Subframe Formulas and Stable Formulas in Intuitionistic Logic

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NNIL

NNIL-formulas are propositional formulas that do not allow nesting of implication to the left (e.g. $(p \to q) \to r$ is forbidden).

These formulas were introduced by VvBdJR995, where it was shown that NNIL-formulas are exactly the formulas that are closed under taking submodels of Kripke models.

Today we show that the set of NNIL-formulas represents (up to frame equivalence) the set of subframe formulas and that subframe logics can be axiomatized by NNIL-formulas (NBdiss, 2006).

ONNILLI

We also introduce ONNILLI-formulas, only NNIL to the left of implications, and show that ONNILLI-formulas are formulas that are closed under order-preserving images of (descriptive and Kripke) frames.

We obtain ss a result that the set of ONNILLI-formulas represents (up to frame equivalence) the set of stable formulas, introduced by $\mathrm{B}^2 2013$.

The J-de J-formula of finite frame $\mathfrak{F}$ axiomatizes the least intermediate logic that does not have $\mathfrak{F}$ as its frame. A descriptive frame $\mathfrak{G}$ refutes such a formula iff $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{G}$.

Subframe Formulas

Zakharyaschev 1989,1996 introduced subframe formulas. For each finite rooted frame $\mathfrak{F}$ the subframe formula of $\mathfrak{F}$ is refuted in a frame $\mathfrak{G}$ iff $\mathfrak{F}$ is a p-morphic image of a subframe of $\mathfrak{G}$.

These subframe logics are exactly those logics whose frames are closed under taking subframes.

There are continuum many of them and each has the finite model property. An intermediate logic L is a subframe logic iff it is axiomatized by $(\wedge, \to)$-formulas.

Stable Formulas

B and B introduced stable formulas.

For each finite rooted frame $\mathfrak{F}$ the stable formula of $\mathfrak{F}$ is refuted in a frame $\mathfrak{G}$ iff $\mathfrak{F}$ is an order-preserving image of $\mathfrak{G}$ ($\mathrm{B}^2 2013$).

Stable logics are intermediate logics for which its frame class is closed under order-preserving images. They are axiomatized by stable formulas. There is a continuum of stable logics and all stable logics have the finite model property.

A good syntactic characterization remained an open problem.

NNIL Formulas

The VvBdJR result implies that NNIL-formulas are also preserved under taking subframes. Moreover, for each finite rooted frame $\mathfrak{F}$, NBdiss (2006) constructs a NNIL-formula that is its subframe formula.

Hence, an intermediate logic is a subframe logic iff it is axiomatized by NNIL-formulas. This also implies that each NNIL-formula is frame-equivalent to a $(\wedge, \rightarrow)$-formula and vice versa.

ONNILLI

We introduce ONNILLI-formulas, only NNIL to the left of implications, and show that ONNILLI-formulas are formulas that are preserved under order-preserving images of (descriptive and Kripke) frames.

We also obtain that that the set of ONNILLI-formulas represents (up to frame equivalence) the set of stable formulas.

Examples of ONNILLI-formulas are LC: $(p \to q) \vee (q \to p)$ (also NNIL), KC: $\neg p \vee \neg\neg p$.

Notations

Let $\mathfrak{F} = (W, R)$ be a Kripke frame. For every $w \in W$ and $U \subseteq W$ let

$$R(w) = \{v \in W : wRv\},$$
$$R^{-1}(w) = \{v \in W : vRw\},$$
$$R(U) = \bigcup_{w \in U} R(w),$$
$$R^{-1}(U) = \bigcup_{w \in U} R^{-1}(w).$$

Subframes

1. Let $\mathfrak{F} = (W, R)$ be a Kripke frame. A frame $\mathfrak{F}' = (W', R')$ is called a subframe of $\mathfrak{F}$ if $W' \subseteq W$ and R' is the restriction of R to W'.
2. Let $\mathfrak{F} = (W, R, \mathcal{P})$ be a descriptive frame. A descriptive frame $\mathfrak{F}' = (W', R', \mathcal{P}')$ is called a subframe of $\mathfrak{F}$ if (W', R') is a subframe of (W, R), $\mathcal{P}' = \{U \cap W' : U \in \mathcal{P}\}$ and the topo-subframe condition, is satisfied:

$$\forall U \subseteq W' \; (W' \setminus U \in \mathcal{P}' \Longrightarrow W \setminus R^{-1}(U) \in \mathcal{P})$$

Operations on descriptive frames III

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Proposition

Let $\mathfrak{F} = (W, R, \mathcal{P})$ and $\mathfrak{F}' = (W', R', \mathcal{P}')$ be descriptive frames. If $\mathfrak{F}'$ is a subframe of $\mathfrak{F}$, then for every descriptive valuation V' on $\mathfrak{F}'$ there exists a descriptive valuation V on $\mathfrak{F}$ such that the restriction of V to W' is V'.

