

Algebraic Theories and Control Effects, Back and Forth

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an example of
the unreasonable effectiveness of mathematics
in computer science and logic

ALCOP
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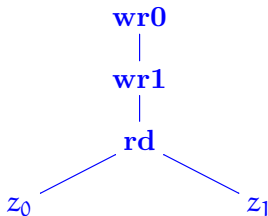
Universal Algebra

[Birkhoff (1935)]

► Signatures

$\mathbf{wr0} : 1$, $\mathbf{wr1} : 1$, $\mathbf{rd} : 2$

► Free algebras



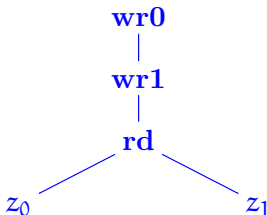
Universal Algebra

[Birkhoff (1935)]

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$\mathbf{wr0} : 1$, $\mathbf{wr1} : 1$, $\mathbf{rd} : 2$

► Free algebras



► Equational theories

$$\begin{aligned} \mathbf{wr0}(\mathbf{wr1}(z)) &\equiv \mathbf{wr1}(z) & , & & \mathbf{wr1}(\mathbf{wr0}(z)) &\equiv \mathbf{wr0}(z) \\ \mathbf{wr0}(\mathbf{rd}(z_0, z_1)) &\equiv \mathbf{wr0}(z_0) & , & & \mathbf{wr1}(\mathbf{rd}(z_0, z_1)) &\equiv \mathbf{wr1}(z_1) \end{aligned}$$

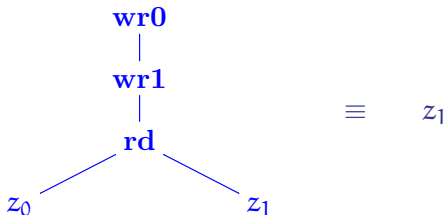
Universal Algebra

[Birkhoff (1935)]

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wr0 : 1 , **wr1** : 1 , **rd** : 2

► Free algebras



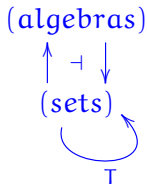
► Equational theories

$$\begin{array}{l} \text{wr0}(\text{wr1}(z)) \equiv \text{wr1}(z) \quad , \quad \text{wr1}(\text{wr0}(z)) \equiv \text{wr0}(z) \\ \text{wr0}(\text{rd}(z_0, z_1)) \equiv \text{wr0}(z_0) \quad , \quad \text{wr1}(\text{rd}(z_0, z_1)) \equiv \text{wr1}(z_1) \end{array}$$

Notions of Computation

[Moggi (1990); Plotkin, Power (2002)]

- ▶ Free-algebra monads



- ▶ Computational metalanguage

$$\frac{\Gamma \vdash_{\mathbf{c}} t : \tau}{\Gamma \vdash_{\mathbf{c}} \mathbf{wr0}_{\tau}(t) : \tau} \qquad \frac{\Gamma \vdash_{\mathbf{c}} t : \tau}{\Gamma \vdash_{\mathbf{c}} \mathbf{wr1}_{\tau}(t) : \tau}$$

$$\frac{\Gamma \vdash_{\mathbf{c}} t_0 : \tau \quad \Gamma \vdash_{\mathbf{c}} t_1 : \tau}{\Gamma \vdash_{\mathbf{c}} \mathbf{rd}_{\tau}(t_0, t_1) : \tau}$$

► Denotational semantics

$$\llbracket \Gamma \vdash_c t : \tau \rrbracket : \llbracket \Gamma \rrbracket \rightarrow T[\tau]$$

