wedge\) is precisely a cut on \(N\).
The 2-category of polarity frames

- objects - separated frames
- 1-cells - cut-reflecting morphisms
- 2-cells

\[(p_1, f_1) \sqsubseteq (p_2, f_2) \iff f_1 \leq f_2 \text{ and } p_2 \leq p_1\]

Factorisation

- \textit{N-embeddings} are \textit{order-mono}
- \textit{N-separating} morphisms are \textit{order-epi}
- Every frame morphism has an \textit{N-separating-N-embedding} factorisation
Lattices and polarity frames

The dual picture:

\[ \text{Fr}^{\text{op}} \xrightarrow{\text{Pred}} \xleftarrow{\text{Stone}} \text{Lat} \]

Explanation:

1. **Pred**: \( F \mapsto [F, 2 \land] \).
   - The **predicates** on \( F \) are the **cuts** of \( F \) with
     \[
     (L, U) \land (L', U') = (L \cap L', U'') \\
     (L, U) \lor (L', U') = (L'', U \cap U')
     \]
   - This is a lattice.

2. **Stone**: \( A \mapsto (\mathcal{F}, \mathcal{I}, N) \).
   - The **Stone polarity frame** of \( A \) is based on **filters** on \( A \), **ideals** on \( A \), related by
     \[ F N I \equiv (F \cap I \neq \emptyset). \]
   - This is a separated frame.
Lattices and polarity frames

The dual picture:

\[
\begin{array}{ccc}
\text{Fr}^{\text{op}} & \xrightarrow{\text{Stone}} & \text{Lat} \\
\uparrow_{?} & & \downarrow_{\text{Pred}} \\
\end{array}
\]

On morphisms:

1. For \((p, f) : F_2 \to F_1\) define \(\text{Pred}(p, f) : \text{Pred}(F_1) \to \text{Pred}(F_2)\) as

\[
(L_2, U_2) \mapsto (p^{-1}[L_2], f^{-1}[L_1])
\]

2. For \(h : A \to B\) define \(\text{Stone}(h) : (\mathcal{F}_B, \mathcal{I}_B, N_B) \to (\mathcal{F}_A, \mathcal{I}_A, N_A)\) as

\[
p(F_B) \mapsto h^{-1}[F_B] \\
f(I_B) \mapsto h^{-1}[I_B]
\]
Residuated lattices and residuated frames

The lifted dual picture:

\[
\begin{array}{ccc}
\text{RFr}^{\text{op}} & \xleftarrow{\text{Stone}^{\#}} & \text{RL} \\
\xrightarrow{?} & & \xrightarrow{\text{Pred}^{\#}} \\
\end{array}
\]

Structure of \(\text{Pred}^{\#}(\mathbb{F})\):

\[
\begin{align*}
(L, U) \otimes (L', U') & = (L'', \{y \mid \forall x \in L, x' \in L'.R(x, x', y)\}) \\
(L, U) \rightarrow (L', U') & = (\{x' \mid \forall x \in L, y \in U'.R(x, x', y)\}, U'') \\
(L', U') & \leftarrow (L, U) = (\{x' \mid \forall x \in L, y \in U'.R(x', x, y)\}, U'') \\
1 & = (O_X, O_Y)
\end{align*}
\]

This is a residuated lattice.
Residuated lattices and residuated frames

The lifted dual picture:

\[ RF_{\text{op}} \xleftarrow{\text{Stone}^\#} \xrightarrow{?} R_{\text{L}} \xleftarrow{\text{Pred}^\#} \]

Structure of \( \text{Stone}^\#(\mathbb{A}) \):

\[
\begin{align*}
O_F &= \{ F \mid 1 \in F \} \\
O_{F'} &= \{ I \mid 1 \in I \} \\
R(F, F', I) &= F \ast F' \subseteq I
\end{align*}
\]

where

\[ F \ast F' = \{ a \mid \exists b \in F, b' \in F'. b \otimes b' \leq a \}. \]

is a residuated polarity frame.
Coproducts

Coproduct of polarity frames:

1. $F_1 \coprod F_2$ is defined on the **disjoint union** of the **underlying sets** as $(X_1 \uplus X_2, Y_1 \uplus Y_2, N)$ with

   $$\neg xNy \equiv \exists i (x \in X_i, y \in Y_i, \neg xN_iy)$$

2. this affects the preorder $N$ generates:

   $$x \leq x' \equiv \begin{cases} \exists i (x \in X_i, x' \in X_i, x \leq_i x') & \text{or} \\ x \text{ is a bottom element in its component} \end{cases}$$
Coproducts

Coproduct of polarity frames:

1. \( F_1 \coprod F_2 \) is defined on the **disjoint union** of the **underlying sets** as 
   \((X_1 \uplus X_2, Y_1 \uplus Y_2, N)\) with
   \[
   \neg xNy \equiv \exists i (x \in X_i, y \in Y_i, \neg xN_i y)
   \]

2. This affects the preorder \( N \) generates:
   \[
   y' \leq y \equiv \begin{cases} 
   \exists i (y' \in Y_i, y \in Y_i, y' \leq_i y) \text{ or } \\ y \text{ is a top element in its component} 
   \end{cases}
   \]
Coproducts of residuated polarity frames is
\((X_1 \uplus X_2, Y_1 \uplus Y_2, N, R, O_X, O_Y)\) with

\[-R(x, x', y) \equiv \exists i( x \in X_i, x' \in X_i, y \in Y_i, \neg R_i(x, x', y))\]

\[O_X = \bigcup O_{X_i}\]

\[O_Y = \bigcup O_{Y_i}\]
Some frame constructions

Coproducts

\[ F_1 \xrightarrow{(\text{inl}_X, \text{inl}_Y)} F_1 \coprod F_2 \xleftarrow{(\text{inr}_X, \text{inr}_Y)} F_2 \]

\[ (p_1, f_1) \quad \xrightarrow{} \quad G \quad \xleftarrow{} \quad (p_2, f_2) \]

