(Dual) Hoops Have Unique Halving

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Abstract. Continuous logic extends the multi-valued Łukasiewicz logic by adding a halving operator on propositions. This extension is designed to give a more satisfactory model theory for continuous structures. The semantics of these logics can be given using specialisations of algebraic structures known as hoops. As part of an investigation into the metatheory of propositional continuous logic, we were indebted to Prover9 for finding a proof of an important algebraic law.

1 Introduction

In 1930, Łukasiewicz and Tarski [1] instigated the study of logics admitting models in which the truth values are real numbers drawn from some subset T of the interval $[0, 1]$. In these models, conjunction is capped addition[1]: $x \dot{+} y = \mathsf{inf}\{x+y, 1\}$. Boolean logic is the special case when $T = \{0, 1\}$. These Łukasiewicz logics have been widely studied, e.g., as instances of fuzzy logics [2]. More recently ben Yaacov has used them as a building block in what is called continuous logic [3]. Continuous logic unifies work of Henson and others [4] that aims to overcome shortfalls of classical first-order model theory when applied to continuous structures such as metric spaces and Banach spaces.

A difficulty with both the Łukasiewicz logics and continuous logic is that the known axiomatisations of their propositional fragments are quite hard to work with. Work on algebraic semantics for Łukasiewicz logic begun by Chang [5,6] has helped greatly with this. This paper reports on ongoing work to gain a better understanding of both the proof theory and the semantics of continuous logic that is benefitting from the use of automated theorem proving technhology to help find counterexamples and to derive algebraic properties.

Our work began with the observation that ben Yaacov's continuous logic, which we call $\mathbf{CL_c}$, is an extension of a very simple intuitionistic substructural logic $\mathbf{AL_i}$. In Section 2 of this paper we show how $\mathbf{CL_c}$ may be built up via a system of extensions of $\mathbf{AL_i}$. We also show how the Brouwer-Heyting intuitionistic propositional logic and Boolean logic fit into this picture. We describe a class of monoids called pocrims, that have been quite widely studied in connection with $\mathbf{AL_i}$ and sketch a proof of a theorem asserting soundness and completeness

[1] We here follow the convention of the literature on continuous logic in ordering the truth values by increasing logical strength so that 0 represents truth and 1 falsehood.

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ [\multimap\mathsf{I}] \qquad \frac{\Gamma \vdash A \quad \Delta \vdash A \multimap B}{\Gamma, \Delta \vdash B}\ [\multimap\mathsf{E}]$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ [\otimes\mathsf{I}] \qquad \frac{\Gamma \vdash A \otimes B \quad \Delta, A, B \vdash C}{\Gamma, \Delta \vdash C}\ [\otimes\mathsf{E}]$$

Fig. 1: Inference Rules

of the eight logics with respect to certain classes of pocrim. The sketch is easy to complete apart from one rather tricky lemma.

In Section 3, we discuss our use of Bill McCune's Mace4 and Prover9 to assist in these investigations, in particular to prove the lemma needed for the theorem of Section 2. Our application seems to be a "sweet spot" for this kind of technology: the automatic theorem prover found a proof of a difficult problem that can readily be translated into a human readable form.

2 The Logics and their Algebraic Semantics

We work in a language $\mathcal{L}_{\frac{1}{2}}$ whose atomic formulas are the propositional constants 0 (truth) and 1 (falsehood) and propositional variables drawn from the set $\mathsf{Var} = \{P, Q, \ldots\}$. If A and B are formulas of $\mathcal{L}_{\frac{1}{2}}$ then so are $A \otimes B$ (conjunction), $A \multimap B$ (implication) and $A/2$ (halving). We denote by $\mathcal{L}_{\mathbf{1}}$ the language without halving. We write $A^{\perp}$ as an abbreviation for $A \multimap 1$. The logics we consider have judgments of the form $\Gamma \vdash A$, where A is an $\mathcal{L}_{\frac{1}{2}}$-formula and Γ is a multiset of $\mathcal{L}_{\frac{1}{2}}$-formulas. We are interested in logics whose inference rules are the introduction and elimination rules for the two connnectives[2] shown in Figure 1.

The axiom schemata for our logics are selected from those shown in Figure 2. These are the axiom of assumption [ASM], ex-falso-quodlibet [EFQ], double negation elimination [DNE], commutative weak conjunction [CWC], commutative strong disjunction [CSD], the axiom of contraction [CON], and two axioms for the halving operator: one for a lower-bound [HLB] and one for an upper bound [HUB].