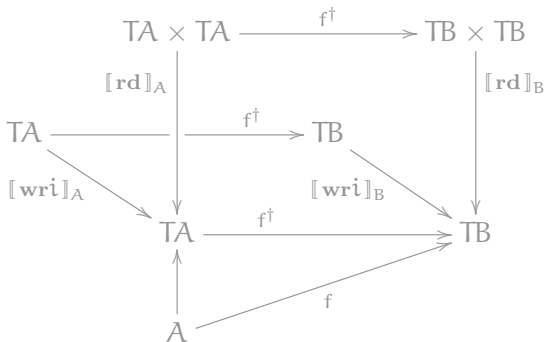
- $\llbracket \mathbf{wr0}_\tau(t) \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{\llbracket t \rrbracket} T[\tau] \xrightarrow{\llbracket \mathbf{wr0} \rrbracket} T[\tau]$
- $\llbracket \mathbf{wr1}_\tau(t) \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{\llbracket t \rrbracket} T[\tau] \xrightarrow{\llbracket \mathbf{wr1} \rrbracket} T[\tau]$
- $\llbracket \mathbf{rd}_\tau(t_0, t_1) \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{\langle \llbracket t_0 \rrbracket, \llbracket t_1 \rrbracket \rangle} T[\tau] \times T[\tau] \xrightarrow{\llbracket \mathbf{rd} \rrbracket} T[\tau]$

► Equational theory

- $\text{let } x = \text{wr0}_\sigma(s) \text{ in } t[x] \equiv \text{wr0}_\tau(\text{let } x = s \text{ in } t[x])$
- $\text{let } x = \text{wr1}_\sigma(s) \text{ in } t[x] \equiv \text{wr1}_\tau(\text{let } x = s \text{ in } t[x])$
- $\text{let } x = \text{rd}_\sigma(s_0, s_1) \text{ in } t[x]$
 $\equiv \text{rd}_\tau(\text{let } x = s_0 \text{ in } t[x], \text{let } x = s_1 \text{ in } t[x])$

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Second-Order Algebraic Theories

- ▶ Binding signatures

[Aczel (1978)]

app : (0, 0) , **abs** : (1)

Second-Order Algebraic Theories

- ▶ Binding signatures

[Aczel (1978)]

app : (0, 0) , **abs** : (1)

- ▶ Free algebras

[Fiore, Plotkin, Turi (1999); Hamana (2004); Fiore (2008)]

$t ::= \text{var}(x) \mid M[t_1, \dots, t_n] \mid \text{app}(t_1, t_2) \mid \text{abs}(x.t')$

► Second-order equational theories

$$\mathbf{app}(\mathbf{abs}(x. M[\mathbf{var}(x)]), N[]) \equiv M[N[]]$$

$$\mathbf{abs}(x. \mathbf{app}(F[], \mathbf{var}(x))) \equiv F[]$$

► Second-order equational theories

$$\mathbf{app}(\mathbf{abs}(x. M[\mathbf{var}(x)]), N[]) \equiv M[N[]]$$

$$\mathbf{abs}(x. \mathbf{app}(F[], \mathbf{var}(x))) \equiv F[]$$

► Free-algebra monads

(second-order algebras)



(context-indexed sets)



Set ^{\mathbb{F}}

- ▶ Context-indexed sets

X_m

$\{x_1, \dots, x_m\}$

Set^F

► Context-indexed sets

$$X_m \xrightarrow{X_g} X_n$$

$$\{x_1, \dots, x_m\} \xrightarrow{g} \{x_1, \dots, x_n\}$$

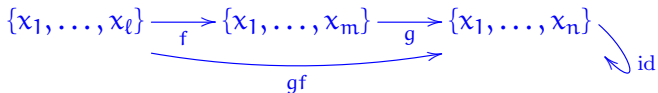
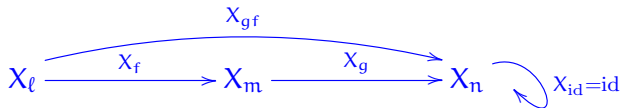
Set^F

► Context-indexed sets

$$X_m \xrightarrow{X_g} X_n \quad \text{with a self-loop on } X_n \text{ labeled } X_{\text{id}=\text{id}}$$

$$\{x_1, \dots, x_m\} \xrightarrow{g} \{x_1, \dots, x_n\} \quad \text{with a self-loop on } \{x_1, \dots, x_n\} \text{ labeled } \text{id}$$

