Subframe Formulas and Stable Formulas in Intuitionistic Logic

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**NNIL**-formulas are propositional formulas that do not allow nesting of implication to the left (e.g. $(p \to q) \to r$ is forbidden).

These formulas were introduced by VvBdJR995, where it was shown that NNIL-formulas are exactly the formulas that are closed under taking submodels of Kripke models.

Today we show that the set of NNIL-formulas represents (up to frame equivalence) the set of subframe formulas and that subframe logics can be axiomatized by NNIL-formulas (NBdiss, 2006).
We also introduce **ONNILLI-formulas**, only NNIL to the left of implications, and show that ONNILLI-formulas are formulas that are closed under order-preserving images of (descriptive and Kripke) frames.

We obtain a result that the set of ONNILLI-formulas represents (up to frame equivalence) the set of stable formulas, introduced by B$^2$2013.
The J-de J-formula of finite frame $\mathcal{F}$ axiomatizes the least intermediate logic that does not have $\mathcal{F}$ as its frame. A descriptive frame $\mathcal{G}$ refutes such a formula iff $\mathcal{F}$ is a p-morphic image of a generated subframe of $\mathcal{G}$. 
Zakharyaschev 1989, 1996 introduced subframe formulas. For each finite rooted frame $\mathcal{F}$ the subframe formula of $\mathcal{F}$ is refuted in a frame $\mathcal{G}$ iff $\mathcal{F}$ is a p-morphic image of a subframe of $\mathcal{G}$.

These subframe logics are exactly those logics whose frames are closed under taking subframes.

There are continuum many of them and each has the finite model property. An intermediate logic $L$ is a subframe logic iff it is axiomatized by $(\land, \to)$-formulas.
B and B introduced stable formulas.

For each finite rooted frame $\mathcal{F}$ the stable formula of $\mathcal{F}$ is refuted in a frame $\mathcal{G}$ iff $\mathcal{F}$ is an order-preserving image of $\mathcal{G}$ ($B^22013$).

Stable logics are intermediate logics for which its frame class is closed under order-preserving images. They are axiomatized by stable formulas. There is a continuum of stable logics and all stable logics have the finite model property.

A good syntactic characterization remained an open problem.
The VvBdJR result implies that NNIL-formulas are also preserved under taking subframes. Moreover, for each finite rooted frame $\mathcal{F}$, NBdiss (2006) constructs a NNIL-formula that is its subframe formula.

Hence, an intermediate logic is a subframe logic iff it is axiomatized by NNIL-formulas. This also implies that each NNIL-formula is frame-equivalent to a ($\land$, $\rightarrow$)-formula and vice versa.
We introduce **ONNILLI-formulas**, only NNIL to the left of implications, and show that ONNILLI-formulas are formulas that are preserved under order-preserving images of (descriptive and Kripke) frames.

We also obtain that the set of ONNILLI-formulas represents (up to frame equivalence) the set of stable formulas.

Examples of ONNILLI-formulas are **LC**: \((p \rightarrow q) \lor (q \rightarrow p)\) (also NNIL), **KC**: \(\neg p \lor \neg \neg p\).
Let $\mathcal{F} = (W, R)$ be a Kripke frame. For every $w \in W$ and $U \subseteq W$ let

$$R(w) = \{ v \in W : wRv \},$$
$$R^{-1}(w) = \{ v \in W : vRw \},$$
$$R(U) = \bigcup_{w \in U} R(w),$$
$$R^{-1}(U) = \bigcup_{w \in U} R^{-1}(w).$$
1. Let $\mathcal{F} = (W, R)$ be a Kripke frame. A frame $\mathcal{F}' = (W', R')$ is called a subframe of $\mathcal{F}$ if $W' \subseteq W$ and $R'$ is the restriction of $R$ to $W'$.

2. Let $\mathcal{F} = (W, R, \mathcal{P})$ be a descriptive frame. A descriptive frame $\mathcal{F}' = (W', R', \mathcal{P}')$ is called a subframe of $\mathcal{F}$ if $(W', R')$ is a subframe of $(W, R)$, $\mathcal{P}' = \{ U \cap W' : U \in \mathcal{P} \}$ and the topo-subframe condition, is satisfied:

$$\forall U \subseteq W' \ (W' \setminus U \in \mathcal{P}' \implies W \setminus R^{-1}(U) \in \mathcal{P})$$
Proposition

Let $\mathcal{F} = (W, R, P)$ and $\mathcal{F}' = (W', R', P')$ be descriptive frames. If $\mathcal{F}'$ is a subframe of $\mathcal{F}$, then for every descriptive valuation $V'$ on $\mathcal{F}'$ there exists a descriptive valuation $V$ on $\mathcal{F}$ such that the restriction of $V$ to $W'$ is $V'$. 
**NNIL–FORMULAS AND SUBMODELS**

NNIL-formulas are known to have the following normal form:

\[ \varphi := \bot | p | \varphi \land \varphi | \varphi \lor \varphi | p \rightarrow \varphi \]

**Theorem (VvBdJR)**

Let \( \mathcal{M} = (W, R, V) \) and \( \mathcal{N} = (W', R', V') \) be (descriptive of Kripke) frames.

