$\mathscr{V}$-cat-ification of functors

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joint work with

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Extensions of functors from sets to preorders

Given $T : \mathsf{Set} \longrightarrow \mathsf{Set}$, find $\overline{T} : \mathsf{Pre} \longrightarrow \mathsf{Pre}$ such that

1. The functor $\overline{T}$ is locally monotone, and
2. The square

$$\begin{array}{ccc} \mathsf{Pre} & \xrightarrow{\overline{T}} & \mathsf{Pre} \\ {\scriptstyle D}\uparrow & & \uparrow{\scriptstyle D} \\ \mathsf{Set} & \xrightarrow[T]{} & \mathsf{Set} \end{array} \qquad\qquad \begin{array}{c} \text{discrete preorder on } X \\ \uparrow \\ X \end{array}$$

commutes (perhaps only up to isomorphism).

Additionally: one may ask for a universal property of the above square (e.g., a left Kan extension).
Such extensions were proved to exist for various[a] (not all) functors, also for posets instead of preorders.

[a] A. Balan, A. Kurz, CALCO 2011

Extensions need not be unique

Both diagrams

$$(X, \leq) \longmapsto \text{connected components of } (X, \leq)$$

$$\begin{array}{ccc} \mathsf{Pre} & \xrightarrow{\mathsf{Id}} & \mathsf{Pre} \\ {\scriptstyle D}\uparrow & & \uparrow{\scriptstyle D} \\ \mathsf{Set} & \xrightarrow[\mathsf{Id}]{} & \mathsf{Set} \end{array} \qquad \begin{array}{ccc} \mathsf{Pre} & \xrightarrow{\qquad\qquad \pi_0 \qquad\qquad} & \mathsf{Pre} \\ {\scriptstyle D}\uparrow & & \uparrow{\scriptstyle D} \\ \mathsf{Set} & \xrightarrow[\qquad\qquad \mathsf{Id} \qquad\qquad]{} & \mathsf{Set} \end{array}$$

commute: Id and π_0 are extensions of Id : Set $\longrightarrow$ Set.

Id : Pre $\longrightarrow$ Pre is the "least extension" (i.e., Id $= Lan_D D$).

Why is the extension problem interesting?

1. Coalgebra: situations where simulations are more interesting than bisimulations. (E.g., J. Hughes, B. Jacobs, TCS, 2004.) Preorders/posets provide an environment for that.
2. Preorders and posets link universal coalgebra and domain theory.
3. Passage from sets to preorders/posets yields positive fragments of coalgebraic modal logic. (E.g., A. Balan, A. Kurz, JV, 2014, submitted.)

Overview of the talk

1. Locally monotone extensions to posets/preorders exist for any $T : \mathsf{Set} \longrightarrow \mathsf{Set}$.

   The technique: a "simplicial representation" of posets/preorders.

2. One can generalise further: for any $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ there is an extension $\overline{T} : \mathscr{V}\text{-}\mathsf{Cat} \longrightarrow \mathscr{V}\text{-}\mathsf{Cat}$, where $\mathscr{V}$ is a (commutative unital) quantale.

   Objects of $\mathscr{V}$-Cat are (rather general) "metric spaces".

   The technique: a "simplicial representation" of metric spaces.

The simplicial resolution of a preorder

For every preorder X, form a diagram

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0$$

of discrete preorders, where

1. X_0 is the set of elements of X,
2. X_1 is the set $\{(x', x) \mid x' \leq x\}$
3. d_0 and d_1 are the projections.

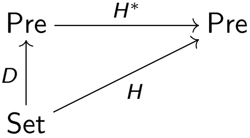
Then X can be recovered as a coinserter (a "2-dimensional coequaliser")

$$\begin{array}{ccccc} & & X_0 & & \\ & \overset{d_0}{\nearrow} & & \overset{c_X}{\searrow} & \\ X_1 & & \rotatebox{90}{$\leq$} & & \mathrm{coins}(d_0, d_1) \\ & \underset{d_1}{\searrow} & & \underset{c_X}{\nearrow} & \\ & & X_0 & & \end{array}$$

The left Kan extension of any $H : \mathsf{Set} \longrightarrow \mathsf{Pre}$

Define $H^*X := \mathrm{coins}(Hd_0, Hd_1)$, for a simplicial resolution (d_0, d_1) of a preorder X. The assignment $X \mapsto H^*X$ can be extended to a locally monotone functor.

Then the diagram



commutes (up to isomorphism) and exhibits H^* as a left Kan extension of H along D, i.e., there is an isomorphism

$$\frac{H^* \Rightarrow K}{H \Rightarrow KD}$$

of preorders, for any locally monotone $K : \mathsf{Pre} \longrightarrow \mathsf{Pre}$.

Example: left Kan extensions for powersets

1. Suppose $H : \mathsf{Set} \longrightarrow \mathsf{Pre}$ sends X to the powerset PX with the discrete preorder.

   Then $H^* : \mathsf{Pre} \longrightarrow \mathsf{Pre}$ sends $(X, \leq)$ to $(PX, \leq_{EM})$, where $\leq_{EM}$ is the Egli-Milner order.

   $$H^* = Lan_D(DP), \quad P : \mathsf{Set} \longrightarrow \mathsf{Set}$$

2. Suppose $H : \mathsf{Set} \longrightarrow \mathsf{Pre}$ sends X to the powerset PX ordered by inclusion.

   Then $H^* : \mathsf{Pre} \longrightarrow \mathsf{Pre}$ sends $(X, \leq)$ to the poset of lowersets on $(X, \leq)$.

   $$H^* = Lan_D H, \quad H : \mathsf{Set} \longrightarrow \mathsf{Pre}$$

A characterisation of left Kan extensions to preorders

For a locally monotone $K : \mathsf{Pre} \longrightarrow \mathsf{Pre}$, TFAE:

1. K is $Lan_D H$ for some $H : \mathsf{Set} \longrightarrow \mathsf{Pre}$.
2. K preserves coinserters of simplicial resolutions.