The definitions of our eight logics are discussed in the next few paragraphs and are summarised in Table 1. In all but $\mathbf{CL_i}$ and $\mathbf{CL_c}$, halving plays no rôle and the logical language may be taken to be the sublanguage $\mathcal{L}_{\mathbf{1}}$ in which halving does not feature.

Intuitionistic affine logic, $\mathbf{AL_i}$, has for its axiom schemata [ASM] and [EFQ]. All our other logics include $\mathbf{AL_i}$. Since the contexts Γ, Δ are multisets, an assumption in the rules of Figure 1 can be used at most once. $\mathbf{AL_i}$ serves as a prototype for substructural logics with this property. It corresponds under the

[2] Omitting disjunction from the logic greatly simplifies the algebraic semantics. While it may be unsatisfactory from the point of view of intuitionistic philosophy, disjunction defined using de Morgan's law is adequate for our purposes.

$$\frac{}{\Gamma, A \vdash A}\ [\mathsf{ASM}] \qquad \frac{}{\Gamma, 1 \vdash A}\ [\mathsf{EFQ}]$$

$$\frac{}{\Gamma, A^{\perp\perp} \vdash A}\ [\mathsf{DNE}] \qquad \frac{}{\Gamma, A \otimes (A \multimap B) \vdash B \otimes (B \multimap A)}\ [\mathsf{CWC}]$$

$$\frac{}{\Gamma, (A \multimap B) \multimap B \vdash (B \multimap A) \multimap A}\ [\mathsf{CSD}] \qquad \frac{}{\Gamma, A \vdash A \otimes A}\ [\mathsf{CON}]$$

$$\frac{}{\Gamma, A/2 \otimes A/2 \vdash A}\ [\mathsf{HLB}] \qquad \frac{}{\Gamma, A/2 \multimap A \vdash A/2}\ [\mathsf{HUB}]$$

Fig. 2: Axiom Schemata

Curry-Howard correspondence to a λ-calculus with pairing (i.e., λ-abstractions of the form $\lambda(x, y) \bullet t$, $\lambda((x, y), z) \bullet t$, $\lambda(x, (y, z)) \bullet t$ etc.) in which no variable is used twice.

Classical affine logic, $\mathbf{AL_c}$, extends $\mathbf{AL_i}$ with the axiom schema [DNE]. It can also be viewed as the extension of the so-called multiplicative fragment of Girard's linear logic [7] by allowing weakening and the axiom schema [EFQ].

Intuitonistic Łukasiewicz logic, $\mathbf{ŁL_i}$, extends $\mathbf{AL_i}$ with the axiom schema [CWC]. In $\mathbf{AL_i}$, $A \otimes (A \multimap B)$ can be viewed as a weak conjunction of A and B. In $\mathbf{ŁL_i}$, we have commutativity of this weak conjunction.

Classical Łukasiewicz logic, $\mathbf{ŁL_c}$, extends $\mathbf{AL_i}$ with the axiom schema [CSD]. In $\mathbf{AL_i}$, $(A \multimap B) \multimap B$ can be viewed as a form of disjunction, stronger than that defined by $(A^\perp \otimes B^\perp)^\perp$. In $\mathbf{AL_c}$ we have commutativity of this strong disjunction. This gives us the widely-studied multi-valued logic of Łukasiewicz.

Logic	Axioms	Models
$\mathbf{AL_i}$	[ASM] + [EFQ]	bounded pocrims
$\mathbf{AL_c}$	as $\mathbf{AL_i}$ + [DNE]	bounded involutive pocrims
$\mathbf{ŁL_i}$	as $\mathbf{AL_i}$ + [CWC]	bounded hoops
$\mathbf{ŁL_c}$	as $\mathbf{AL_i}$ + [CSD]	bounded Wajsberg hoops
IL	as $\mathbf{AL_i}$ + [CON]	bounded idempotent pocrims
BL	as **IL** + [CON]	bounded involutive idempotent pocrims
$\mathbf{CL_i}$	as $\mathbf{ŁL_i}$ + [HLB] + [HUB]	bounded coops
$\mathbf{CL_c}$	as $\mathbf{ŁL_c}$ + [HLB] + [HUB]	bounded involutive coops

Table 1: Different logics and their models

Intuitionistic propositional logic, **IL**, extends $\mathbf{AL_i}$ with the axiom schema [CON], which is equivalent to a contraction rule allowing $\Gamma, A \vdash B$ to be derived from $\Gamma, A, A \vdash B$. This gives us the conjunction-implication fragment of the well-known Brouwer-Heyting intuitionistic propositional logic.

