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Negative Translations for Affine and Łukasiewicz Logic

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Abstract We investigate four well-known negative translations of classical logic into intuitionistic logic within a sub-structural setting. We find that in affine logic the translation schemes due to Kolmogorov and Gödel both satisfy Troelstra's criteria for a negative translation. On the other hand, the schemes of Glivenko and Gentzen both fail to work for affine logic, but for different reasons: one can extend affine logic to make Glivenko work and Gentzen fail and *vice versa*. By contrast, in the setting of Łukasiewicz, we can prove a general result asserting that a wide class of formula translations including those of Kolmogorov, Gödel, Gentzen and Glivenko not only satisfy Troelstra's criteria with respect to a natural intuitionistic fragment of Łukasiewicz logic but are all equivalent. We conclude by proving some De Morgan properties for this intuitionistic fragment.

1 Introduction

Negative translations (aka double negation translations) have a long history in logic and proof theory. Kolmogorov [12] was probably the first one to observe that classical logic can be "embedded" into its intuitionistic fragment. Kolmogorov's translation $A \mapsto A^{\mathsf{K}}$ places double negations in front of every sub-formula of a given formula A so that A is provable classically if and only if A^{K} is provable intuitionistically. Around the same time, Glivenko [8], Gödel [9] and Gentzen [6] defined three other more "economic" translations that also eliminate classical principles from proofs at the cost of introducing extra negations, but not as many as Kolmogorov's.

In the present paper we recast these different negative translations in the setting of sub-structural logic, concentrating on logics lying between intuitionistic affine logic and classical Łukasiewicz logic. This will shed light on the amount of contraction required in order to make each of these translations work.

We start by defining a fragment of classical Łukasiewicz logic $\mathbf{ŁL_c}$, which we will call *intuitionistic Łukasiewicz logic* $\mathbf{ŁL_i}$. Just as Łukasiewicz logic [10] is a subsystem of classical logic $\mathbf{CL}$, intuitionistic Łukasiewicz logic is a subsystem of the usual intuitionistic logic $\mathbf{IL}$ [16]. The paper focuses entirely on propositional logic, leaving a similar study for predicate logic to future work.

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$$\begin{array}{ccccc}
\mathbf{AL_c} & \longrightarrow & \mathbf{ŁL_c} & \longrightarrow & \mathbf{CL} \\
\uparrow & & \uparrow & & \uparrow \\
\mathbf{AL_i} & \longrightarrow & \mathbf{ŁL_i} & \longrightarrow & \mathbf{IL}
\end{array}$$

Figure 1 Relationships between the six logics

The logics $\mathbf{ŁL_i}$ and $\mathbf{ŁL_c}$ are defined here as extensions of the $\{\multimap, \otimes, \bot\}$-fragment of intuitionistic *affine logic* $\mathbf{AL_i}$, i.e. intuitionistic multiplicative linear logic [7] extended by allowing weakening. A similar sequent calculus for Łukasiewicz logic based on classical affine logic has been proposed in [4]. The main differences are that we work on the implication-conjunction fragment of affine logic, and take *intuitionistic* affine logic as the starting point.

Starting from $\mathbf{AL_i}$ one obtains *intuitionistic Łukasiewicz logic* $\mathbf{ŁL_i}$ by adjoining the axiom that we call *commutativity of weak conjunction*

$$A \otimes (A \multimap B) \vdash B \otimes (B \multimap A)$$

which is a simple consequence of contraction, but is strictly weaker than it. The reason we call $A \otimes (A \multimap B)$ a weak form of conjunction can be explained as follows: Note that $A \otimes (A \multimap B)$ implies both A and B, but without contraction (so that A can be used twice), we do not have in general $A \otimes (A \multimap B) \vdash A \otimes B$. On the other hand, due to the presence of weakening in the affine systems, we always have $A \otimes B \vdash A \otimes (A \multimap B)$. Hence, $A \otimes (A \multimap B)$ is strictly weaker than the usual multiplicative conjunction $A \otimes B$. The axiom states that this conjunction is commutative. This is also known as the *axiom of divisibility* in the basic logic literature [11].

The logic $\mathbf{ŁL_i}$ has been studied before, under different names. For instance, Blok and Ferreirim [2] refer to it as $\mathsf{S}_{\mathcal{HO}}$. $\mathbf{ŁL_i}$ can also be viewed as a fragment of Hájek's *basic logic* without the (intuitionistically unacceptable) *axiom of pre-linearity* [10]. The relationship between the various logical systems is depicted in Figure 1, where arrows indicate inclusion.

Finally, from the intuitionistic systems $\mathbf{AL_i}$ and $\mathbf{ŁL_i}$ one can obtain their "classical" counterparts ($\mathbf{AL_c}$ and $\mathbf{ŁL_c}$, respectively) by adding the law of *double negation elimination* [DNE]

$$A^{\bot\bot} \vdash A$$

where $A^{\bot}$ is defined as $A \multimap \bot$. In order to move horizontally in the diagram of Figure 1 from the left-most column (affine system) to the right-most column (intuitionistic $\mathbf{IL}$, and classical logic $\mathbf{CL}$) one adds the *contraction axiom*

$$A \vdash A \otimes A$$

The Łukasiewicz systems sit in between affine systems, where no contraction is permitted, and the full systems which contain the contraction axiom for all formulas.

The main result in this paper is that all four standard negative translations of $\mathbf{CL}$ into $\mathbf{IL}$ are also negative translations of $\mathbf{ŁL_c}$ into $\mathbf{ŁL_i}$ (Section 5). Our result relies on several novel derivations of theorems of $\mathbf{ŁL_i}$ given in Section 4, in particular the result that the double negation mapping $A \mapsto A^{\bot\bot}$ is a homomorphism in $\mathbf{ŁL_i}$.

We also prove that Kolmogorov's and Gödel's translations are even negative translations of $\mathbf{AL_c}$ into $\mathbf{AL_i}$, and point to counter-examples demonstrating that Glivenko and Gentzen are not: in fact $\mathbf{AL_i}$ can be extended so as to make the Glivenko translation a negative translation but not the Gentzen translation or *vice versa*.

In the present paper, whenever we need to show that a formula is provable in one of our logics, we do so constructively. In the case of $\mathbf{ŁL_i}$ most non-trivial derivations involve intricate applications of commutativity of weak conjunction. We express here our gratitude to the late Bill McCune for the development of the automated theorem prover Prover9 and the finite-model finder Mace4, which we have used extensively to find derivations or counter-models to our various conjectures.

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\,[\multimap\mathsf{I}] \qquad \frac{\Gamma \vdash A \quad \Delta \vdash A \multimap B}{\Gamma, \Delta \vdash B}\,[\multimap\mathsf{E}]$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\,[\otimes\mathsf{I}] \qquad \frac{\Gamma \vdash A \otimes B \quad \Delta, A, B \vdash C}{\Gamma, \Delta \vdash C}\,[\otimes\mathsf{E}]$$

Figure 2 Natural deduction (in sequent-style) rules for $\otimes$ and $\multimap$

Most of the $\mathbf{ŁL_i}$ derivations presented here were initially found by Prover9. Perhaps remarkably, it has proved possible to organise and present the derivations in what we believe is a systematic and human-readable style.

2 Definitions of the Logics

2.1 Language We work in a language, $\mathcal{L}$, built from a countable set of propositional variables $\mathsf{Var} = \{V_1, V_2, \ldots\}$, the constant $\bot$ (falsehood) and the binary connectives $\multimap$ (implication) and $\otimes$ (conjunction). We write $A^\perp$ for $A \multimap \bot$ and $\top$ for $\bot \multimap \bot$. Our choice of notation for connectives is that commonly used for affine logic, since all the systems we consider will be extensions of intuitionistic affine logic.

As usual, we adopt the convention that $\multimap$ associates to the right and has lower precedence than $\otimes$, which in turn has lower precedence than $(\cdot)^\perp$. So, for example, the brackets in $(A \otimes (B^\perp)) \multimap (C \multimap (D \otimes F))$ are all redundant, while those in $(((A \multimap B) \multimap C) \otimes D)^\perp$ are all required.

2.2 The logics In this section we give *natural deduction* systems (in sequent style) for the logics we will study. The judgments of the calculi are sequents $\Gamma \vdash A$ where the *context* Γ is a multiset of formulas and A is a formula. The rules of inference for *all* the calculi comprise the sequent formulation of a natural deduction system shown in Figure 2.

The six calculi are defined by adding to the rules of Figure 2 some or all of the following axiom schemata: *assumption* [ASM], *contraction* [CON], *ex falso quodlibet* [EFQ], *double negation elimination* [DNE], and *commutativity of weak conjunction* [CWC], defined in Figure 3. The six calculi and their axiom schemata are as defined in Table 1.

The systems $\mathbf{AL_i}$, $\mathbf{AL_c}$, $\mathbf{ŁL_i}$ and $\mathbf{ŁL_c}$ are intuitionistic and classical variants of affine logic and Łukasiewicz logic. **IL** and **CL** as we shall see shortly are the usual intuitionistic and classical logic. The relationship between the six logics is depicted in Figure 1.