► Context-indexed sets



► Renaming-invariant functions

$$\begin{array}{c} Y \\ \downarrow \varphi \\ X \end{array}$$

$$\begin{array}{ccc} Y_m & \xrightarrow{Y_g} & Y_n \\ \downarrow \varphi_m & & \downarrow \varphi_n \\ X_m & \xrightarrow{X_g} & X_n \end{array}$$

$$\{x_1, \dots, x_m\} \xrightarrow{g} \{x_1, \dots, x_n\}$$

► Renaming-invariant functions

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★ Examples

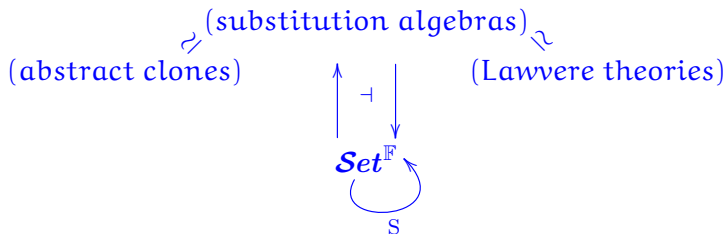
$$1 \quad \{*\} \equiv \{*\} \equiv \dots \quad \{*\} \quad \dots$$

$$V \quad \{\} \longrightarrow \{x_1\} \begin{array}{l} \longrightarrow \\ \longrightarrow \end{array} \dots \quad \{x_1, \dots, x_n\} \quad \dots$$

Substitution Algebras

[Fiore, Plotkin, Turi (1999)]

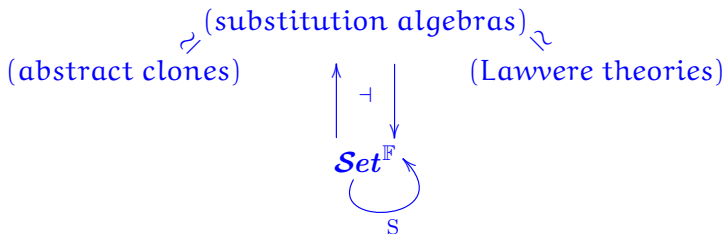
- Free substitution-algebra monad



Substitution Algebras

[Fiore, Plotkin, Turi (1999)]

- ▶ Free substitution-algebra monad



★ Definition

A substitution algebra is a structure

$$V \xrightarrow{\text{var}} A \xleftarrow{\text{sub}} A^V \times A$$

subject to four axioms: substitution, weakening, extensionality, and associativity.

Computational Interpretation

[Fiore, Staton (2014)]

► Computational metalanguage

$$\frac{\Gamma \vdash_v e : \mathbf{V}}{\Gamma \vdash_c \mathbf{var}_\tau(e) : \tau} \quad \frac{\Gamma, a : \mathbf{V} \vdash_c t : \tau \quad \Gamma \vdash_c u : \tau}{\Gamma \vdash_c \mathbf{sub}(a.t, u) : \tau}$$

Computational Interpretation

[Fiore, Staton (2014)]

► Computational metalanguage

$$\frac{\Gamma \vdash_v e : \mathbf{V}}{\Gamma \vdash_c \mathbf{var}_\tau(e) : \tau} \qquad \frac{\Gamma, a : \mathbf{V} \vdash_c t : \tau \quad \Gamma \vdash_c u : \tau}{\Gamma \vdash_c \mathbf{sub}(a.t, u) : \tau}$$

► Denotational semantics

computations

values

$$\mathbf{Kl}(S) \begin{array}{c} \xrightarrow{\quad \top \quad} \\ \xleftarrow{\quad \top \quad} \end{array} \mathbf{Set}^{\mathbb{F}}$$

► Equational theory (I)

- $\text{let } x = \text{sub}(a. M[a], N[]) \text{ in } P[x]$
 $\equiv \text{sub}(a. \text{let } x = M[a] \text{ in } P[x], \text{let } x = N[] \text{ in } P[x])$

► Equational theory (I)

- $\text{let } x = \text{sub}(\alpha. M[\alpha], N[]) \text{ in } P[x]$
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- Weakening axiom
 $\text{sub}(\alpha. M[] , N[]) \equiv M[]$

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- Substitution axiom
 $\text{sub}(\alpha. \text{var}(\alpha) , N[]) \equiv N[]$

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- Weakening axiom
 $\text{sub}(a. M[], N[]) \equiv M[]$
- Substitution axiom
 $\text{sub}(a. \text{var}(a), N[]) \equiv N[]$
- Abort law
 $(\text{let } x = \text{var}(a) \text{ in } P[x]) \equiv \text{var}(a)$

► Equational theory (I)

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- Abort law
 $(\text{let } x = \text{var}(a) \text{ in } P[x]) \equiv \text{var}(a)$

► Operational intuition

In $\text{sub}(a.t[a], u)$ the *jump point* a is declared and the computation proceeds as in $t[a]$ leading to a value if this produces one, or *aborting and restarting* with the computation of u if $\text{var}(a)$ is invoked.