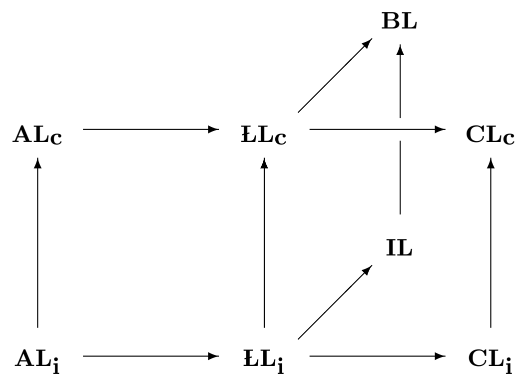

Fig. 3: Relationships between the Logics

Boolean logic, **BL**, extends **IL** with the axiom schema [DNE]. This is the familiar two-valued logic of truth tables.

Intuitionistic Continuous logic, **CL**$_\mathbf{i}$, allows the halving operator and extends **ŁL**$_\mathbf{i}$ with the axiom schemas [HLB] and [HUB], which effectively give lower and upper bounds on the logical strength of $A/2$. They imply the surprisingly strong condition that $A/2$ is equivalent to $A/2 \multimap A$. This is an intuitionistic version of the continuous logic of ben Yaacov [3].

Classical Continuous logic, **CL**$_\mathbf{c}$ extends **CL**$_\mathbf{i}$ with the axiom schema [DNE]. This gives ben Yaacov's continuous logic. The motivating example takes truth values to be real numbers between 0 and 1 with the logical operations defined to be certain continuous functions.

Our initial goal was to understand the relationships amongst **AL**$_\mathbf{i}$, **ŁL**$_\mathbf{c}$ and **CL**$_\mathbf{c}$. The other logics came into focus when we tried to decompose the somewhat intractable axiom [CSD] into a combination of [DNE] and an intuitionistic component. It can be shown that the eight logics are related as shown in Figure 3. In the figure, an arrow from T_1 to T_2 means that T_1 extends T_2, i.e., the set of provable sequents of T_2 contains that of T_1. In each square, the north-east logic is the least extension of the south-west logic that contains the other two. For human beings, at least, the proof of this fact is quite tricky for the **AL**$_\mathbf{i}$-**ŁL**$_\mathbf{c}$ square, see [2, chapters 2 and 3].

The routes in Figure 3 from **AL**$_\mathbf{i}$ to **IL** and **BL** have been quite extensively studied [8,9]. We are not aware of any work on **CL**$_\mathbf{i}$, but it is clearly a natural object of study in connection with ben Yaacov's continuous logic. It should be noted that **IL** and **CL**$_\mathbf{i}$ are incompatible: given [CON], $A/2$ and $A/2 \otimes A/2$ are equivalent, so that from [HLB] and [HUB] one finds that $A/2 \multimap A$ and $A/2$ are both provable; which proves A, for arbitrary formulas A.

We give an algebraic semantics to our logics using pocrims: partially ordered, commutative, residuated, integral monoids. A *pocrim*[3], is a structure for the signature $(0, +, \rightarrow; \leq)$ of type $(0, 2, 2; 2)$ satisfying the following laws:

[3] Strictly speaking, this is a *dual pocrim*, since we order it by increasing logical strength and write it additively, whereas in much of the literature the opposite order and multiplicative notation is used (so halves would be square roots). We follow the ordering convention of the continuous logic literature.

$$
\begin{array}{lr}
(x+y)+z = x+(y+z) & [\mathsf{m}_1] \\
x+y = y+x & [\mathsf{m}_2] \\
x+0 = x & [\mathsf{m}_3] \\
x \geq x & [\mathsf{o}_1] \\
\text{if } x \geq y \text{ and } y \geq z, \text{ then } x \geq z & [\mathsf{o}_2] \\
\text{if } x \geq y \text{ and } y \geq x, \text{ then } x = y & [\mathsf{o}_3] \\
\text{if } x \geq y, \text{ then } x+z \geq y+z & [\mathsf{o}_4] \\
x \geq 0 & [\mathsf{b}] \\
x+y \geq z \text{ iff } x \geq y \to z & [\mathsf{r}]
\end{array}
$$