As our axiom schemata all allow additional premisses Γ in the context, the following rule of weakening

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}\,[\mathsf{WK}]$$

is admissible in all our logics, since given a proof tree with $\Gamma \vdash B$ at the root, we may obtain a proof of $\Gamma, A \vdash B$ by adding A to the context of every sequent on *some* path from the root to a leaf (axiom). Also, note that in intuitionistic affine logic $\mathbf{AL_i}$, and hence in all the logics, the contraction axiom [CON] is inter-derivable with the contraction rule

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,[\mathsf{CON}_r]$$

Thus with [CON] we have the structural rules of weakening and contraction, which proves our claim that **IL** and **CL** are the usual intuitionistic and classical propositional logics.

$$\frac{}{\Gamma, A \vdash A}\,[\mathsf{ASM}] \qquad \frac{}{\Gamma, A \vdash A \otimes A}\,[\mathsf{CON}]$$

$$\frac{}{\Gamma, \bot \vdash A}\,[\mathsf{EFQ}] \qquad \frac{}{\Gamma, \neg\neg A \vdash A}\,[\mathsf{DNE}]$$

$$\frac{}{\Gamma, A, A \multimap B \vdash B \otimes (B \multimap A)}\,[\mathsf{CWC}]$$

Figure 3 Sequent Calculus Axioms

Most of the results in this paper involve the derivability of a sequent in one of our calculi above (mainly $\mathbf{\L L_i}$). When deriving these, instead of writing proof trees we will often find it convenient to adopt a form of equational reasoning, using the notations of the following definition.

Definition 2.1 *Let T be an extension of $\mathbf{AL_i}$.*

- *We write $A \rightarrow_T B$ if $A \vdash B$ can be derived in T.*
- *We write $A \leftrightarrow_T B$ if $A \rightarrow_T B$ and $B \rightarrow_T A$.*

When the T in question is clear from the context we just write $A \rightarrow B$ and $A \leftrightarrow B$.

Lemma 2.1 *Let T be an extension of $\mathbf{AL_i}$. Then $\rightarrow_T$ is symmetric and transitive, $\leftrightarrow_T$ is an equivalence relation, and, for any formulas A, B and C, such that $A \rightarrow_T B$, the following hold:*

$$\begin{aligned} B \multimap C &\rightarrow_T A \multimap C \\ C \multimap A &\rightarrow_T C \multimap B \\ A \otimes C &\rightarrow_T B \otimes C \\ C \otimes A &\rightarrow_T C \otimes B. \end{aligned}$$

Hence, $\leftrightarrow_T$ is a congruence with respect to both $\multimap$ and $\otimes$, i.e., if $A \leftrightarrow_T B$ then

$$\begin{aligned} B \multimap C &\leftrightarrow_T A \multimap C \\ C \multimap A &\leftrightarrow_T C \multimap B \\ A \otimes C &\leftrightarrow_T B \otimes C \\ C \otimes A &\leftrightarrow_T C \otimes B. \end{aligned}$$

Moreover, we have:

$$\begin{aligned} A \otimes (B \otimes C) &\leftrightarrow_T (A \otimes B) \otimes C \\ A \otimes B &\leftrightarrow_T B \otimes A \\ A \otimes \top &\leftrightarrow_T A \end{aligned}$$

and A is provable iff $\top \rightarrow_T A$ iff $A \leftrightarrow_T \top$.

Proof: Straightforward, recalling that $\top$ abbreviates $\bot \multimap \bot$. ■

When deriving an "equation" such as $A \leftrightarrow_T B$ or an "inequality" such as $A \rightarrow_T B$ we will often use the fact that $A \leftrightarrow A \otimes \top$. For instance, assume we are given some formula B which is provable in T, so that $B \leftrightarrow_T \top$. We can then conclude $A \otimes B \leftrightarrow_T A$.

Notation. Whenever, in a chain of equalities or inequalities, we make use of $B \leftrightarrow_T \top$ in the manner above, we will highlight the formula B which (dis)appears by underlining it. Moreover, so that we can omit the subscript T from $\leq_T$ and $\leftrightarrow_T$, we will generally specify T in brackets at the beginning of the statement of each lemma or theorem.

Calculus	Axiom Schemata
$\mathbf{AL_i}$	[ASM], [EFQ]
$\mathbf{AL_c}$	[ASM], [EFQ], [DNE]
$\mathbf{ŁL_i}$	[ASM], [CWC], [EFQ]
$\mathbf{ŁL_c}$	[ASM], [CWC], [EFQ], [DNE]
$\mathbf{IL}$	[ASM], [CON], [EFQ]
$\mathbf{CL}$	[ASM], [CON], [EFQ], [DNE]

Table 1 The Six Calculi

The following lemma and its proof illustrates these notational conventions. Since we have (even in $\mathbf{AL_i}$) that $A \otimes (A \multimap B) \vdash A$ and $A \otimes (A \multimap B) \vdash B$, the lemma shows that (over $\mathbf{ŁL_i}$) $A \otimes (A \multimap B)$ is the weakest formulas that is stronger than both A and B.

Lemma 2.2 ($\mathbf{ŁL_i}$) *If $C \to A$ and $C \to B$ then $C \to A \otimes (A \multimap B)$.*

Proof: Assume (a) $C \to A$ (and hence $C \multimap A \leftrightarrow \top$) and (b) $C \to B$. We derive the conclusion as follows:

$$\begin{aligned} C &\leftrightarrow C \otimes (\underline{C \multimap A}) && \text{(a)} \\ &\leftrightarrow A \otimes (A \multimap C) && \text{([CWC])} \\ &\to A \otimes (A \multimap B) && \text{(b)} \end{aligned}$$

invoking Lemma 2.1 to use (b) in the last step. ■

We contrast this concise sequence of steps with the full natural deduction derivation of the same lemma:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{C \vdash C}\,[\mathsf{ASM}] \quad \dfrac{C \vdash A}{\vdash C \multimap A}\,[\multimap\mathsf{I}]}{C \vdash C \otimes (C \multimap A)}\,[\otimes\mathsf{I}]}{C \vdash A \otimes (A \multimap C)}\,[\mathsf{CWC}] \quad \dfrac{\dfrac{}{A \vdash A}\,[\mathsf{ASM}] \quad \dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,[\mathsf{ASM}] \quad \dfrac{}{A \multimap C \vdash A \multimap C}\,[\mathsf{ASM}]}{A, A \multimap C \vdash C}\,[\multimap\mathsf{E}] \quad \dfrac{C \vdash B}{\vdash C \multimap B}\,[\multimap\mathsf{I}]}{A, A \multimap C \vdash B}\,[\multimap\mathsf{E}]}{A \multimap C \vdash A \multimap B}\,[\multimap\mathsf{I}]}{A, A \multimap C \vdash A \otimes (A \multimap B)}\,[\otimes\mathsf{I}]}{C \vdash A \otimes (A \multimap B)}\,[\otimes\mathsf{E}]$$

where we use a rule-like version of [CWC] which is clearly derivable from the [CWC] axiom. The natural deduction system $\mathbf{ŁL_i}$ comes into its own in Section 3 where we prove properties of the system by induction on the proof rules.

The rules of Figure 2 and the axioms of Figure 3 are closed under substitution of formulas for variables. Hence a substitution instance of a theorem in any of our logics is again a theorem of that logic. When reading a result such as Lemma 2.2, it is immaterial whether one views the letters A, B and C as metavariables ranging over $\mathcal{L}$ or as specific variables in $\mathsf{Var} \subset \mathcal{L}$.

As mentioned in the introduction, in [4] one can find an earlier proposal of viewing Łukasiewicz logic as an extension of affine logic. In that context, a rule called $(\&, l_c)$ was added to classical affine logic in order to obtain classical Łukasiewicz logic. It is easy to check that in the presence of weakening the premise of that rule is derivable, which means that over affine logic the rule $(\&, l_c)$ should be viewed as the axiom schema $A \,\&\, B \vdash A \otimes (A \multimap B)$. Let us assume, for the sake of argument, that we extend $\mathbf{ŁL_i}$ with the additive conjunction $A \,\&\, B$, with the usual rules, as in [4]. By Lemma 2.2 above, and since $A \,\&\, B \vdash A$ and $A \,\&\, B \vdash B$, we have that over $\mathbf{AL_i}$ the axiom $(\&, l_c)$ follows from [CWC]. Conversely, since $A \,\&\, B$ is commutative, and $A \otimes (A \multimap B) \vdash A \,\&\, B$, it follows that over $\mathbf{AL_i}$ the axiom [CWC] follows from $(\&, l_c)$. Therefore, one can indeed obtain $\mathbf{ŁL_i}$ by adjoining either of the axioms to $\mathbf{AL_i}$. However, by working with [CWC] we could restrict ourselves to the fragment of affine logic without the additive conjunction $A \& B$, but only containing implication ($\multimap$) and the multiplicative conjunction ($\otimes$).