► Equational theory (II)

- Extensionality axiom

$$\text{sub}(a.M[a], \text{var}(b)) \equiv M[b]$$

- Associativity axiom

$$\begin{aligned} & \text{sub}(a.\text{sub}(b.L[a, b], M[a]), N[]) \\ & \equiv \text{sub}(b.\text{sub}(a.L[a, b], N[]), \text{sub}(a.M[a], N[])) \end{aligned}$$

★ Substitution, Jumps, and Algebraic Effects

[Fiore, Staton (2014)]

- Computational interpretation of substitution algebras as a code-jumping mechanism.
- Adequate denotational semantics, equational theory, and abstract machine.
- Representation of first-order virtual effects.
- Extension incorporating effect handlers.

CPS Semantics

★ Theorem

[Lawvere (1969), Kock (1970)]

In the context of strong monads, T -algebra structures on objects A are in bijective correspondence with monad morphisms $T \rightarrow K_A$ for K_A the double-dualization or continuation monad relative to A .

- ▶ CPS translation: $S \rightarrow K_V$

CPS Semantics

★ Theorem

[Lawvere (1969), Kock (1970)]

In the context of strong monads, T -algebra structures on objects A are in bijective correspondence with monad morphisms $T \rightarrow K_A$ for K_A the double-dualization or continuation monad relative to A .

▶ CPS translation: $S \rightarrow K_V$

- $\text{var}_X \mapsto \lambda \alpha : V. \lambda k : V^X. \alpha$
- $\text{sub}_X \mapsto \lambda \langle M : (SX)^V, N : SX \rangle. \lambda k : V^X. M (N k) k$

Direct-Style CPS Semantics

```
signature SubstitutionAlgebra
= sig
  type V
  val var : V -> 'a
  val sub : ( V -> 'a ) * 'a -> 'a
end
```


Direct-Style CPS Semantics

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signature SubstitutionAlgebra
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end

structure sa :> SubstitutionAlgebra
= struct
  type V = unit -> (unit cont)
  fun var( a ) = throw (a()) ()
  fun sub( M , N )
    = callcc( fn k => ( M( fn() => throw k N ) ) )
end
```

Inception Algebras

★ **Definition**

An (L, P) -inception algebra is a structure

$$L \times P \xrightarrow{\text{recall}} A \xleftarrow{\text{incept}} A^L \times A^P$$

Inception Algebras

★ Definition

An (L, P) -inception algebra is a structure

$$L \times P \xrightarrow{\text{recall}} A \xleftarrow{\text{incept}} A^L \times A^P$$

subject to:

- Substitution axiom

$$\text{inc}(l.\text{rec}(l, P[]), x.L[x]) \equiv L[P]$$

- Extensionality axiom

$$\text{inc}(l.P[l], x.\text{rec}(k, x)) \equiv P[k]$$

- Weakening axiom

$$\text{inc}(l.M[], x.L[x]) \equiv M[]$$

- Associativity axiom

$$\begin{aligned} &\text{inc}(l.\text{inc}(k.P[l, k], x.K[l, x]), y.L[y]) \\ &\equiv \text{inc}(k.\text{inc}(l.P[l, k], y.L[y]), x.\text{inc}(l.K[l, x], y.L[y])) \end{aligned}$$

★ Examples

- Substitution algebras = $(V, 1)$ -inception algebras

★ Examples

- Substitution algebras = $(V, 1)$ -inception algebras
- More generally, for a set D , (V, D) -inception algebras model a code-jumping with data-passing mechanism:

$$\frac{\Gamma \vdash_V e : V \quad \Gamma \vdash_V d : D}{\Gamma \vdash_C \mathbf{rec}_\tau(e, d) : \tau}$$

$$\frac{\Gamma, l : V \vdash_C t : \tau \quad \Gamma, x : D \vdash_C u : \tau}{\Gamma \vdash_C \mathbf{inc}_\tau(l, t, x, u) : \tau}$$

Inception Algebras in Logic

► $L = \neg P$

\neg elimination

$$\frac{\neg P \quad P}{A}$$

excluded middle

$$\frac{\begin{array}{c} [\neg P] \\ \vdots \\ A \end{array} \quad \begin{array}{c} [P] \\ \vdots \\ A \end{array}}{A}$$

[de Groote (1995)]

► **NB** Inceptions are not local exceptions.

► Counterexample

► Programming idiom