1. If \( \mathcal{N} \) is a submodel of \( \mathcal{M} \), then for each \( \varphi \in \text{NNIL} \) and \( w \in W' \), \( \mathcal{M}, w \models \varphi \implies \mathcal{N}, w \models \varphi \).

2. If for all \( w \) in submodels \( \mathcal{N} \) of \( \mathcal{M} \), \( \mathcal{M}, w \models \varphi \) implies \( \mathcal{N}, w \models \varphi \), then \( \exists \psi \in \text{NNIL} \) (IPC \( \vdash \psi \leftrightarrow \varphi \)).

(1) implies that NNIL-formulas are preserved under taking subframes of (Kripke and descriptive) frames.
**Definition**

Let $\mathcal{M} = (\mathcal{F}, V)$ be a descriptive model for $p_1, \ldots, p_n$. If $w$ in $\mathcal{M}$, $\text{col}(w)$ (the color of $w$) = $i_1 \ldots i_n$ such that:

$$i_k = \begin{cases} 
1, & \text{if } w \models p_k, \\
0, & \text{if } w \nvDash p_k.
\end{cases}$$

A finite model $\mathcal{M} = (W, R, V)$ is colorful if

$$\forall w \in W \exists! p_w(V(p_w) = R(w)).$$
**Lemma**

Let $(\mathfrak{F}, V)$ be a colorful model. Then for every $w, v \in W$ we have:

1. $w = v$ iff $\text{col}(w) = \text{col}(v)$,
2. $w \neq v$ and $w R v$ iff $\text{col}(w) < \text{col}(v)$. 
For finite rooted frames $\mathcal{F}$ we inductively define the subframe formula $\beta(\mathcal{F})$ in NNIL.

\[
\text{prop}(v) := \{p_k \mid v \models p_k, k \leq n\}, \quad \text{notprop}(v) := \{p_k \mid v \not\models p_k, k \leq n\}.
\]

If $v$ is a maximal, then

\[
\beta(v) := \bigwedge \text{prop}(v) \rightarrow \bigvee \text{notprop}(v)
\]

Let $w_1, \ldots, w_m$ be all the immediate successors of $w$.

\[
\beta(w) := \bigwedge \text{prop}(w) \rightarrow \bigvee \text{notprop}(w) \lor \bigvee_{i=1}^{m} \beta(w_i).
\]

Finally, $\beta(\mathcal{F}) := \beta(r)$, where $r$ is the root of $\mathcal{F}$. 
Crucial Property of Subframe Formulas

Theorem
Let \( G = (W', R', \mathcal{P}') \) be a descriptive frame and let \( \mathcal{F} = (W, R) \) be a finite rooted frame. Then

\[ G \not\models \beta(\mathcal{F}) \iff \mathcal{F} \text{ is a } p\text{-morphic image of a subframe of } G. \]

The proof depends on the fact that, if \( \bigwedge \text{prop}(v) \rightarrow \bigvee \text{notprop}(v) \) is false anywhere, then some node above will need to have the color of \( v \) (with \( \text{prop}(v) \) true and \( \text{notprop}(v) \) false). If \( \bigwedge \text{prop}(v) \rightarrow \bigvee \text{notprop}(v) \lor \bigvee_{i=1}^{m} \beta(w_i) \) is false anywhere, then some node above will need to have the color of \( w \) with above it nodes of the colors of the \( w_i \). Falsity of \( \beta(\mathcal{F}) \) will then guarantee nodes of the right colors in the proper order.
Theorem

1. An intermediate logic $L$ is a subframe logic iff $L$ is axiomatized by NNIL-formulas.
2. The class of NNIL-formulas is up to frame-equivalence the class of subframe formulas.
3. Each NNIL-formula is frame-equivalent to a $(\land, \rightarrow)$-formula.

A direct syntactic transformation of NNIL-formulas into frame-equivalent $(\land, \rightarrow)$-formulas can be found in Fanthesis2008. No way is known to transform a $(\land, \rightarrow)$-formula directly syntactically into a NNIL-formula.
We construct a new class of formulas, ONNILLI, preserved by order-preserving maps. 