Particularly in the literature on Łukasiewicz logic the systems that we have presented above as sequent calculi are traditionally presented as Hilbert-style systems with *modus ponens* as the only rule of inference (see [10], Def. 3.1.3, for a Hilbert-style presentation of $\mathbf{LL_C}$). It can be shown that the two presentations are equivalent in the sense that $A_1, A_2, \ldots, A_n \vdash A$ is provable in one of the sequent calculi iff $A_1 \multimap A_2 \multimap \ldots \multimap A_n \multimap A$ is provable in the corresponding Hilbert style system

2.3 Derived connectives In additon to the primitive connectives $\otimes$ and $\multimap$, we will make extensive use of the following four *derived* binary connectives $A \wedge B$, $A \vee B$, $A \Rightarrow B$, $A \downarrow B$ defined as follows:

$$
\begin{array}{llll}
A \wedge B & \equiv & A \otimes (A \multimap B) & \text{(weak conjunction)} \\
A \vee B & \equiv & (B \multimap A) \multimap A & \text{(strong disjunction)} \\
A \Rightarrow B & \equiv & A \multimap A \otimes B & \text{(strong implication)} \\
A \downarrow B & \equiv & A^{\perp} \otimes (B \multimap A) & \text{(NOR binary connective)}
\end{array}
$$

Recall that we are assuming conjunction binds more strongly than the implication, so that $A \Rightarrow B$ is $A \multimap (A \otimes B)$. For the new connectives we will also use the convention that $\wedge, \vee$ and $\downarrow$ all bind more strongly than $\Rightarrow$. So $(A \vee B) \Rightarrow (C \wedge D)$, for instance, may be written as $A \vee B \Rightarrow C \wedge D$.

We justify our notation by observing that when $\otimes$ and $\multimap$ are replaced by the standard conjunction and implication of *classical logic* then $A \wedge B, A \vee B, A \Rightarrow B$ are indeed equivalent to the standard conjunction, disjunction and implication. Moreover, $A \downarrow B$ is equivalent to the NOR binary connective. In affine linear logic, however, it is not even possible to prove that $A \wedge B$ and $A \vee B$ are *commutative*.

We have chosen our notation so that in each of the derived connectives the left operand appears both positively and negatively while the right operand appears only positively in $\wedge, \vee$ and $\Rightarrow$ and only negatively in $\downarrow$.

Let us conclude this section with a short list of basic theorems of $\mathbf{AL_i}$ which will prove very useful in the sequel. Recall that $A \ \rightarrow \ B$ stands for $A \vdash B$; and that $A \ \leftrightarrow \ B$ stands for $A \vdash B$ and $B \vdash A$.

Lemma 2.3 *In* $\mathbf{AL_i}$*:*

- ***(i)*:** $A \ \rightarrow \ B \vee A$, *in particular, taking* $B = \perp$, $A \ \rightarrow \ A^{\perp\perp}$
- ***(ii)*:** $A \ \rightarrow \ B \Rightarrow A$
- ***(iii)*:** $A \otimes B \ \leftrightarrow \ A \otimes (A \Rightarrow B)$
- ***(iv)*:** $C \vee (A \multimap B) \ \rightarrow \ (C \vee A) \multimap (C \vee B)$
- ***(v)*:** $(A^{\perp} \multimap B^{\perp})^{\perp\perp} \ \leftrightarrow \ A^{\perp} \multimap B^{\perp}$
- ***(vi)*:** $(A^{\perp\perp} \multimap B^{\perp\perp})^{\perp} \ \leftrightarrow \ (A \multimap B^{\perp\perp})^{\perp}$
- ***(vii)*:** $(A^{\perp\perp} \otimes B^{\perp\perp})^{\perp} \ \leftrightarrow \ (A \otimes B)^{\perp}$
- ***(viii)*:** $(A \otimes B)^{\perp\perp} \ \leftrightarrow \ (A^{\perp\perp} \multimap B^{\perp})^{\perp}$

Proof: Easy, making much use of $A^{\perp\perp\perp} \ \leftrightarrow \ A^{\perp}$ and $A \multimap B^{\perp} \ \leftrightarrow \ (A \otimes B)^{\perp} \ \leftrightarrow \ B \multimap A^{\perp}$. ■

3 Negative Translations

In [15], Troelstra identifies certain requirements on a translation of classical logic into intuitionistic logic and shows that any two translations satisfying these requirements are intuitionistically equivalent. To set up the analogue of this characterisation in sub-structural setting, we first define the notion of *negative formula* in the language $\mathcal{L}$.

Definition 3.1 *The set* $\mathcal{N}$ *of* negative formulas *is defined inductively as*

- $\perp \in \mathcal{N}$,
- *if* $A \in \mathcal{N}$ *and* $B \in \mathcal{N}$ *then* $A \otimes B \in \mathcal{N}$,

- *if $B \in \mathcal{N}$ then $A \multimap B \in \mathcal{N}$.*

We can now formulate an adaptation of Troelstra's requirements:

Definition 3.2 *Let* **A** *be a fragment of intuitionistic logic* **IL** *over the language $\mathcal{L}$. A formula translation $(\cdot)^\dagger \colon \mathcal{L} \to \mathcal{L}$ is called a* negative translation *for* **A** *if the following holds for every formula A in the language of* **A**

(NT1): **A** *proves $A^\dagger \vdash B$ and $B \vdash A^\dagger$, for some $B \in \mathcal{N}$.*
(NT2): **A** + [DNE] *proves $A^\dagger \vdash A$ and $A \vdash A^\dagger$.*
(NT3): *if* **A** + [DNE] *proves $\vdash A$ then* **A** *proves $\vdash A^\dagger$.*

Conditions **(NT1)**, **(NT2)** and **(NT3)** correspond to Troelstra's [15, Section 10] (*iii*), (*i*) and (*ii*), respectively. We have rearranged them as we will show that in $\mathbf{ŁL_i}$ condition **(NT3)** is redundant. It is often the case in practice that $A^\dagger \in \mathcal{N}$, so that **(NT1)** holds trivially.

In this section we shall consider the four well-known negative translations for **IL**, namely, Kolmogorov, Gödel, Gentzen and Glivenko, in the context of affine logic $\mathbf{AL_i}$ (cf. [5] for an analysis of the relationship between these translations). We prove that both Kolmogorov and Gödel are negative translation for $\mathbf{AL_i}$, and give counter-examples to show that Gentzen and Glivenko fail to satisfy **(NT3)**. Moreover, in Example 3.1 we show that in the context of $\mathbf{AL_i}$ not all negative translations are equivalent, by presenting a variant of the Kolmogorov translation which satisfies **(NT1)** – **(NT3)** but is distinct from the original Kolmogorov translation. In Section 5, however, we will see that in $\mathbf{ŁL_i}$ the situation is similar to the setting of full intuitionistic logic **IL**.

3.1 Kolmogorov and Gödel translations

First of all, we show that both the Kolmogorov and the Gödel translations are in fact negative translations for *affine* logic, i.e. no contraction is necessary to prove Troelstra's three requirements. Let $\mathcal{L}$ be the language of the theories $\mathbf{AL_c}$ and $\mathbf{AL_i}$.

Definition 3.3 (Kolmogorov translation [12]) *For each formula $A \in \mathcal{L}$ associate a formula $A^{\mathsf{K}} \in \mathcal{L}$ inductively as follows:*

$$\begin{array}{rcll} P^{\mathsf{K}} & \equiv & P^{\perp\perp} & (P \text{ atomic}) \\ \perp^{\mathsf{K}} & \equiv & \perp & \\ (A \otimes B)^{\mathsf{K}} & \equiv & (A^{\mathsf{K}} \otimes B^{\mathsf{K}})^{\perp\perp} & \\ (A \multimap B)^{\mathsf{K}} & \equiv & (A^{\mathsf{K}} \multimap B^{\mathsf{K}})^{\perp\perp}. & \end{array}$$

We will also consider the following negative translation which can be distilled from [9]. In Gödel's presentation an implication $A \multimap B$ is translated as $(A \otimes B^\perp)^\perp$. We use here that in $\mathbf{AL_i}$ this is equivalent to $A \multimap B^{\perp\perp}$. The translation often referred to as the Gödel-Gentzen translation will be treated in the following section, where we attribute it to Gentzen. It will become clear that in the sub-structural setting the Gödel translation is not the same as the Gentzen one.

Definition 3.4 (Gödel translation [9]) *For each formula $A \in \mathcal{L}$ we first associate a formula $A^* \in \mathcal{L}$ inductively as follows:*

$$\begin{array}{rcll} P^* & \equiv & P & (P \text{ atomic}) \\ \perp^* & \equiv & \perp & \\ (A \otimes B)^* & \equiv & A^* \otimes B^* & \\ (A \multimap B)^* & \equiv & A^* \multimap (B^*)^{\perp\perp}. & \end{array}$$

Then we define $A^{\mathsf{G\ddot{o}}} = (A^*)^{\perp\perp}$. Gödel [9] in fact does not need this final double negation since in Heyting arithmetic one can already prove $(A^*)^{\perp\perp} \vdash A^*$. Hence in that context we can even take $A^{\mathsf{G\ddot{o}}} = A^*$. In $\mathbf{AL_i}$, however, we need the outermost double negation to make the proof of the following theorem go through.

