From Admissible Rules to (New) Unification Types

George Metcalfe

Mathematics Institute
University of Bern

Work in progress with Leonardo Cabrer

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Unifiability and Admissibility

$x \vee y \approx \top$ is **unifiable** in the class of *Boolean algebras*, e.g. by

$$\sigma(x) = x; \quad \sigma(y) = \neg x,$$

and also in the class $\mathcal{HA}$ of *Heyting algebras*, e.g. by

$$\sigma_1(x) = \top;\ \sigma_1(y) = y \qquad \text{or} \qquad \sigma_2(x) = x;\ \sigma_2(y) = \top.$$

Moreover, the "disjunction property" is **admissible** in $\mathcal{HA}$:

$$\sigma \text{ unifies } x \vee y \approx \top \text{ in } \mathcal{HA} \quad \Longrightarrow \quad \sigma \text{ unifies } x \approx \top \text{ or } y \approx \top \text{ in } \mathcal{HA}.$$

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Question

Can **admissibility** be determined by comparing **unifiers**?

Equational Unification

Let us fix an equational class $\mathcal{V}$ of $\mathcal{L}$-algebras for a language $\mathcal{L}$, and denote by $\mathbf{Fm}_{\mathcal{L}}(X)$ the formula algebra of $\mathcal{L}$ over $X \subseteq \omega$.

A $\mathcal{V}$-**unifier** of a set of $\mathcal{L}$-identities Σ over $X \supseteq \mathrm{Var}(\Sigma)$ is a substitution

$$\sigma \colon \mathbf{Fm}_{\mathcal{L}}(X) \to \mathbf{Fm}_{\mathcal{L}}(\omega)$$

satisfying

$$(\varphi \approx \psi) \in \Sigma \qquad \Longrightarrow \qquad \mathcal{V} \models \sigma(\varphi) \approx \sigma(\psi).$$

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Admissibility

A **clause** (an ordered pair of finite sets of $\mathcal{L}$-identities)

$$\Sigma \Rightarrow \Delta$$

is $\mathcal{V}$**-admissible** if each $\mathcal{V}$-unifier of Σ over $\mathrm{Var}(\Sigma \cup \Delta)$ is a $\mathcal{V}$-unifier of some $\varphi \approx \psi$ in Δ.

Note. For any finite set Σ of $\mathcal{L}$-identities

$$\Sigma \text{ is } \mathcal{V}\text{-unifiable} \quad \Longleftrightarrow \quad \Sigma \Rightarrow \emptyset \text{ is not } \mathcal{V}\text{-admissible.}$$

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μ-Sets

A subset M of a preordered set $\mathbf{P} = \langle P, \leq \rangle$ is called

- **complete** for $\mathbf{P}$ if for all $x \in P$, there exists $y \in M$ such that $x \leq y$
- and a **μ-set** for $\mathbf{P}$ if also $x \not\leq y$ and $y \not\leq x$ for all distinct $x, y \in M$.

Note. All μ-sets for $\mathbf{P}$ have the same cardinality.

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Types

A preordered set $\mathbf{P} = \langle P, \leq \rangle$ is said to be

- **unitary** (type 1) if it has a μ-set of cardinality 1
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Unification Type of an Equational Class

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Examples

- The class of *Boolean algebras* is **unary** (Büttner & Simonis 1987).
  If unifiable in the class, $\{\varphi \approx \top\}$ has a **most general unifier** defined by $\sigma(x) = \neg\varphi \vee x$ for each $x \in \mathrm{Var}(\varphi)$.
- The class of *Heyting algebras* is **finitary** (Ghilardi 1999).
- The class of *semigroups* is **infinitary** (Plotkin 1972).
  E.g., $\{x \cdot y \approx y \cdot x\}$ has a μ-set $\{\sigma_{m,n} \mid \gcd(m,n) = 1\}$ where $\sigma_{m,n}(x) = z^m$ and $\sigma_{m,n}(y) = z^n$.
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if σ_1 $\mathcal{V}$-unifies *fewer identities than* σ_2:

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Unification Type and Exact Type

Equational Class	**Unification Type**	**Exact Type**
Boolean Algebras	*Unitary*	*Unitary*
Heyting Algebras	*Finitary*	*Finitary*
Semigroups	*Infinitary*	*Infinitary* or *Nullary*
Modal algebras	*Nullary*	*Nullary*
Distributive Lattices	*Nullary*	*Unitary*
Stone Algebras	*Nullary*	*Unitary*
Bounded Distributive Lattices	*Nullary*	*Finitary*
Idempotent Semigroups	*Nullary*	*Finitary*
MV-algebras	*Nullary*	*Finitary*

Identities Algebraically

We identify a finite set of identities Σ with a **finitely presented algebra**

$$\mathbf{Fp}_{\mathcal{V}}(\Sigma) \;=\; \frac{\mathbf{F}_{\mathcal{V}}(\mathrm{Var}(\Sigma))}{\mathrm{Cg}(\Sigma)}$$

and denote the class of finitely presented algebras in $\mathcal{V}$ by $\mathrm{FP}(\mathcal{V})$.

Unifiers Algebraically

Given $\mathbf{A}, \mathbf{B} \in \mathrm{FP}(\mathcal{V})$, a homomorphism

$$u\colon \mathbf{A} \to \mathbf{B}$$

is called

- a **unifier** for $\mathbf{A}$ if $\mathbf{B}$ is *projective* in $\mathcal{V}$ (i.e., a retract of $\mathbf{F}_{\mathcal{V}}(\omega)$)
- a **coexact unifier** for $\mathbf{A}$ if $\mathbf{B}$ embeds into $\mathbf{F}_{\mathcal{V}}(\omega)$.

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Ordering Homomorphisms

For $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathrm{FP}(\mathcal{V})$ and homomorphisms $u_1 \colon \mathbf{A} \to \mathbf{B}_1$, $u_2 \colon \mathbf{A} \to \mathbf{B}_2$,

$$u_2 \leq u_1 \quad \text{“}u_1 \text{ is } \textbf{more general than } u_2\text{”},$$

if there exists a homomorphism $f \colon \mathbf{B}_1 \to \mathbf{B}_2$ such that

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Ghilardi's Theorem

Let the $\mathcal{V}$**-unification type** of $\mathbf{A} \in \mathrm{FP}(\mathcal{V})$ be the type of the $\leq$-preordered set of unifiers of **A**.

Theorem (Ghilardi 1997)

For any finite $\mathcal{V}$-unifiable set of identities Σ:

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Our Theorem

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Corollary

If $\mathcal{V}$ is locally finite, then it has unitary or finitary exact unification type.

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Corollary

If $\mathcal{V}$ is locally finite, then it has unitary or finitary exact unification type.

Our Theorem

Let the $\mathcal{V}$**-exact type** of $\mathbf{A} \in \mathrm{FP}(\mathcal{V})$ be the type of the $\leq$-preordered set of coexact unifiers of **A**.

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Further Questions

- Are there equational classes with…
  - …unification type ω and exact type 1?
  - …unification type ∞ and exact type 1, ω, or 0?
  - …unification type 0 and exact type ∞?
- How does the new hierarchy relate to finding axiomatizations for admissible rules, determining structural completeness, etc.?
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