\((X, R), (Y', R')\) Kripke frames. \(f : X \rightarrow Y\) is order-preserving if \(u R v\) implies \(f(u) R' f(v)\) and is admissible\(^2\) if appropriate.

Applied to models we assume \(f\) to be valuation preserving as well.

**Proposition**

Let \(\mathcal{M} = (X, R, V)\) and \(\mathcal{N} = (Y, R', V')\) be two (Kripke or descriptive) models and \(f : X \rightarrow Y\) an order-preserving map. Then,

\[
\forall u \in X, \varphi \in \text{NNIL} \quad (f(u) \models \varphi \Rightarrow u \models \varphi)
\]

\(^2\) \(W \setminus f^{-1}(W \setminus U') \in \mathcal{P}\)
**Order preserving functions and NNIL-formulas II**

*Proof.* Only the last inductive step is non-trivial. Assume $f(u) \models \varphi \Rightarrow u \models \varphi$ for all $u \in X$ (IH). Suppose $f(u) \models p \rightarrow \varphi$, and let $u R v$ with $v \models p$. Then $f(u) R f(v)$ and $f(v) \models p$. So, $f(v) \models \varphi$. By IH, $v \models \varphi$. So $u \models p \rightarrow \varphi$. □

Note that the identity function from a submodel into the larger model is obviously an order-preserving function. Thus this shows that NNIL-formulas are also exactly the ones that are preserved backwards by order-preserving functions on models.
ONNILLI-FORMULAS

Definition

1. BASIC is the closure of the set of the atoms plus $\top$ and $\bot$ under conjunctions and disjunctions.

2. The class ONNILLI (only NNIL to the left of implications) is defined as the closure of $\{\varphi \rightarrow \psi \mid \varphi \in \text{NNIL}, \psi \in \text{BASIC}\}$ under conjunctions and disjunctions.

So, no iterations of implications in ONNILLI-formulas except inside the NNIL-part. Note:

If $\psi \in \text{BASIC}$, $f$ valuation-preserving, then $f(v) \models \psi \iff v \models \psi$. 
KC is ONNILLI

Example

$\neg p \lor \neg \neg p$ is ONNILLI. To see this, write it as

$((p \rightarrow \bot) \lor (\neg p \rightarrow \bot))$, and note that $\neg p$ is in NNIL.

$\neg p \lor \neg \neg p$ is not preserved under taking subframes. So, it cannot be frame-equivalent to a NNIL-formula. Thus, ONNILLI $\not\subseteq$ NNIL. We will see later that also NNIL $\not\subseteq$ ONNILLI.
Order-preserving functions and ONNILLI-formulas I

Let $\mathcal{M} = (X, R, V)$ and $\mathcal{N} = (Y, R', V')$ be Kripke or descriptive, $f : X \rightarrow Y$ surjective, order-preserving:
If $\varphi \in \text{ONNILLI}$, then $\mathcal{M} \models \varphi \implies \mathcal{N} \models \varphi$.

Proof.
Let $\varphi = \psi \rightarrow \chi$ with $\psi \in \text{NNIL}, \chi \in \text{BASIC}$, $\mathcal{M} \models \psi \rightarrow \chi$, i.e. $u \models \psi \rightarrow \chi$ for all $u \in X$.
f is surjective: all elements of $Y$ are of the form $f(u), u \in X$.
Assume $f(u) \models \psi$. By previous, $u \models \psi$.
$u \models \psi \rightarrow \chi \implies u \models \chi \implies f(u) \models \chi$.
Hence, $f(u) \models \psi \rightarrow \chi$. Thus, $\mathcal{N} \models \psi \rightarrow \chi$.

Validity in models is needed, truth in a node insufficient. Also surjectivity is an essential.
Corollary
Let \( \mathcal{F} = (X, R) \) and \( \mathcal{G} = (Y, R') \) be (Kripke or descriptive) frames and \( f : X \to Y \) an order-preserving map from \( \mathcal{F} \) onto \( \mathcal{G} \). Then, for each \( \varphi \in \text{ONNILLI} \), \( \mathcal{F} \models \varphi \implies \mathcal{G} \models \varphi \).
Stable formulas and ONNILLI

Definition

1. If \( c \) is an \( n \)-color we write \( \psi_c \) for \( p_1 \land \cdots \land p_k \to q_1 \lor \cdots \lor q_m \) if \( p_1 \ldots p_k \) are the propositional variables that are 1 in \( c \) and \( q_1 \ldots q_m \) the ones that are 0 in \( c \).

2. If \( M \) is colorful and \( w \in W \), we write \( Col(M_w) \) for the formula \( \text{prop}(w) \land \land \{ \psi_c \mid c \text{ a color that is not in } M_w \} \).