Notice:

\[ \text{Pred}(\coprod_{i \in I} F_i) \cong \prod_{i \in I} (\text{Pred}F_i) \]
Coproducts

\[ F_1 \xrightarrow{(\text{inl}_X,\text{inl}_Y)} F_1 \coprod F_2 \xleftarrow{(\text{inr}_X,\text{inr}_Y)} F_2 \]

\[ (p_1,f_1) \xrightarrow{G} (p_2,f_2) \]

Notice:

\[ \text{Pred}^\#(\coprod_{i \in I} F_i) \cong \prod_{i \in I}(\text{Pred}^\# F_i) \]
Subframes

We say that \( F_1 \) is (isomorphic to) a subframe of \( F_2 \)

\[
F_1 \xrightarrow{(p,f)} F_2
\]

if \((p, f)\) is an \(N\)-embedding.

Example - pair generated polarity subframes

For \( F \) and a pair \((x, y)\) with \( \neg xNy \) we define the pair generated subframe \( F_{(x,y)} \) as the smallest subframe containing \((x, y)\) and closed under finite iterations of \( \neg N \).

Notice:

1. \((p, f)\) need not be injective. It is an order-mono.
2. preserves validity of sequents from \( F_2 \) to \( F_1 \).
3. Each polarity frame is a morphic image of its pair-generated subframes.
Images of frames

We say that $F_2$ is a $N$-separating image of $F_1$

$$F_1 \xrightarrow{(p,f)} F_2$$

if $(p, f)$ is an $N$-separating morphism.

Notice:

1. $(p, f)$ need not be surjective. It is an order-epi.
2. preserves validity of sequents from $F_1$ to $F_2$. 
From the dual picture:

1. If \( F_1 \xrightarrow{(p,f)} F_2 \) then \( \overset{\text{Pred}(p,f)}{\text{Pred}}F_2 \xrightarrow{\text{Pred}(p,f)} \text{Pred}F_1 \)

2. If \( F_1 \xrightarrow{(p,f)} F_2 \) then \( \text{Pred}F_2 \xrightarrow{\text{Pred}(p,f)} \text{Pred}F_1 \)

3. If \( A_1 \xrightarrow{h} A_2 \) then \( \overset{\text{Stone}(h)}{\text{Stone}}A_2 \xrightarrow{\text{Stone}(h)} \text{Stone}A_1 \)

4. If \( A_1 \xrightarrow{h} A_2 \) then \( \text{Stone}A_2 \xrightarrow{\text{Stone}(h)} \text{Stone}A_1 \)

holds for both polarity and residuated frames (both \( \text{Stone}, \text{Pred} \) and \( \text{Stone}^\#, \text{Pred}^\# \)).
Goldblatt-Thomason Theorem for classes of residuated polarity frames

Suppose \( C \) is a class of frames closed under the canonical extensions (\( F \in C \) implies that \( \text{Stone}^\# \text{Pred}^\# F \in C \)). Then the following are equivalent:

1. \( C \) is modally definable (by a set of sequents).
2. \( C \) has the following closure properties:
   1. If \( F_1 \) is in \( C \), \( (p, f) : F_1 \xrightarrow{\text{N}} F_2 \) is \( N \)-separating, then \( F_2 \) is in \( C \).
   2. If \( F_2 \) is in \( C \), \( (p, f) : F_1 \xrightarrow{\text{N}} F_2 \) is \( N \)-embedding, then \( F_1 \) is in \( C \).
   3. If \( F_i \) for all \( i \in I \) are in \( C \), then \( \bigsqcup_{i \in I} F_i \) is in \( C \).
   4. If \( \text{Stone}^\# \text{Pred}^\# (F) \) is in \( C \), then \( F \) is in \( C \).
A proof of the theorem

1. Assume $F$ satisfies the logic of $\mathcal{C}$. Then $Pred(F)$ satisfies the corresponding equational theory of the variety generated by the complex algebras of $\mathcal{C}$.

2. Therefore $PredF$ is in $HSP(Cm(\mathcal{C}))$.

3. $Pred(F) \leftarrow B \rightarrow \prod(Pred F_i) \cong Pred \bigsqcup F_i$ with all $F_i \in \mathcal{C}$.

4. $StonePred(F) \leftarrow StoneB \leftarrow StonePred \bigsqcup F_i$.
A model-theoretic proof of the theorem

- Assume $\mathcal{C}$ is closed under ultraproducts. Assume $F$ validates the logic of the class. Assume w.l.o.g. that $F$ is generated by $\neg x Ny$.

- Put $At_F = \{p_{(L,U)} \mid (L,U) \in \text{Pred}F\}$ and generate language $L(At)_F$. Consider $F$ with the obvious valuation as the model $\mathcal{M}$.

- Define $\Delta = \{\alpha \Rightarrow \beta \mid \mathcal{M} \models^x \alpha, \mathcal{M} \models^y \beta\}$.

- Each $\Delta' \subseteq \omega \Delta$ is refutable in $\mathcal{C}$, w.l.o.g. in a pair-generated frame (model).

- Therefore $\Delta$ is refutable in $\mathcal{C}$, w.l.o.g. in a pair-generated frame (model). Consider a countably saturated ultrapower $\mathcal{N}$ of this model, on a frame $G$ in $\mathcal{C}$.

- Show that $G \rightarrow \text{StonePred}F$