```
signature LogicalInceptionAlgebra
= sig
  type 'a li
  val rec : 'a li * 'a -> 'b
  val inc : ( 'a li -> 'b ) * ( 'a -> 'b ) -> 'b
end
```

► Logical Inception Algebra Structure

► Programming idiom

```
signature LogicalInceptionAlgebra
= sig
  type 'a li
  val rec : 'a li * 'a -> 'b
  val inc : ( 'a li -> 'b ) * ( 'a -> 'b ) -> 'b
end
```

- **NB** `inc` generalises and introduces a level of abstraction over `callcc`:

$$\text{callcc}(f) = \text{inc}(f, \text{id})$$

$$\text{inc}(f, h) = \text{callcc}(f \circ \neg h)$$

► Programming idiom

```
signature LogicalInceptionAlgebra
= sig
  type 'a li
  val rec : 'a li * 'a -> 'b
  val inc : ( 'a li -> 'b ) * ( 'a -> 'b ) -> 'b
end
```

★ Example

```
val DeMorgan : ( 'a * 'b ) li -> ('a li,'b li) sum
= fn c =>
  inc( fn a => left a ,
      fn x => inc( fn b => right b ,
                  fn y => rec( c , (x,y) ) ) ) )
```

► Programming idiom

```
signature LogicalInceptionAlgebra
= sig
  type 'a li
  val rec : 'a li * 'a -> 'b
  val inc : ( 'a li -> 'b ) * ( 'a -> 'b ) -> 'b
end
```

► **Applications**

- Encoding of local and global exception mechanisms
- Safe exception handling in program modules
- Coroutines

Untyped Inception Algebras

- ▶ (V, V^n) -inception algebras

$$V \times V^n \xrightarrow{\text{rec}} A \xleftarrow{\text{inc}} A^V \times A^{V^n}$$

Untyped Inception Algebras

- ▶ (V, V^n) -inception algebras

$$V \times V^n \xrightarrow{\text{rec}} A \xleftarrow{\text{inc}} A^V \times A^{V^n}$$

model the untyped CPS calculus [Appel (1992); Thielecke (1997)].

- ▶ Conversion table:

$\text{rec}(l, \vec{x})$	$l\langle \vec{x} \rangle$
$\text{inc}(l. P[l], \vec{x}. L[\vec{x}])$	$P[l] \{ l\langle \vec{x} \rangle = L[\vec{x}] \}$

Nullary/Unary Untyped Inception Algebras

- ▶ Self recall

$$\mathbf{inc}(\ell.\mathbf{rec}(\ell, \ell), x.M[x]) \equiv \mathbf{inc}(\ell.M[\ell], x.M[x])$$

Nullary/Unary Untyped Inception Algebras

- ▶ Self recall

$$\mathbf{inc}(\ell.\mathbf{rec}(\ell, \ell), x.M[x]) \equiv \mathbf{inc}(\ell.M[\ell], x.M[x])$$

- ▶ Recursion

For

$$Y_t = \mathbf{inc}(\ell.\mathbf{rec}(\ell, \ell), \ell.\mathbf{sub}(a.t[a], \mathbf{rec}(\ell, \ell)))$$

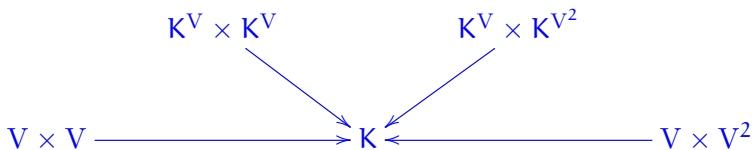
we have

$$Y_t \equiv \mathbf{sub}(a.t[a], Y_t)$$

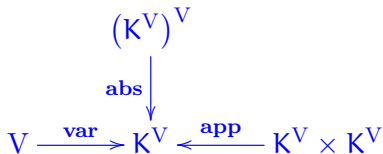
Unary/Binary Untyped Inception Algebras