3. \( \gamma(M) = \lor \{ Col(M_w) \to p_{w_1} \lor \cdots \lor p_{w_m} \mid w \in W, w_1, \ldots w_m \) are all the proper successors of \( w \} \).

Let \( \mathcal{F} \) be a finite rooted frame. We define a valuation \( V \) on \( \mathcal{F} \) such that \( M = (\mathcal{F}, V) \) is colorful and define \( \gamma(\mathcal{F}) \) by

\[ \gamma(\mathcal{F}) := \gamma(M). \]

We call \( \gamma(\mathcal{F}) \) the stable formula of \( \mathcal{F} \). \( \gamma(\mathcal{F}) \) is an ONNILLI-formula.
**Lemmas**

**Lemma**
Assume $\mathcal{M} = (W, R, V)$ is colorful, $w \in W$, $u'$ and $v'$ are nodes in an arbitrary (Kripke or descriptive) model $\mathcal{M}' = (W', R', V')$ such that $u' R' v'$. Then

1. If $\text{col}(u') = \text{col}(u)$ and $\text{col}(v') = \text{col}(v)$ for $u, v \in W$, then $u R v$.
2. If $u' \models \text{Col}(\mathcal{M}_u)$, then $u'$ and $v'$ both have one of the colors available in $\mathcal{M}_u$.
3. If $u' \not\models \text{Col}(\mathcal{M}_w) \rightarrow p_{w_1} \lor \cdots \lor p_{w_m}$, then there is $v'' \in W'$ such that $u' R v'$ and $\text{col}(v'') = \text{col}(w)$.

**Lemma**
Let $\mathcal{F}$ be a finite rooted frame. Then $\mathcal{F} \not\models \gamma(\mathcal{F})$. 
The basic ONNILLI theorem

Corollary

Let $\mathcal{F} = (W, R)$ be a finite rooted frame and let $\mathcal{G}$ a (Kripke or descriptive) frame. Then

1. $\mathcal{G} \not\models \gamma(\mathcal{F})$ iff there is a surjective order-preserving map from a generated subframe of $\mathcal{G}$ onto $\mathcal{F}$.

2. $\mathcal{G} \not\models \gamma(\mathcal{F})$ iff there is a surjective order-preserving map from $\mathcal{G}$ onto $\mathcal{F}$. 
Proof of the basic ONNILLI theorem

(1) ⇒: We know that $\mathcal{F} \not
\gamma(\mathcal{F})$. Since $\gamma(\mathcal{F})$ is ONNILLI, it is preserved under order-preserving images. So, $\mathcal{G} \not
\gamma(\mathcal{F})$.

⇐: Let $\mathcal{N}$ on $\mathcal{G}$, $\mathcal{N}$, $u \not
\gamma(\mathcal{F})$. Then $\forall w \in W \exists w', u R w'$ with $Col(M_w)$ true and $p_{w_1}, \ldots, p_{w_m}$ false. Thus, $w'$ has the color of $w$ and its successors have colors of successors of $w$. Let $W'$ be the set of the chosen $w'$s. As $W$ is finite, $W'$ is also finite.

Let $\mathcal{N}' = M_{R(W')}$. 

Now define $f : R(W') \to W$ by $f(u) = w$ if $col(u) = col(w)$.

If $u'R v' \in R(W')$, then there are $u R v \in W$ such that $col(u') = col(u)$ and $col(v') = col(v)$. So, $f$ is order-preserving.

Finally, $\forall w \in W \exists u \in R(W')(col(u) = col(w))$. Thus, $f(u) = w$ and $f$ is also surjective.
Stable Logics and ONNILLI

Theorem

1. An intermediate logic \( L \) is stable iff \( L \) is axiomatized by ONNILLI-formulas.

2. The class of ONNILLI-formulas is up to frame-equivalence the class of stable formulas.
Example
NNIL-formulas that are not equivalent to an ONNILLI-formula.

For each $n$, the logic $BD_n$ of frames of depth $\leq n$ is preserved under taking subframes. Thus, it is a subframe logic axiomatized by NNIL formulas.

But there are frames of depth $n$ having frames of depth $m > n$ as order-preserving images. So $BD_n$ is not a stable logic and cannot be axiomatized by ONNILLI formulas. Thus, the class of ONNILLI-formulas does not contain the class of NNIL-formulas.
Some examples of stable logics

$LC_n$ be the logic of all linear rooted frames of depth $\leq n$,

$BW_n$ be the logic of all rooted frames of width $\leq n$,

$BTW_n$ be the logic of all rooted descriptive frames of cofinal width $\leq n$,

OPEN QUESTION

It is an open problem whether ONNILLI-formulas are exactly the ones that are preserved under order-preserving preserving maps of models.
THE END

THANKS!