Theorem 3.1 *Both the Kolmogorov translation $(\cdot)^{\mathsf{K}}$ and the Gödel translation $(\cdot)^{\mathsf{G\ddot{o}}}$ are negative translations for $\mathbf{AL_i}$.*

Proof: In the case of Kolmogov we have:
(NT1). Trivial since $A^{\mathsf{K}} \in \mathcal{N}$.
(NT2). Clearly $\mathbf{AL_c} = \mathbf{AL_i} + [\mathsf{DNE}]$ proves $A \leftrightarrow A^{\mathsf{K}}$.
(NT3). Finally, we show that if $\Gamma \vdash A$ is provable in $\mathbf{AL_c}$ then $\Gamma^{\mathsf{K}} \vdash A^{\mathsf{K}}$ is provable in $\mathbf{AL_i}$, where Γ^{K} abbreviates ${B_0}^{\mathsf{K}}, \ldots {B_n}^{\mathsf{K}}$. This can be shown by induction on the derivation of the sequent $\Gamma \vdash A$. The cases of the axioms [ASM] and [EFQ] are trivial. In the case of [DNE] we just need to observe that $A^{\perp\perp\perp} \leftrightarrow A^{\perp}$ holds in $\mathbf{AL_i}$. The case of [$\multimap$E] we need to derive $\Gamma^{\mathsf{K}}, \Delta^{\mathsf{K}} \vdash B^{\mathsf{K}}$ from $\Gamma^{\mathsf{K}} \vdash A^{\mathsf{K}}$ and $\Delta^{\mathsf{K}} \vdash (A \multimap B)^{\mathsf{K}}$. This can be done as

$$\dfrac{\Gamma^{\mathsf{K}} \vdash A^{\mathsf{K}} \qquad \dfrac{\dfrac{\Delta^{\mathsf{K}} \vdash (A \multimap B)^{\mathsf{K}}}{\Delta^{\mathsf{K}} \vdash (A^{\mathsf{K}} \multimap B^{\mathsf{K}})^{\perp\perp}}\,(\text{def})}{\Delta^{\mathsf{K}} \vdash A^{\mathsf{K}} \multimap B^{\mathsf{K}}}\,(\text{Lemma 2.3 } (v))}{\Gamma^{\mathsf{K}}, \Delta^{\mathsf{K}} \vdash B^{\mathsf{K}}}\,[\multimap\mathsf{E}]$$

[$\multimap$I] can also be easily shown as

$$\dfrac{\dfrac{\dfrac{\Gamma^{\mathsf{K}}, A^{\mathsf{K}} \vdash B^{\mathsf{K}}}{\Gamma^{\mathsf{K}} \vdash A^{\mathsf{K}} \multimap B^{\mathsf{K}}}\,[\multimap\mathsf{I}]}{\Gamma^{\mathsf{K}} \vdash (A^{\mathsf{K}} \multimap B^{\mathsf{K}})^{\perp\perp}}\,(\text{Lemma 2.3 } (v))}{\Gamma^{\mathsf{K}} \vdash (A \multimap B)^{\mathsf{K}}}\,(\text{def})$$

The case of [$\otimes$I] is easy once we observe that $A \vdash A^{\perp\perp}$ is provable in $\mathbf{AL_i}$.
Finally, the case of [$\otimes$E] can be show as

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma^{\mathsf{K}} \vdash (A \otimes B)^{\mathsf{K}}}{\Gamma^{\mathsf{K}} \vdash (A^{\mathsf{K}} \otimes B^{\mathsf{K}})^{\perp\perp}}\,(\text{def}) \qquad \dfrac{\Delta^{\mathsf{K}}, A^{\mathsf{K}}, B^{\mathsf{K}} \vdash C^{\mathsf{K}}}{\Delta^{\mathsf{K}}, (C^{\mathsf{K}})^{\perp} \vdash (A^{\mathsf{K}} \otimes B^{\mathsf{K}})^{\perp}}}{\Gamma^{\mathsf{K}}, \Delta^{\mathsf{K}}, (C^{\mathsf{K}})^{\perp} \vdash \perp}\,[\multimap\mathsf{E}]}{\Gamma^{\mathsf{K}}, \Delta^{\mathsf{K}} \vdash (C^{\mathsf{K}})^{\perp\perp}}\,[\multimap\mathsf{I}]}{\Gamma^{\mathsf{K}}, \Delta^{\mathsf{K}} \vdash C^{\mathsf{K}}}$$

In the final step above we are using that $(A^{\mathsf{K}})^{\perp\perp} \leftrightarrow A^{\mathsf{K}}$, which is easy to show.
For the Gödel translation, it is enough to show that $A^{\mathsf{K}} \leftrightarrow A^{\mathsf{G\ddot{o}}}$ in $\mathbf{AL_i}$. We do that by induction on the structure of A. The non-trivial cases are implication and conjunction. Recall that $A^{\mathsf{G\ddot{o}}} = (A^*)^{\perp\perp}$. For implication we have

$$\begin{aligned}
(A \multimap B)^{\mathsf{K}} &\leftrightarrow (A^{\mathsf{K}} \multimap B^{\mathsf{K}})^{\perp\perp} && (\text{def } (\cdot)^{\mathsf{K}}) \\
&\leftrightarrow ((A^*)^{\perp\perp} \multimap (B^*)^{\perp\perp})^{\perp\perp} && (\text{IH}) \\
&\leftrightarrow (A^* \multimap (B^*)^{\perp\perp})^{\perp\perp} && (\text{Lemma 2.3 } (vi)) \\
&\leftrightarrow (A \multimap B)^{\mathsf{G\ddot{o}}}. && (\text{def } (\cdot)^{\mathsf{G\ddot{o}}})
\end{aligned}$$

Similarly for conjunction

$$\begin{aligned}
(A \otimes B)^{\mathsf{K}} &\leftrightarrow (A^{\mathsf{K}} \otimes B^{\mathsf{K}})^{\perp\perp} && (\text{def } (\cdot)^{\mathsf{K}}) \\
&\leftrightarrow ((A^*)^{\perp\perp} \otimes (B^*)^{\perp\perp})^{\perp\perp} && (\text{IH}) \\
&\leftrightarrow (A^* \otimes B^*)^{\perp\perp} && (\text{Lemma 2.3 } (vii)) \\
&\leftrightarrow (A \otimes B)^{\mathsf{G\ddot{o}}}. && (\text{def } (\cdot)^{\mathsf{G\ddot{o}}})
\end{aligned}$$

■

Example 3.1 *Consider the following variant of the Kolmogorov translation where the translation of atoms P is taken to be*

$$P^{\tilde{\mathsf{K}}} \equiv (P^{\perp\perp} \otimes (P^{\perp\perp} \multimap P))^{\perp\perp}$$

for all fixed atom Q. Clearly this variant still satisfies **(NT1)** *and* **(NT2)**. *It is easy to check that it also satisfies* **(NT3)**, *as done for Kolmogorov above. However, it can be shown that, over* $\mathbf{AL_i}$, *P^{K} does not imply $P^{\tilde{\mathsf{K}}}$ ($\mathbf{P}_4$ in [1, Example 2.2.6] provides a counter-model).*

3.2 Gentzen and Glivenko translations For both the Gentzen and the Glivenko translations (defined below) a corresponding Theorem 3.1 no longer holds for $\mathbf{AL_i}$. These translations rely on uses of contraction which are not available in affine logic. Nevertheless, we will find that the amount of contraction available in Łukasiewicz logic, via [CWC], is sufficient for these translations to go through. The Gentzen negative translation works by adding double negations on all the atoms of a given formula:

Definition 3.5 (Gentzen translation [6]) *For each formula $A \in \mathcal{L}$ associate a formula $A^{\mathsf{Gen}} \in \mathcal{L}$ inductively as follows:*

$$\begin{array}{rcll} P^{\mathsf{Gen}} & \equiv & P^{\perp\perp} & (P \text{ atomic}) \\ \perp^{\mathsf{Gen}} & \equiv & \perp & \\ (A \otimes B)^{\mathsf{Gen}} & \equiv & A^{\mathsf{Gen}} \otimes B^{\mathsf{Gen}} & \\ (A \multimap B)^{\mathsf{Gen}} & \equiv & A^{\mathsf{Gen}} \multimap B^{\mathsf{Gen}}. & \end{array}$$

As $A^{\perp} = A \multimap \perp$, we have that $(A^{\perp})^{\mathsf{Gen}}$ is equivalent to $(A^{\mathsf{Gen}})^{\perp}$.