► CPS lambda structure

The initial unary/binary untyped inception algebra



provides a CPS lambda structure



thereby inducing an homomorphic CPS interpretation

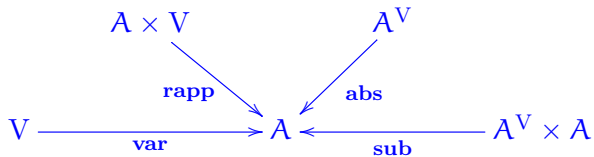
$$\Lambda \rightarrow K^V$$

of lambda terms.

► CPS interpretation

- $\mathbf{var} : x[e] \mapsto \mathbf{rec}(e, \langle x \rangle)$
- $\mathbf{abs} : M[x][e] \mapsto \mathbf{inc}(f.\mathbf{rec}(e, \langle f \rangle) , \langle a, e \rangle.M[a][e])$
- $\mathbf{app} : M[e], N[e] \mapsto \mathbf{inc}(m.M[m] , \langle f \rangle.\mathbf{inc}(n.N[n] , \langle a \rangle.\mathbf{rec}(f, \langle a, e \rangle)))$

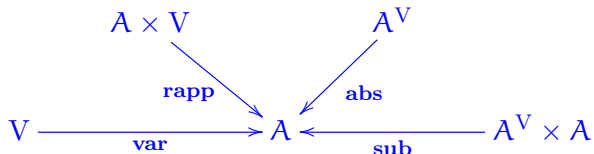
Right Lambda Algebras



$$(\beta_r) \quad \mathbf{rapp}(\mathbf{abs}(x.M[x]), a) \equiv M[a]$$

[Hirschowitz, Maggesi (2010); Hyland (2013)]

Right Lambda Algebras



$$\underbrace{(\beta_r)}_{\text{summon}} \quad \underbrace{\text{rapp}(\overbrace{\text{abs}(x. M[x])}^{\text{push } a}, a)}_{\text{pop in } x} \equiv M[a]$$

- ▶ Computational interpretation:

[Fiore, Staton (2014)]

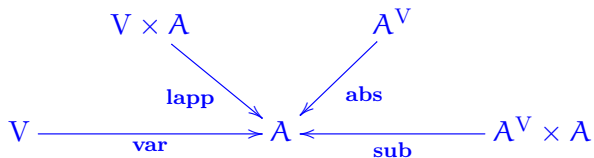
A mechanism for stack manipulation of code pointers.

- ▶ Application:

Stack abstract machine for CPS calculus.

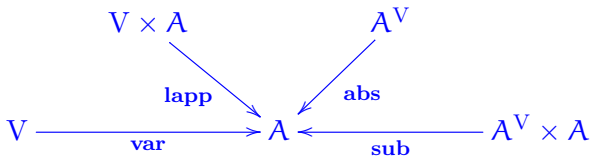
▶ CPS Structure

Left Lambda Algebras



$$(\beta_1) \quad \text{sub}(\ell. \text{lapp}(\ell, N[]), \text{abs}(k. M[k])) \equiv \text{sub}(k. M[k], N[])$$

Left Lambda Algebras



$$\underbrace{(\beta_l)}_{\text{resume}} \quad \text{sub} \left(\underbrace{\ell. \text{lapp}(\ell, N[])}_{\text{yield } N}, \underbrace{\text{abs}(k. M[k])}_{\text{thunk } M} \right) \equiv \text{sub}(k. M[k], N[])$$

- Computational interpretation:

A synchronous coroutine mechanism

- Application:

Producer-Consumer

► Producer-Consumer

Final Remarks

► Conclusions

- The first algebraic axiomatisation of control effects.
- Foundational analysis of the principles of programming with algebraic effects.

► Directions

- Models
- Classical/intermediate logics
- Algebraic theories

Appendix

Substitution Algebra Axioms

► Substitution

$$\text{sub}(a.\text{var}(a), M[]) \equiv M[]$$

$$\begin{array}{ccc} 1 \times A & & \\ \widehat{\text{var}} \times \text{id} \downarrow & \searrow \pi_2 & \\ A^V \times A & \xrightarrow{\text{sub}} & A \end{array}$$

► Extensionality