And K is $Lan_D(DT)$ for some $T : \mathsf{Set} \longrightarrow \mathsf{Set}$, if K, in addition, preserves discrete preorders.

Why does this hold?

Coinserters of simplicial resolutions form a density presentation of $D : \mathsf{Set} \longrightarrow \mathsf{Pre}$. This means:

1. Pre is the closure of Set under these coinserters.
2. These coinserters are preserved by every $\mathsf{Pre}(DX, -)$.

Remarks

1. There is a finer characterisation for finitary functors.
2. Essentially the same reasoning works for posets.

In any case, one has to employ (not very deep) enriched category theory.

$\mathscr{V}$-Cat for a commutative unital quantale $\mathscr{V}$

1. A quantale $\mathscr{V} = (V, e, \otimes, [-,-])$, where V is a complete lattice, $\otimes$ is monotone, associative, has e as a unit, and $a \otimes v \leq b$ iff $a \leq [v, b]$.
2. A small $\mathscr{V}$-category $\mathscr{K}$: a set of objects x, y, ..., and every $\mathscr{K}(x, y)$ is in V.
   Moreover: $e \leq \mathscr{K}(x, x)$ and $\mathscr{K}(y, z) \otimes \mathscr{K}(x, y) \leq \mathscr{K}(x, z)$.
3. A $\mathscr{V}$-functor $f : \mathscr{K} \longrightarrow \mathscr{L}$: the object assignment $x \mapsto fx$ and $\mathscr{K}(x, y) \leq \mathscr{L}(fx, fy)$.
4. A $\mathscr{V}$-category of $\mathscr{V}$-functors $[\mathscr{K}, \mathscr{L}]$: objects are $\mathscr{V}$-functors from $\mathscr{K}$ to $\mathscr{L}$ and $[\mathscr{K}, \mathscr{L}](f, g) = \bigwedge_x \mathscr{L}(fx, gx)$.

All of this: $\mathscr{V}$-Cat, enriched in itself.

For $\mathscr{V} = 2$ we have $\mathscr{V}$-Cat $=$ Pre.

Further examples of $\mathscr{V}$-Cat

For $\mathscr{V}$ being the interval $[0; +\infty]$ with reversed order and $+$ as the tensor: a $\mathscr{V}$-category $\mathscr{K}$ is a (generalised) metric space:

1. $\mathscr{K}(x, y)$ is the "amount of work to get from x to y" (due to nonsymmetry: $\mathscr{K}(x, y) = \mathscr{K}(y, x)$ does not hold in general).
2. $\mathscr{K}(x, x) = 0$, $\mathscr{K}(y, z) + \mathscr{K}(x, y) \geq \mathscr{K}(x, z)$.

Analogously, a $\mathscr{V}$-functor is a non-expanding map: $\mathscr{K}(x, y) \geq \mathscr{L}(fx, fy)$.

Such an intuition works for any quantale $\mathscr{V}$.

The "simplicial resolution" of a $\mathscr{V}$-category $\mathscr{K}$

Define $N_{\mathscr{K}} : V^+ \longrightarrow \mathsf{Set}$, where

1. V^+ has as objects: all elements of V plus an extra element $*$, morphisms $d_0^v : v \longrightarrow *$ and $d_1^v : v \longrightarrow *$.
2. $N_{\mathscr{K}}$ works as follows:
    1. $* \mapsto$ set of objects of $\mathscr{K}$, $v \mapsto \{(x', x) \mid v \leq \mathscr{K}(x', x)\}$
    2. Nd_0^v is the first projection, Nd_1^v is the second projection.

Further, one can define a weight $\varphi : (V^+)^{op} \longrightarrow \mathscr{V}\text{-Cat}$ such that

$$\mathscr{K} \cong \text{colimit of } N_{\mathscr{K}} \text{ weighted by } \varphi$$

Corollary: the extension result

$H^* = Lan_D H$ exists for every $H : \mathsf{Set} \longrightarrow \mathscr{V}\text{-Cat}$.

Extension of the powerset

For $P : \mathsf{Set} \longrightarrow \mathsf{Set}$, the left Kan extension

$$P_{\mathscr{V}} = Lan_D(DP) : \mathscr{V}\text{-Cat} \longrightarrow \mathscr{V}\text{-Cat}$$

is the "Egli-Milner" construction:

$$(\mathcal{P}_V \mathscr{K})(A, B) \;=\; \bigvee \{v \mid \forall a \in A\, \exists b \in B.\, v \leq \mathscr{K}(a, b) \text{ and } \forall b \in B\, \exists a \in A.\, v \leq \mathscr{K}(a, b)\}$$

In case $\mathscr{V} = [0; +\infty]$ (i.e., when $\mathscr{V}$-Cat are metric spaces), the extension gives the Hausdorff distance.