Theorem 3.2 *The translation $(\cdot)^{\mathsf{Gen}}$ is not a negative translation for $\mathbf{AL_i}$.*

Proof: We show that **(NT3)** fails for the Gentzen translation on $\mathbf{AL_i}$. Let P, Q be atomic formulas and take $A \equiv (P \otimes Q)^{\perp\perp} \multimap (P \otimes Q)$. Obviously $\mathbf{AL_c}$ proves A, since A is an instance of [DNE]. However the Gentzen translation of A is

$$(P^{\perp\perp} \otimes Q^{\perp\perp})^{\perp\perp} \multimap P^{\perp\perp} \otimes Q^{\perp\perp}$$

which is not provable in $\mathbf{AL_i}$ (see [1, Theorem 5.2.4] for a counter-model). ■

The Glivenko negative translation simply doubly negates the whole formula:

Definition 3.6 (Glivenko translation [8]) *Given a formula $A \in \mathcal{L}$ define its Glivenko translation A^{Gli} as $A^{\mathsf{Gli}} \equiv A^{\perp\perp}$.*

Theorem 3.3 *The Glivenko translation is not a negative translation for $\mathbf{AL_i}$.*

Proof: As with the Gödel translation, we also show that **(NT3)** fails in the Glivenko translation for $\mathbf{AL_i}$. Let P be an atomic formula. The Glivenko translation of $P^{\perp\perp} \multimap P$ (an instance of [DNE]) is $(P^{\perp\perp} \multimap P)^{\perp\perp}$, which is not provable in $\mathbf{AL_i}$ (see [1, Theorem 5.2.4] for a counter-model). ■

We conclude by noting that the Gentzen and the Glivenko translations do not have to fail simultaneously, i.e. there are extensions of $\mathbf{AL_i}$ for which one translation works but the other does not.

Theorem 3.4 *There are extensions $\mathbf{A}_1$ and $\mathbf{A}_2$ of $\mathbf{AL_i}$ such that*

- *$(\cdot)^{\mathsf{Gli}}$ is a negative translation for $\mathbf{A}_1$ but $(\cdot)^{\mathsf{Gen}}$ is not.*
- *$(\cdot)^{\mathsf{Gen}}$ is a negative translation for $\mathbf{A}_2$ but $(\cdot)^{\mathsf{Gli}}$ is not.*

Proof: See [1], where we construct two finite models whose theories have the property above. ■

$\mathbf{AL_i}$ can be presented using a Gentzen-style sequent calculus that admits cut-elimination. This leads to a relatively efficient decision procedure for $\mathbf{AL_i}$ which one can use as an alternative to semantic methods to decide unprovability where needed in the proofs of Theorems 3.2 and 3.3. However, we do not know of a proof based on cut-elimination for Theorem 3.4.

4 Homomorphism Properties of Double Negation in $\mathbf{ŁL_i}$

Our goal in this section is to find $\mathbf{ŁL_i}$ derivations of some important theorems about the primitive and derived connectives. These include:

- a derivation of $[\mathsf{DNE}]^{\perp\perp}$ (Corollary 4.8);
- a duality property between $\vee$ and $\downarrow$ (Theorem 4.9);
- homomorphism properties of double negation with respect to both implication (Section 4.3) and conjunction (Section 4.4); and,
- a collection of "De Morgan" properties (Section 6).

As we have already remarked, the derivations we will give have been extracted by analysis of computer-generated proofs found by the Prover9 automated theorem-prover. Our contribution was to propose conjectures to Prover9, to study the machine-oriented proofs it found and to present the proofs in a human-intelligible form by breaking them down into structurally interesting lemmas. This was an iterative process since often Prover9 was able to find simpler proofs of a lemma when presented with it as a conjecture in isolation. In cases when Prover9 was unable to find a proof, Mace4 was often able to find a counter-model: a finite model of the logic in question in which the conjecture can be seen to fail. See also [1] for examples of models found by Mace4.

It follows from work on commutative GBL-algebras that $\mathbf{ŁL_i}$ is decidable [3]. However the problem is PSPACE-complete. In [1], we give a simple indirect method for demonstrating that a formula is valid in intuitionistic Łukasiewicz Logic, a heuristic method which we have used extensively in parallel with attempts to find explicit proofs and counter-examples with Prover9 and Mace4.

4.1 Basic identities on $\vee, \Rightarrow, \wedge$

We start by proving in $\mathbf{ŁL_i}$ several useful results about the derived connectives $\vee, \wedge$ and $\Rightarrow$.

Lemma 4.1 ($\mathbf{ŁL_i}$) $A \;\leftrightarrow\; (A \vee B) \otimes (B \multimap A)$

Proof: We have

$$
\begin{aligned}
A \;&\leftrightarrow\; A \otimes \underline{(A \multimap (B \multimap A))} && ([\mathsf{WK}])\\
&\leftrightarrow\; ((B \multimap A) \multimap A) \otimes (B \multimap A) && ([\mathsf{CWC}])\\
&\leftrightarrow\; (A \vee B) \otimes (B \multimap A). && (\text{def } \vee)
\end{aligned}
$$

Recall that we underline easily proved conjuncts which are either inserted or deleted. ∎

Lemma 4.2 ($\mathbf{ŁL_i}$) $A \otimes B \;\leftrightarrow\; A \otimes (B \vee (A \Rightarrow B))$

Proof: Let $X = ((A \Rightarrow B) \multimap B) \multimap A$. We have

$$
\begin{aligned}
A \otimes B \;&\leftrightarrow\; (A \Rightarrow B) \otimes A && (\text{Lemma 2.3 } (iii))\\
&\leftrightarrow\; (A \Rightarrow B) \otimes A \otimes \underline{(A \multimap ((A \Rightarrow B) \multimap B))} && (\text{easy})\\
&\leftrightarrow\; (A \Rightarrow B) \otimes ((A \Rightarrow B) \multimap B) \otimes X && ([\mathsf{CWC}])\\
&\leftrightarrow\; B \otimes \underline{(B \multimap (A \Rightarrow B))} \otimes X && ([\mathsf{CWC}])\\
&\leftrightarrow\; B \otimes X && (\text{Lemma 2.3 } (ii))\\
&\leftrightarrow\; (B \vee (A \Rightarrow B)) \otimes ((A \Rightarrow B) \multimap B) \otimes X && (\text{Lemma 4.1})\\
&\leftrightarrow\; (B \vee (A \Rightarrow B)) \otimes A \otimes \underline{(A \multimap ((A \Rightarrow B) \multimap B))} && ([\mathsf{CWC}])\\
&\leftrightarrow\; A \otimes (B \vee (A \Rightarrow B)). && (\text{easy})
\end{aligned}
$$

∎

The following lemma is used in Section 4.4.

Lemma 4.3 ($\mathbf{ŁL_i}$) $(A \multimap C) \otimes (C \multimap B) \;\rightarrow\; (A \multimap B) \otimes (A \wedge B \multimap C)$

Proof: We have

$$
\begin{aligned}
(A \multimap C) \otimes (C \multimap B) &\;\rightarrow\; (A \multimap C) \otimes ((A \multimap C) \multimap (A \multimap B)) && \text{(easy)}\\
&\;\leftrightarrow\; (A \multimap B) \otimes ((A \multimap B) \multimap (A \multimap C)) && \text{([CWC])}\\
&\;\leftrightarrow\; (A \multimap B) \otimes (A \otimes (A \multimap B) \multimap C) && \text{(easy)}\\
&\;\leftrightarrow\; (A \multimap B) \otimes (A \wedge B \multimap C). && \text{(def)}
\end{aligned}
$$

■

So far, we have not used the constant $\perp$. We now prove a few basic properties of $(\cdot)^\perp$ and its relation to the derived connectives.

Lemma 4.4 ($\mathbf{ŁL_i}$) $A^\perp \otimes (B \vee A) \;\leftrightarrow\; A^\perp \otimes B$

Proof: The right-to-left direction follows directly from $B \;\rightarrow\; B \vee A$. For the other direction, note that by [EFQ] we have $A \;\rightarrow\; A^\perp \Rightarrow B$. Hence, $B \vee A \;\rightarrow\; B \vee (A^\perp \Rightarrow B)$. Therefore, the result follows directly from Lemma 4.2. ■

It turns out that many intuitionistically unacceptable equivalences become provable in $\mathbf{ŁL_i}$ "under" a negation. For example, our first important result is that in $\mathbf{ŁL_i}$ strong implication $\Rightarrow$ is a dual of a weak conjunction $\wedge$ in the sense that $(A \wedge B)^\perp \;\leftrightarrow\; A \Rightarrow B^\perp$. This is akin to the relation between conjunction and implication $(A \otimes B)^\perp \;\leftrightarrow\; A \multimap B^\perp$ which one obtains in $\mathbf{AL_i}$ simply by currying and uncurrying.