$$\text{sub}(a.M[a], \text{var}(b)) \equiv M[b]$$

$$\begin{array}{ccc} A^V \times V & & \\ \text{id} \times \text{var} \downarrow & \searrow \varepsilon & \\ A^V \times A & \xrightarrow{\text{sub}} & A \end{array}$$

► Weakening

$$\text{sub}(a.M[], N[]) \equiv M[]$$

$$\begin{array}{ccc} A^1 \times A & \xrightarrow{\cong} & A \times A \\ A^! \times \text{id} \downarrow & & \downarrow \pi_1 \\ A^V \times A & \xrightarrow{\text{sub}} & A \end{array}$$

► Associativity

$$\begin{aligned} & \text{sub}(a.\text{sub}(b.L[a, b], M[a]), N[]) \\ & \equiv \text{sub}(b.\text{sub}(a.L[a, b], N[]), \text{sub}(a.M[a], N[])) \end{aligned}$$

$$\begin{array}{ccc} (A^V \times A)^V \times A & \xrightarrow{\text{sub}^V \times \text{sub}} & (A^V \times A)^V \times (A^V \times A) & \xrightarrow{\text{sub}^V \times \text{sub}} & A^V \times A \\ \text{sub}^V \times \text{id} \downarrow & & & & \downarrow \text{sub} \\ A^V \times A & \xrightarrow{\text{sub}} & & & A \end{array}$$

Contraction Laws

- ▶ $\text{sub}(a. M[a, b], \text{var}(b)) \equiv M[b, b]$
- ▶ $\text{sub}(a. \text{sub}(b. M[a, b], N[]), N[]) \equiv \text{sub}(c. M[c, c], N[])$
- ▶ $\text{sub}(c. \text{let } x = M[c] \text{ in } N[x, c], L[])$
 $\equiv \text{sub}(a. \text{let } x = M[a] \text{ in } \text{sub}(b. N[x, b], L[]), L[])$

Inception Algebra Axioms

► Substitution

$$\text{inc}(\ell.\text{rec}(\ell, P[]), x.L[x]) \equiv L[P]$$

$$\begin{array}{ccc} P \times A^P & & \\ \downarrow & \searrow & \\ A^L \times A^P & \longrightarrow & A \end{array}$$

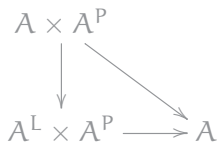
► Extensionality

$$\text{inc}(\ell.P[\ell], x.\text{rec}(k, x)) \equiv P[k]$$

$$\begin{array}{ccc} A^L \times L & & \\ \downarrow & \searrow & \\ A^L \times A^P & \longrightarrow & A \end{array}$$

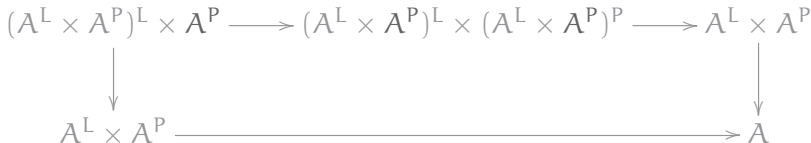
► Weakening

$$\mathbf{inc}(\ell. M[], x. L[x]) \equiv M[]$$



► Associativity

$$\begin{aligned}
 & \mathbf{inc}(\ell. \mathbf{inc}(k. P[\ell, k], x. K[\ell, x]), y. L[y]) \\
 & \equiv \mathbf{inc}(k. \mathbf{inc}(\ell. P[\ell, k], y. L[y]), x. \mathbf{inc}(\ell. K[\ell, x], y. L[y]))
 \end{aligned}$$



Inceptions Are Not Local Exceptions

► **NB**

$\text{sub}(e_1.\text{let } \ell = \text{sub}(e_2.\text{ret } e_2, \text{ret } e_1) \text{ in var } \ell, \langle \rangle) \equiv \langle \rangle$

however