NNIL–formulas and submodels

NNIL-formulas are known to have the following normal form:

$$\varphi := \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid p \to \varphi$$

Theorem (VvBdJR)

Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{N} = (W', R', V')$ be (descriptive of Kripke) frames.

1. *If $\mathfrak{N}$ is a submodel of $\mathfrak{M}$, then for each $\varphi \in \mathsf{NNIL}$ and $w \in W'$, $\mathfrak{M}, w \models \varphi \implies \mathfrak{N}, w \models \varphi$.*
2. *If for all w in submodels $\mathfrak{N}$ of $\mathfrak{M}$, $\mathfrak{M}, w \models \varphi$ implies $\mathfrak{N}, w \models \varphi$, then $\exists \psi \in \mathsf{NNIL}\,(\mathsf{IPC} \vdash \psi \leftrightarrow \varphi)$.*

(1) implies that NNIL-formulas are preserved under taking subframes of (Kripke and descriptive) frames.

Colors

Definition

Let $\mathfrak{M} = (\mathfrak{F}, V)$ be a descriptive model for $p_1, \ldots, p_n$.
If w in $\mathfrak{M}$, col(w) (the color of w) $= i_1 \ldots i_n$ such that:

$$i_k = \begin{cases} 1 & \textit{if } w \models p_k, \\ 0, & \textit{if } w \not\models p_k. \end{cases}$$

A finite model $\mathfrak{M} = (W, R, V)$ is colorful if

$$\forall w \in W \, \exists! p_w (V(p_w) = R(w)).$$

Lemma

Let $(\mathfrak{F}, V)$ be a colorful model. Then for every $w, v \in W$ we have:

1. $w = v$ *iff* $col(w) = col(v)$,
2. $w \neq v$ *and* $w\,R\,v$ *iff* $col(w) < col(v)$.

NNIL–type Subframe Formulas

For finite rooted frames $\mathfrak{F}$ we inductively define the subframe formula $\beta(\mathfrak{F})$ in NNIL.

$prop(v) := \{p_k \mid v \models p_k, k \leq n\}$, $notprop(v) := \{p_k \mid v \not\models p_k, k \leq n\}$.

If v is a maximal, then

$$\beta(v) := \bigwedge prop(v) \to \bigvee notprop(v)$$

Let $w_1, \ldots, w_m$ be all the immediate successors of w.

$$\beta(w) := \bigwedge prop(w) \to \bigvee notprop(w) \vee \bigvee_{i=1}^{m} \beta(w_i).$$

Finally, $\beta(\mathfrak{F}) := \beta(r)$, where r is the root of $\mathfrak{F}$.

Crucial Property of Subframe formulas

Theorem

Let $\mathfrak{G} = (W', R', \mathcal{P}')$ be a descriptive frame and let $\mathfrak{F} = (W, R)$ be a finite rooted frame. Then

$$\mathfrak{G} \not\models \beta(\mathfrak{F}) \text{ iff } \mathfrak{F} \text{ is a p-morphic image of a subframe of } \mathfrak{G}.$$

The proof depends on the fact that, if $\bigwedge prop(v) \to \bigvee notprop(v)$ is false anywhere, then some node above will need to have the color of v (with $prop(v)$ true and $notprop(v)$ false). If $\bigwedge prop(v) \to \bigvee notprop(v) \vee \bigvee_{i=1}^{m} \beta(w_i)$ is false anywhere, then some node above will need to have the color of w with above it nodes of the colors of the w_i. Falsity of $\beta(\mathfrak{F})$ will then guarantee nodes of the right colors in the proper order.

Theorem

1. *An intermediate logic L is a subframe logic iff L is axiomatized by* NNIL*-formulas.*
2. *The class of* NNIL*-formulas is up to frame-equivalence the class of subframe formulas.*
3. *Each* NNIL*-formula is frame-equivalent to a* $(\wedge, \rightarrow)$*-formula.*

A direct syntactic transformation of NNIL-formulas into frame-equivalent $(\wedge, \rightarrow)$-formulas can be found in Fanthesis2008. No way is known to transform a $(\wedge, \rightarrow)$-formula directly syntactically into a NNIL-formula.

Order preserving functions and NNIL–formulas I

We construct a new class of formulas, ONNILLI, preserved by order-preserving maps.
(X, R), (Y', R') Kripke frames. $f : X \to Y$ is order-preserving if $u\,R\,v$ implies $f(u)\,R'\,f(v)$ and is admissible[2] if appropriate.