Theorem 4.5 ($\mathbf{ŁL_i}$) $(A \wedge B)^\perp \;\leftrightarrow\; A \Rightarrow B^\perp$

Proof: Observe that by Lemma 4.4 (with B and A interchanged) it follows that $(*)$ $B^\perp \;\rightarrow\; A \vee B \multimap A \otimes B^\perp$. Hence

$$
\begin{aligned}
(A \wedge B)^\perp &\;\leftrightarrow\; (A \otimes (A \multimap B))^\perp && \text{(def } \wedge\text{)}\\
&\;\leftrightarrow\; (B \otimes (B \multimap A))^\perp && \text{([CWC])}\\
&\;\leftrightarrow\; (B \multimap A) \multimap B^\perp && \text{(easy)}\\
&\;\rightarrow\; (B \multimap A) \multimap (A \vee B) \multimap A \otimes B^\perp && (*)\\
&\;\leftrightarrow\; A \multimap (\underline{A \multimap (B \multimap A)}) \multimap A \otimes B^\perp && \text{([CWC])}\\
&\;\leftrightarrow\; A \multimap A \otimes B^\perp. && \text{([WK])}\\
&\;\leftrightarrow\; A \Rightarrow B^\perp. && \text{(def } \Rightarrow\text{)}
\end{aligned}
$$

The converse implication is straightforward. ■

Corollary 4.6 ($\mathbf{ŁL_i}$) $A \Rightarrow B^\perp \;\leftrightarrow\; B \Rightarrow A^\perp$

Proof: Direct from Theorem 4.5, since $\wedge$ is commutative (i.e. [CWC]). ■

4.2 Symmetries of $\vee$ and $\downarrow$ and **[DNE]** Although the commutativity of $\vee$ is clearly a classical principle, it is perhaps surprising that commutativity of $B \downarrow A$ can be proved intuitionistically.

Theorem 4.7 ($\mathbf{ŁL_i}$) $B \downarrow A \;\leftrightarrow\; A \downarrow B$

Proof: By symmetry it is enough to prove $B \downarrow A \;\rightarrow\; A \downarrow B$. We have

$$\begin{array}{rcll} (B \multimap A) \otimes A^{\perp} & \leftrightarrow & (B \multimap A) \otimes A^{\perp} \otimes \underline{(A^{\perp} \multimap (A \multimap B))} & (\textsf{[EFQ]}) \\ & \leftrightarrow & (B \multimap A) \otimes (A \multimap B) \otimes ((A \multimap B) \multimap A^{\perp}) & (\textsf{[CWC]}) \\ & \leftrightarrow & (B \multimap A) \otimes (A \multimap B) \otimes ((A \multimap B) \otimes A)^{\perp} & (\text{easy}) \\ & \leftrightarrow & (B \multimap A) \otimes (A \multimap B) \otimes (B \otimes (B \multimap A))^{\perp} & (\textsf{[CWC]}) \\ & \leftrightarrow & (A \multimap B) \otimes (B \multimap A) \otimes ((B \multimap A) \multimap B^{\perp}) & (\text{easy}) \\ & \rightarrow & (A \multimap B) \otimes B^{\perp}. & (\text{easy}) \end{array}$$

Recall that we underline easily proved conjuncts which are either inserted or deleted. ■

A corollary of the above theorem is that the double negation of the classical axiom [DNE] is provable in $\mathbf{ŁL_i}$.

Corollary 4.8 ($\mathbf{ŁL_i}$) $(A^{\perp\perp} \multimap A)^{\perp\perp}$

Proof: Note that, since $\perp \leftrightarrow\; A \otimes A^{\perp}$ we have $(*)$ $A^{\perp\perp} \;\leftrightarrow\; A^{\perp} \Rightarrow A$. Moreover, it is easy to check that $(**)$ $X \downarrow (Y \multimap X) \;\leftrightarrow\; X^{\perp} \otimes (X \vee Y)$, for all X and Y. Hence

$$\begin{array}{rcll} (A^{\perp\perp} \multimap A)^{\perp} & \leftrightarrow & ((A^{\perp} \Rightarrow A) \multimap A)^{\perp} & (*) \\ & \leftrightarrow & ((A^{\perp} \Rightarrow A) \multimap A)^{\perp} \otimes \underline{(A \multimap ((A^{\perp} \Rightarrow A) \multimap A))} & (\textsf{[WK]}) \\ & \leftrightarrow & ((A^{\perp} \Rightarrow A) \multimap A) \downarrow A & (\text{def} \downarrow) \\ & \leftrightarrow & A \downarrow ((A^{\perp} \Rightarrow A) \multimap A) & (\text{Theorem 4.7}) \\ & \leftrightarrow & A^{\perp} \otimes (A \vee (A^{\perp} \Rightarrow A)) & (**) \\ & \leftrightarrow & A^{\perp} \otimes A & (\text{Lemma 4.2}) \\ & \leftrightarrow & \perp . & \end{array}$$

■

It is well known that this is provable in full intuitionistic logic **IL**. The usual proof making apparently essential use of the full contraction axiom goes as follows. Assuming (1) $(A^{\perp\perp} \multimap A)^{\perp}$ we must derive a contradiction. First use (1) to derive $A^{\perp}$, by [WK]. Assume also (2) $A^{\perp\perp}$. From (2) and $A^{\perp}$ we obtain $\perp$, and hence A. Discharging the assumption (2) we have $A^{\perp\perp} \multimap A$, which by (1) gives a contradiction. Note, however, that assumption (1) was used twice. The corollary above gives us a proof using only the weak form of contraction permitted by [CWC].

Next we have a theorem showing that the NOR connective $A \downarrow B$ is indeed the negation of the disjunction $\vee$, a fact which holds in full intuitionistic logic **IL**, but again, via a simple proof that appears to make essential use of the full contraction axiom.

Theorem 4.9 ($\mathbf{ŁL_i}$) $(A \vee B)^{\perp} \;\leftrightarrow\; A \downarrow B$

Proof: By Lemma 2.3 (i) we have $B \multimap A \vee B$; and by [EFQ] we have $\perp \multimap A$. Hence, $(*)$ $(A \vee B)^{\perp} \multimap (B \multimap A)$. Therefore

$$\begin{array}{rcll} (A \vee B)^{\perp} & \leftrightarrow & (A \vee B)^{\perp} \otimes ((A \vee B)^{\perp} \multimap (B \multimap A)) & (*) \\ & \leftrightarrow & (B \multimap A) \otimes ((B \multimap A) \multimap (A \vee B)^{\perp}) & (\textsf{[CWC]}) \\ & \leftrightarrow & (B \multimap A) \otimes ((B \multimap A) \otimes (A \vee B))^{\perp} & (\text{easy}) \\ & \leftrightarrow & (B \multimap A) \otimes (A \otimes (\underline{A \multimap (B \multimap A)}))^{\perp} & (\textsf{[CWC]}) \\ & \leftrightarrow & (B \multimap A) \otimes A^{\perp}. & (\textsf{[WK]}) \end{array}$$

■

The above above imply the commutativity of $\vee$ under a negation:

Theorem 4.10 ($\mathbf{ŁL_i}$) $(A \vee B)^{\perp} \;\leftrightarrow\; (B \vee A)^{\perp}$

Proof: Direct from Theorems 4.7 and 4.9. ■

4.3 Double negation homomorphism: Implication We now show that (in $\mathbf{ŁL_i}$) the double negation operation $(\cdot)^{\perp\perp}$ is a homomorphism for implication, i.e.

$$(A \multimap B)^{\perp\perp} \;\leftrightarrow\; A^{\perp\perp} \multimap B^{\perp\perp}.$$

We will show the same for conjunction in Section 4.4. Note that $\perp \vee A = A^{\perp\perp}$. Hence, it follows from Lemma 2.3 (iv) that

$$(A \multimap B)^{\perp\perp} \;\to\; A^{\perp\perp} \multimap B^{\perp\perp}$$

and hence $(A \multimap B)^{\perp\perp} \;\to\; A \multimap B^{\perp\perp}$ is provable already in $\mathbf{AL_i}$. We will now see that the converse implication holds in $\mathbf{ŁL_i}$. Again, the fact that this holds in full intuitionistic logic is well known. See [15], page 9, for instance, for an **IL**-derivation of Theorem 4.11. That derivation, however, uses the assumption $(A \multimap B)^{\perp}$ twice, and hence cannot be formalised in $\mathbf{ŁL_i}$.

Theorem 4.11 ($\mathbf{ŁL_i}$) $A^{\perp\perp} \multimap B^{\perp\perp} \;\leftrightarrow\; (A \multimap B)^{\perp\perp}$

Proof: By the remarks above, we have only the left-to-right direction to prove:

$$\begin{array}{rlr}
A^{\perp\perp} \multimap B^{\perp\perp} & \to\; A \multimap B^{\perp\perp} & (A \;\to\; A^{\perp\perp}) \\
& \to\; (B^{\perp\perp} \multimap B) \multimap (A \multimap B) & \text{(easy)} \\
& \to\; (A \multimap B)^{\perp} \multimap (B^{\perp\perp} \multimap B)^{\perp} & \text{(easy)} \\
& \leftrightarrow\; (A \multimap B)^{\perp} \multimap \perp & \text{(Corollary 4.8)} \\
& \leftrightarrow\; (A \multimap B)^{\perp\perp}. & (\text{def } (\cdot)^{\perp})
\end{array}$$

■

4.4 Double negation homomorphism: Conjunction As done in Section 4.3 for implication, we now show that (in $\mathbf{ŁL_i}$) the double negation operation $(\cdot)^{\perp\perp}$ is also a homomorphism for conjunction, i.e.

$$(A \otimes B)^{\perp\perp} \;\leftrightarrow\; A^{\perp\perp} \otimes B^{\perp\perp}.$$

This result will follow immediately from a duality between implication ($\multimap$) and conjunction ($\otimes$) – Theorem 4.13 below.