```
let exception e1
in ( let val l
      = let exception e2
          in e2
          handle e2 => e1 end
      in
        raise l
      end )
handle e1 => () end
```

outputs `uncaught exception e2.`

Logical Inception Algebra Structure

```
signature LogicalInceptionAlgebra
= sig
  type 'a li
  val rec : 'a li * 'a -> 'b
  val inc : ( 'a li -> 'b ) * ( 'a -> 'b ) -> 'b
end

structure lia :> LogicalInceptionAlgebra
= struct
  type 'a li = 'a -> (unit cont)
  fun rec( a , x ) = throw (a x) ()
  fun inc( M , N )
    = callcc( fn k => ( M( fn x => throw k (N x) ) ) )
end
```

Sorted Inception Algebras

- ▶ Sorted sets

[Milner (1991)]

$$|-| : S \rightarrow S^*$$

- ▶ Structures

$$\begin{array}{ccc} V_\sigma \times \prod_i V_{|\sigma|_i} & & A^{V_\sigma} \times A^{\prod_i V_{|\sigma|_i}} \\ & \searrow & \swarrow \\ & A & \end{array}$$

in $\mathbf{Set}^{\mathbb{R}[S]}$

subject to the inception algebra axioms.

▶ Untyped Inception Algebras

Right Lambda Algebras

► Stack CPS structure

- $V \times V^n \longrightarrow A$
 $a, \langle a_1, \dots, a_n \rangle \mapsto \text{push}(\dots \text{push}(\text{var}(a), a_1) \dots, a_n)$
- $A^V \times A^{V^n} \longrightarrow A$
 $M[a], N[a_1, \dots, a_n] \mapsto \text{sub}(a.M[a] , \text{pop}(a_1 \dots \text{pop}(a_n.N[a_1, \dots, a_n])))$

► Parameterised fixpoint

- For

$$T[f, x] = \mathbf{in}(\mathbf{yield}(x, \mathbf{thunk}(z. \mathbf{push}(z, \mathbf{push}(f, \mathbf{var} f))))))$$

and

$$\mathbf{In}[x] = \mathbf{sub}(f. T[f, x], \mathbf{pop}(f. \mathbf{pop}(x. t[f, x])))$$

we have

$$\mathbf{In}[x] \equiv \mathbf{in}(\mathbf{yield}(x, \mathbf{thunk}(z. \mathbf{In}[z])))$$

Left Lambda Algebras

► Axioms

- $\text{sub}(\ell.\text{yield}(\ell, N[]), \text{thunk}(k.M[k])) \equiv \text{sub}(k.M[k], N[])$
- $\text{sub}(\ell.\text{thunk}(x.M[\ell, x]), N[])$
 $\equiv \text{thunk}(x.\text{sub}(\ell.M[\ell, x], N[]))$
- $\text{sub}(e.\text{yield}(L[e], M[e]), N[])$
 $\equiv \text{sub}(\ell.\text{yield}(\ell, \text{sub}(e.M[e], N[])),$
 $\text{sub}(e.\text{var}(L[e]), N[]))$

► Fixpoints

- For a computation $t(\cdot)$,

$$Y_t = \text{sub}(x. t(\text{yield}(x, \text{var } x)), \text{thunk}(x. t(\text{yield}(x, \text{var } x))))$$

we have

$$Y_t \equiv t(Y_t)$$

- For a parameterised computation $t[x](c.M[c])$, let

$$T = \text{thunk}(f. \text{thunk}(x. t[x](c. \text{sub}(p. \text{yield}(p, \text{var } c), \text{yield}(f, \text{var } f))))))$$

For

$$Y_t[x] = \text{sub}(f. t[x](c. \text{sub}(p. \text{yield}(p, \text{var } c), \text{yield}(f, T))), T)$$

we have

$$Y_t[x] \equiv t[x](c. Y_t[c])$$

► Producer-Consumer

Let

$$\mathbf{In}[a] \equiv \mathbf{in}(\mathbf{yield}(a, \mathbf{thunk}(x.\mathbf{In}[x])))$$

$$\mathbf{Out} \equiv \mathbf{thunk}(y.\mathbf{out}(\mathbf{out}(\mathbf{yield}(y, \mathbf{Out}))))$$

Then

$$\mathbf{sub}(a.\mathbf{In}[a], \mathbf{Out}) \equiv \mathbf{in}(\mathbf{out}(\mathbf{out}(\mathbf{sub}(x.\mathbf{In}[x], \mathbf{Out}))))$$