Applied to models we assume f to be valuation preserving as well.

Proposition

Let $\mathfrak{M} = (X, R, V)$ and $\mathfrak{N} = (Y, R', V')$ be two (Kripke or descriptive) models and $f : X \to Y$ an order-preserving map. Then,

$$\forall u \in X, \varphi \in \text{NNIL} \quad (f(u) \models \varphi \Rightarrow u \models \varphi)$$

[2] $W \setminus f^{-1}(W \setminus U') \in \mathcal{P}$

Order preserving functions and NNIL–formulas II

Proof. Only the last inductive step is non-trivial. Assume $f(u) \models \varphi \Rightarrow u \models \varphi$ for all $u \in X$ (IH). Suppose $f(u) \models p \to \varphi$, and let $u\,R\,v$ with $v \models p$. Then $f(u)\,R\,f(v)$ and $f(v) \models p$. So, $f(v) \models \varphi$. By IH, $v \models \varphi$. So $u \models p \to \varphi$. □

Note that the identity function from a submodel into the larger model is obviously an order-preserving function. Thus this shows that NNIL-formulas are also exactly the ones that are preserved backwards by order-preserving functions on models.

Definition

1. BASIC is the closure of the set of the atoms plus $\top$ and $\bot$ under conjunctions and disjunctions.
2. The class ONNILLI (only NNIL to the left of implications) is defined as the closure of $\{\varphi \rightarrow \psi \mid \varphi \in \text{NNIL}, \psi \in \text{BASIC}\}$ under conjunctions and disjunctions.

So, no iterations of implications in ONNILLI-formulas except inside the NNIL-part. Note:

If $\psi \in$ BASIC, f valuation-preserving, then $f(v) \models \psi \Leftrightarrow v \models \psi$.

KC IS ONNILLI

EXAMPLE

$\neg p \vee \neg\neg p$ is ONNILLI. To see this, write it as $(p \to \bot) \vee (\neg p \to \bot)$, and note that $\neg p$ is in NNIL.

$\neg p \vee \neg\neg p$ is not preserved under taking subframes. So, it cannot be frame-equivalent to a NNIL-formula. Thus, ONNILLI $\not\subseteq$ NNIL. We will see later that also NNIL $\not\subseteq$ ONNILLI.

Order-preserving functions and ONNILLI-formulas I

Let $\mathfrak{M} = (X, R, V)$ and $\mathfrak{N} = (Y, R', V')$ be Kripke or descriptive,
$f : X \to Y$ surjective, order-preserving:
If $\varphi \in$ ONNILLI, then $\mathfrak{M} \models \varphi \Longrightarrow \mathfrak{N} \models \varphi$.

Proof.
Let $\varphi = \psi \to \chi$ with $\psi \in$ NNIL, $\chi \in$ BASIC,
$\mathfrak{M} \models \psi \to \chi$, i.e. $u \models \psi \to \chi$ for all $u \in X$.
f is surjective: all elements of Y are of the form $f(u), u \in X$.
Assume $f(u) \models \psi$. By previous, $u \models \psi$.
$u \models \psi \to \chi \implies u \models \chi \implies f(u) \models \chi$.
Hence, $f(u) \models \psi \to \chi$. Thus, $\mathfrak{N} \models \psi \to \chi$. $\square$

Validity in models is needed, truth in a node insufficient. Also surjectivity is an essential.

ORDER-PRESERVING FUNCTIONS AND ONNILLI-FORMULAS II

COROLLARY

Let $\mathfrak{F} = (X, R)$ *and* $\mathfrak{G} = (Y, R')$ *be (Kripke or descriptive) frames and* $f : X \to Y$ *an order-preserving map from* $\mathfrak{F}$ *onto* $\mathfrak{G}$*. Then, for each* $\varphi \in \mathsf{ONNILLI}$, $\mathfrak{F} \models \varphi \Longrightarrow \mathfrak{G} \models \varphi$.

DEFINITION

1. If c is an n-color we write ψ_c for $p_1 \wedge \cdots \wedge p_k \to q_1 \vee \cdots \vee q_m$ if $p_1 \ldots p_k$ are the propositional variables that are 1 in c and $q_1 \ldots q_m$ the ones that are 0 in c.
2. If $\mathfrak{M}$ is colorful and $w \in W$, we write $Col(\mathfrak{M}_w)$ for the formula $prop(w) \wedge \bigwedge\{\psi_c \mid c$ a color that is not in $\mathfrak{M}_w\}$.
3. $\gamma(\mathfrak{M}) = \bigvee\{Col(\mathfrak{M}_w) \to p_{w_1} \vee \cdots \vee p_{w_m} \mid w \in W,\ w_1, \ldots w_m$ are all the proper successors of $w\}$.