Lemma 4.12 ($\mathbf{ŁL_i}$) $A^{\perp} \;\leftrightarrow\; (A \multimap B) \otimes (A \wedge B)^{\perp}$

Proof: Left-to-right follows directly from Lemma 4.3, taking $C = \perp$. For the converse observe that $(A \wedge B)^{\perp} \;\leftrightarrow\; A \multimap (A \multimap B)^{\perp}$. ■

Theorem 4.13 ($\mathbf{ŁL_i}$) $(A^{\perp} \multimap B)^{\perp} \;\leftrightarrow\; A^{\perp} \otimes B^{\perp}$

Proof: The implication from right to left is easy. Since, $A^{\perp} \multimap \perp \to\; A^{\perp} \multimap B$ we obtain

$$(*)\; A^{\perp\perp} \multimap (A^{\perp\perp} \otimes (A^{\perp} \multimap B)^{\perp}) \;\to\; A^{\perp\perp} \multimap (A^{\perp\perp} \otimes A^{\perp\perp\perp}) \;\leftrightarrow\; A^{\perp\perp\perp}.$$

Hence, taking $A' = A^{\perp} \multimap B$ and $B' = A^{\perp\perp}$ in Lemma 4.12, we have the first line of the following chain

$$\begin{array}{rlr}
(A^{\perp} \multimap B)^{\perp} & \leftrightarrow\; ((A^{\perp} \multimap B) \multimap A^{\perp\perp}) \otimes ((A^{\perp} \multimap B) \wedge A^{\perp\perp})^{\perp} & \\
& \leftrightarrow\; ((A^{\perp} \multimap B) \multimap A^{\perp\perp}) \otimes (A^{\perp\perp} \Rightarrow (A^{\perp} \multimap B)^{\perp}) & \text{(Theorem 4.5)} \\
& \to\; ((A^{\perp} \multimap B) \multimap A^{\perp\perp}) \otimes A^{\perp\perp\perp} & (*) \\
& \leftrightarrow\; (A^{\perp} \otimes (A^{\perp} \multimap B))^{\perp} \otimes A^{\perp} & \text{(easy)}
\end{array}$$

$$
\begin{aligned}
&\leftrightarrow\ (A^{\perp} \wedge B)^{\perp} \otimes A^{\perp} && (\text{def } \wedge)\\
&\leftrightarrow\ (B \Rightarrow A^{\perp\perp}) \otimes A^{\perp} && (\text{Theorem 4.5})\\
&\leftrightarrow\ (A^{\perp} \Rightarrow B^{\perp}) \otimes A^{\perp} && (\text{Corollary 4.6 } (i))\\
&\rightarrow\ A^{\perp} \otimes B^{\perp}. && (\text{easy})
\end{aligned}
$$

■

Theorem 4.14 ($\mathbf{ŁL_i}$) $(A \otimes B)^{\perp\perp} \leftrightarrow A^{\perp\perp} \otimes B^{\perp\perp}$

Proof: By Theorem 4.13 and Lemma 2.3 (*viii*). ■

5 Negative Translations of Łukasiewicz Logic

In this section we show that all four translations considered (Kolmogorov, Gödel, Gentzen and Glivenko) are negative translations for $\mathbf{ŁL_i}$. In fact, as it is the case in **IL**, it turns out that any two negative translations for $\mathbf{ŁL_i}$ are equivalent. This is a non-trivial result, since, as we have shown, the Gentzen and Glivenko translations fail for $\mathbf{AL_i}$. The crucial property we will need here is that the double negation mapping $A \mapsto A^{\perp\perp}$ is a homomorphism in $\mathbf{ŁL_i}$, as proven in Sections 4.3 and 4.4.

Theorem 5.1 *The Glivenko translation* $(\cdot)^{\mathsf{Gli}}$ *is a negative translation for* $\mathbf{ŁL_i}$.

Proof: We show by induction on the structure of A that $A^{\mathsf{K}} \leftrightarrow A^{\perp\perp}$ in $\mathbf{ŁL_i}$. This is similar to the proof of Theorem 3.1, where we showed that Gödel's translation is equivalent to Kolmogorov's in $\mathbf{AL_i}$. In here we need a slightly stronger version of Lemma 2.3 (*vi*) which in fact follows from Theorem 4.11. Again, the non-trivial cases are implication and conjunction. For implication we have

$$
\begin{aligned}
(A \multimap B)^{\mathsf{K}} &\leftrightarrow (A^{\mathsf{K}} \multimap B^{\mathsf{K}})^{\perp\perp} && (\text{def } (\cdot)^{\mathsf{K}})\\
&\leftrightarrow (A^{\perp\perp} \multimap B^{\perp\perp})^{\perp\perp} && (\text{IH})\\
&\leftrightarrow (A \multimap B)^{\perp\perp\perp\perp} && (\text{Theorem 4.11})\\
&\leftrightarrow (A \multimap B)^{\perp\perp}. && (\text{easy})
\end{aligned}
$$

Similarly for conjunction

$$
\begin{aligned}
(A \otimes B)^{\mathsf{K}} &\leftrightarrow (A^{\mathsf{K}} \otimes B^{\mathsf{K}})^{\perp\perp} && (\text{def } (\cdot)^{\mathsf{K}})\\
&\leftrightarrow (A^{\perp\perp} \otimes B^{\perp\perp})^{\perp\perp} && (\text{IH})\\
&\leftrightarrow (A \otimes B)^{\perp\perp\perp\perp} && (\text{Lemma 2.3 } (vi))\\
&\leftrightarrow (A \otimes B)^{\perp\perp}. && (\text{easy})
\end{aligned}
$$

■

But note that we have not yet used the full strength of our homomorphism properties for double negation. We will make use of them now to show that any translation for $\mathbf{ŁL_i}$ which satisfies **(NT1)** and **(NT2)** will in fact also satisfy **(NT3)**.

Lemma 5.2 *For any formula* $A \in \mathcal{N}$ *we have that* $A \leftrightarrow_{\mathbf{ŁL_i}} A^{\perp\perp}$.

Proof: By induction on $A \in \mathcal{N}$. We need to consider three cases:

If $A = \perp$ the result is trivial.

If $A = B \multimap C$ with $C \in \mathcal{N}$ then, by the inductive hypothesis, $C \leftrightarrow C^{\perp\perp}$. Hence $B \multimap C \leftrightarrow B \multimap C^{\perp\perp}$. Since $B \multimap C^{\perp\perp} \leftrightarrow B^{\perp\perp} \multimap C^{\perp\perp}$, even in $\mathbf{AL_i}$, by Theorem 4.11 we have that $B \multimap C \leftrightarrow (B \multimap C)^{\perp\perp}$.

If $A = B \otimes C$ with $B, C \in \mathcal{N}$ then, by the inductive hypothesis, we have that $B \leftrightarrow B^{\perp\perp}$ and $B \leftrightarrow B^{\perp\perp}$, hence $B \otimes C \leftrightarrow (B \otimes C)^{\perp\perp}$ by Theorem 4.14. ∎

Theorem 5.3 *Any translation $(\cdot)^\dagger$ for* $\mathbf{ŁL_i}$ *which satisfies* **(NT1)** *and* **(NT2)** *is equivalent to* $(\cdot)^{\mathsf{Gli}}$ *and hence is a negative translation, i.e.,* $(\cdot)^\dagger$ *also satisfies* **(NT3)**.

Proof: Fix a formula A. By **(NT1)**, $A^\dagger \leftrightarrow B$ with $B \in \mathcal{N}$. That $A^\dagger \leftrightarrow_{\mathbf{ŁL_i}} A^{\mathsf{Gli}}$ can be shown as

$$\begin{aligned} A^\dagger &\leftrightarrow B \\ &\leftrightarrow (B)^{\perp\perp} && \text{(Lemma 5.2)} \\ &\leftrightarrow (A^\dagger)^{\perp\perp} && \text{(since } A^\dagger \leftrightarrow B) \\ &\leftrightarrow A^{\perp\perp}. && \text{(by \textbf{(NT2)} and Theorem 5.1)} \end{aligned}$$

By Theorem 5.1, $(\cdot)^{\mathsf{Gli}}$ satisfies **(NT3)**, hence so does $(\cdot)^\dagger$. ∎

Corollary 5.4 *The Gentzen translation* $(\cdot)^{\mathsf{Gen}}$ *is a negative translation for* $\mathbf{ŁL_i}$.

Proof: Since $(\cdot)^{\mathsf{Gen}}$ satisfies **(NT1)** and **(NT2)** in $\mathbf{ŁL_i}$. ∎

Theorem 5.3 can be used to conclude that several other formula translations are also negative translations for $\mathbf{ŁL_i}$.

Example 5.1 *Define a variant of the Gödel translation whereby the definition of $(A \multimap B)^*$ is modified as*

$$(A \multimap B)^* \equiv A \multimap (B^*)^{\perp\perp},$$

i.e. the premise of the implication is not inductively translated. It is easy to see that this "simplification" still satisfies **(NT1)** *and* **(NT2)** *and hence, by Theorem 5.3, is a negative translation for* $\mathbf{ŁL_i}$. *A similar simplification can be considered for the Gentzen translation, leading, again, to a negative translation for* $\mathbf{ŁL_i}$.

Example 5.2 *Define $A^\dagger \in \mathcal{L}$ inductively as follows:*

$$\begin{aligned} P^\dagger &\equiv P^\perp \qquad (P \textit{ atomic}) \\ \perp^\dagger &\equiv \top \\ (A \otimes B)^\dagger &\equiv A^\dagger \multimap (B^\dagger)^\perp \\ (A \multimap B)^\dagger &\equiv A^\dagger \otimes (B^\dagger)^\perp. \end{aligned}$$

Then we define the Krivine translation *of A as $A^{\mathsf{Kr}} = (A^\dagger)^\perp$. The formula A^{Kr} is clearly a negative formula. It is also easy to check that $A^{\mathsf{Kr}} \leftrightarrow_{\mathbf{ŁL_c}} A$. Therefore, by Theorem 5.3, it is a negative translation for* $\mathbf{ŁL_i}$. *This translation is inspired by the negative translation behind Krivine's classical realizability interpretation [13, 14].*

6 Some De Morgan Laws for $\mathbf{ŁL_i}$

The proof of the homomorphism property for $A \otimes B$ (Theorem 4.14) we made essential use of Theoorem 4.13, a kind of De Morgan duality between $\multimap$ and $\otimes$. Let us conclude the paper with a list of $\mathbf{ŁL_i}$ De Morgan laws for formulas of the form $(A \circ B)^\perp$ with $\circ$ ranging over each of our connectives (primitive and derived).

Theorem 6.1 *The following "De Morgan dualities" hold in intuitionistic Łukasiewicz logic* $\mathbf{ŁL_i}$

$$\begin{array}{rcl} (A \otimes B)^\perp & \leftrightarrow & A \multimap B^\perp \\ (A \multimap B)^\perp & \leftrightarrow & A^{\perp\perp} \otimes B^\perp \\ (A \wedge B)^\perp & \leftrightarrow & A \Rightarrow B^\perp \\ (A \Rightarrow B)^\perp & \leftrightarrow & A^{\perp\perp} \wedge B^\perp \\ (A \wedge B)^\perp & \leftrightarrow & A^\perp \vee B^\perp \\ (A \vee B)^\perp & \leftrightarrow & A^\perp \wedge B^\perp \\ (A \downarrow B)^\perp & \leftrightarrow & A^\perp \Rightarrow B^{\perp\perp} \end{array}$$

Proof: The first equation $(A \otimes B)^\perp \leftrightarrow A \multimap B^\perp$ follows directly from currying and uncurrying. For the second equation we calculate as follows

$$\begin{array}{rcll} (A \multimap B)^\perp & \leftrightarrow & (A \multimap B)^{\perp\perp\perp} & \text{(easy)} \\ & \leftrightarrow & (A^{\perp\perp} \multimap B^{\perp\perp})^\perp & \text{(Theorem 4.11)} \\ & \leftrightarrow & (B^\perp \multimap A^\perp)^\perp & \text{(easy)} \\ & \leftrightarrow & A^{\perp\perp} \otimes B^\perp. & \text{(Theorem 4.13)} \end{array}$$

The third equation follows from Theorem 4.5 and [CWC]. The fourth equation can be derived as:

$$\begin{array}{rcll} (A \Rightarrow B)^\perp & \leftrightarrow & (A \multimap A \otimes B)^\perp & \text{(def} \Rightarrow\text{)} \\ & \leftrightarrow & A^{\perp\perp} \otimes (A \otimes B)^\perp & \text{(duality of} \multimap\text{)} \\ & \leftrightarrow & A^{\perp\perp} \otimes (B \multimap A^\perp) & \text{(duality of} \otimes\text{)} \\ & \leftrightarrow & A^{\perp\perp} \otimes (B \multimap A^{\perp\perp\perp}) & (A^\perp \leftrightarrow A^{\perp\perp\perp}) \\ & \leftrightarrow & A^{\perp\perp} \otimes (A^{\perp\perp} \multimap B^\perp) & \text{(easy)} \\ & \leftrightarrow & A^{\perp\perp} \wedge B^\perp. & \text{(def} \wedge\text{)} \end{array}$$

The fifth equation follows by:

$$\begin{array}{rcll} (A \wedge B)^\perp & \leftrightarrow & (A \otimes (A \multimap B))^\perp & \text{(easy)} \\ & \leftrightarrow & (A \multimap B)^{\perp\perp} \multimap A^\perp & \text{(easy)} \\ & \leftrightarrow & (A^{\perp\perp} \multimap B^{\perp\perp}) \multimap A^\perp & \text{(Theorems 4.11)} \\ & \leftrightarrow & (B^\perp \multimap A^\perp) \multimap A^\perp & \text{(easy)} \\ & \leftrightarrow & A^\perp \vee B^\perp. & \end{array}$$

For the sixth equation we proceed as follows:

$$\begin{array}{rcll} (B \vee A)^\perp & \leftrightarrow & ((A \multimap B) \multimap B)^\perp & \text{(def} \vee\text{)} \\ & \leftrightarrow & (A \multimap B)^{\perp\perp} \otimes B^\perp & \text{(duality of} \multimap\text{)} \\ & \leftrightarrow & (A^{\perp\perp} \otimes B^\perp)^\perp \otimes B^\perp & \text{(duality of} \multimap\text{)} \\ & \leftrightarrow & (B^\perp \multimap A^{\perp\perp\perp}) \otimes B^\perp & \text{(duality of} \otimes\text{)} \\ & \leftrightarrow & (B^\perp \multimap A^\perp) \otimes B^\perp & (A^\perp \leftrightarrow A^{\perp\perp\perp}) \\ & \leftrightarrow & B^\perp \wedge A^\perp. & \text{(def} \wedge\text{)} \end{array}$$

Finally, the last equation follows from Theorem 4.9 and the laws for ∧ and ∨. ■

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References

[1] Rob Arthan and Paulo Oliva. On pocrims and hoops. `http://arXiv.org/abs/1404.0816`, 2014. 9, 10

[2] W. J. Blok and I. M. A. Ferreirim. On the structure of hoops. *Algebra Universalis*, 43(2-3):233–257, 2000. 2

[3] Simone Bova and Franco Montagna. The consequence relation in the logic of commutative GBL-algebras is PSPACE-complete. *Theor. Comput. Sci.*, 410(12-13):1143–1158, March 2009. 10, 17

[4] Agata Ciabattoni and Duccio Luchi. Two connections between linear logic and Lukasiewicz logics. In *Proceedings of the 5th Kurt Gödel Colloquium on Computational Logic and Proof Theory*, pages 128–139, London, UK, 1997. Springer-Verlag. 2, 5

[5] G. Ferreira and P. Oliva. On the relation between various negative translations. *Logic, Construction, Computation, Ontos-Verlag Mathematical Logic Series*, 3:227–258, 2012. 7

[6] G. Gentzen. Über das Verhältnis zwischen intuitionistischer und klassischer Arithmetik, galley proof (received in 1933). *Mathematische Annalen*, 1933. 1, 9

[7] J.-Y. Girard. Linear logic. *Theoretical Computer Science*, 50(1):1–102, 1987. 2

[8] V. I. Glivenko. Sur quelques points de la logique de M. Brouwer. *Bulletin de la Société Mathématique de Belgique*, 15, 1929. 1, 9

[9] K. Gödel. Zur intuitionistischen Arithmetik und Zahlentheorie. *Ergebnisse eines Mathematischen Kolloquiums*, 4:34–38, 1933. 1, 7

[10] Petr Hájek. *Metamathematics of Fuzzy Logic.* Kluwer Academic Publishers, 1998. 1, 2, 6

[11] P. Jipsen and F. Montagna. On the structure of generalized BL-algebras. *Algebra Univers.*, 55(2-3):227–238, 2006. 2

[12] A. N. Kolmogorov. On the principle of the excluded middle (Russian). *Mat. Sb.*, 32:646–667, 1925. 1, 7

[13] J. Krivine. Dependent choice, 'quote' and the clock. *Th. Comp. Sc.*, 308:259–276, 2003. 15

[14] P Oliva and T. Streicher. On Krivine's realizability interpretation of classical second-order arithmetic. *Fundamenta Informaticae*, 84(2):207–220, 2008. 15

[15] A. S. Troelstra. *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*, volume 344 of *Lecture Notes in Mathematics*. Springer, Berlin, 1973. 6, 7, 13

[16] A. S. Troelstra and H. Schwichtenberg. *Basic Proof Theory.* Cambridge University Press, Cambridge (2nd edition), 2000. 1

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