Let $\mathfrak{F}$ be a finite rooted frame. We define a valuation V on $\mathfrak{F}$ such that $\mathfrak{M} = (\mathfrak{F}, V)$ is colorful and define $\gamma(\mathfrak{F})$ by

$$\gamma(\mathfrak{F}) := \gamma(\mathfrak{M}).$$

We call $\gamma(\mathfrak{F})$ the stable formula of $\mathfrak{F}$. $\gamma(\mathfrak{F})$ is an ONNILLI-formula.

Lemmas

Lemma

Assume $\mathfrak{M} = (W, R, V)$ is colorful, $w \in W$, u' and v' are nodes in an arbitrary (Kripke or descriptive) model $\mathfrak{M}' = (W', R', V')$ such that $u' R' v'$. Then

1. *If $col(u') = col(u)$ and $col(v') = col(v)$ for $u, v \in W$, then $u \, R \, v$.*
2. *If $u' \models Col(\mathfrak{M}_u)$, then u' and v' both have one of the colors available in $\mathfrak{M}_u$.*
3. *If $u' \not\models Col(\mathfrak{M}_w) \to p_{w_1} \vee \cdots \vee p_{w_m}$, then there is $v'' \in W'$ such that $u' R v'$ and $col(v'') = col(w)$.*

Lemma

Let $\mathfrak{F}$ be a finite rooted frame. Then $\mathfrak{F} \not\models \gamma(\mathfrak{F})$.

The basic ONNILLI theorem

Corollary

Let $\mathfrak{F} = (W, R)$ be a finite rooted frame and let $\mathfrak{G}$ a (Kripke or descriptive) frame. Then

1. $\mathfrak{G} \not\models \gamma(\mathfrak{F})$ *iff there is a surjective order-preserving map from a generated subframe of* $\mathfrak{G}$ *onto* $\mathfrak{F}$.
2. $\mathfrak{G} \not\models \gamma(\mathfrak{F})$ *iff there is a surjective order-preserving map from* $\mathfrak{G}$ *onto* $\mathfrak{F}$.

Proof of the basic ONNILLI theorem

(1) $\Rightarrow$: We know that $\mathfrak{F} \not\models \gamma(\mathfrak{F})$. Since $\gamma(\mathfrak{F})$ is ONNILLI, it is preserved under order-preserving images. So, $\mathfrak{G} \not\models \gamma(\mathfrak{F})$.

$\Leftarrow$: Let $\mathfrak{N}$ on $\mathfrak{G}$, $\mathfrak{N}, u \not\models \gamma(\mathfrak{F})$. Then $\forall w \in W\, \exists w', u\, R\, w'$ with $Col(\mathfrak{M}_w)$ true and $p_{w_1}, \ldots, p_{w_m}$ false. Thus, w' has the color of w and its successors have colors of successors of w. Let W' be the set of the chosen w's. As W is finite, W' is also finite.

Let $\mathfrak{N}' = \mathfrak{M}_{R(W')}$.

Now define $f : R(W') \to W$ by $f(u) = w$ if $col(u) = col(w)$.

If $u'\, R\, v' \in R(W')$, then there are $u\, R\, v \in W$ such that $col(u') = col(u)$ and $col(v') = col(v)$. So, f is order-preserving.

Finally, $\forall w \in W\, \exists u \in R(W')\, (col(u) = col(w))$.
Thus, $f(u) = w$ and f is also surjective.

Stable Logics and ONNILLI

Theorem

1. *An intermediate logic L is stable iff L is axiomatized by* ONNILLI*-formulas.*
2. *The class of* ONNILLI*-formulas is up to frame-equivalence the class of stable formulas.*

NNIL and ONNILLI

Example

NNIL-formulas that are not equivalent to an ONNILLI-formula.

For each n the logic BD_n of frames of depth $\leq n$ is preserved under taking subframes. Thus, it is a subframe logic axiomatized by NNIL formulas.

But there are frames of depth n having frames of depth $m > n$ as order-preserving images. So BD_n is not a stable logic and cannot be axiomatized by ONNILLI formulas. Thus, the class of ONNILLI-formulas does not contain the class of NNIL-formulas.

Some examples of stable logics

LC_n be the logic of all linear rooted frames of depth $\leq n$,

BW_n be the logic of all rooted frames of width $\leq n$,

BTW_n be the logic of all rooted descriptive frames of cofinal width $\leq n$,

OPEN QUESTION
It is an open problem whether ONNILLI-formulas are exactly the ones that are preserved under order-preserving preserving maps of models.

THE END

THANKS!