Let $\mathbf{M} = (M, 0, +, \to; \leq)$ be a pocrim. The laws $[\mathsf{m}_i]$, $[\mathsf{o}_j]$ and $[\mathsf{b}]$ say that $(M, 0, +; \leq)$ is a partially ordered commutative monoid with the identity 0 as least element. Axiom $[\mathsf{r}]$, the *residuation property*, says that for any x and z the set $\{y \mid x+y \geq z\}$ is non-empty and has $x \to z$ as least element. $\mathbf{M}$ is said to be *bounded* if it has a necessarily unique *annihilator*, i.e., an element 1 such that for every x we have:

$$x + 1 = 1 \qquad [\mathsf{ann}]$$

Let us assume $\mathbf{M}$ is bounded. Then $0 \leq x \leq x+1 = 1$ for any x and $(M; \leq)$ is indeed a bounded ordered set. Let $\alpha : \mathsf{Var} \to M$ be an interpretation of logical variables as elements of M and extend α to a function $v_\alpha : \mathcal{L}_{\mathbf{1}} \to M$ by interpreting 0, 1, $\otimes$ and $\multimap$ as 0, 1, $+$ and $\to$ respectively. If $\Gamma = C_1, \ldots, C_n$, we say that α *satisfies* the sequent $\Gamma \vdash A$, iff $v_\alpha(C_1) + \ldots + v_\alpha(C_n) \geq v_\alpha(A)$. We say that $\Gamma \vdash A$ is *valid* in $\mathbf{M}$ iff it is satisfied by every assignment $\alpha : \mathsf{Var} \to M$. If $\mathcal{C}$ is a class of procrims, we say $\Gamma \vdash A$ is valid iff it is valid in every $\mathbf{M} \in \mathcal{C}$.

We will need some special classes of pocrim. Writing $\neg x$ as an abbreviation for $x \to 1$, we say a bounded pocrim is *involutive* if it satisfies $\neg\neg x = x$. We say a pocrim is *idempotent* if it is idempotent as a monoid, i.e., it satisfies $x + x = x$. A *hoop* is a pocrim that is *naturally ordered*, i.e., whenever $x \geq y$, there is z such that $x = y + z$. It is a nice exercise in the use of the residuation property to show that a pocrim is a hoop iff it satisfies the identity

$$x + (x \to y) = y + (y \to x) \qquad [\mathsf{cwc}]$$

A *Wajsberg hoop* is a hoop satisfying the identity

$$(x \to y) \to y = (y \to x) \to x \qquad [\mathsf{csd}]$$

See [8] for more information on hoops.

We adopt the term, *continuous hoop*, or *coop* for short, for a hoop where, for every x, there is a unique y, such that $y = y \to x$, in which case we write $y = x/2$. If $\mathbf{M}$ is a coop, we extend the function $v_\alpha : \mathcal{L}_{\mathbf{1}} \to M$ induced by an interpration $\alpha : \mathsf{Var} \to M$ to a function $v_\alpha : \mathcal{L}_{\frac{\mathbf{1}}{\mathbf{2}}} \to M$ by interpreting $A/2$ as $v_\alpha(A)/2$. The notions of validity and satisfaction extend to interpretations of $\mathcal{L}_{\frac{\mathbf{1}}{\mathbf{2}}}$ in a coop in the evident way.

We say that a logic L is sound for a class of pocrims $\mathcal{C}$ if every sequent that is provable in L is valid in $\mathcal{C}$. We say that L is complete for $\mathcal{C}$ if the converse holds. We then have:

Theorem 1 *Each of our logics is sound and complete for the class of pocrims indicated in Table 1.*

Proof: Apart from one detail, this is standard. Soundness is a routine exercise. For the completeness, one defines an equivalence relation $\simeq$ on formulas such that $A \simeq B$ holds iff both $A \vdash B$ and $B \vdash A$ are provable in the logic. One then shows that the set of equivalence classes becomes a pocrim, called the *term model* in the indicated class under operators $+$ and $\rightarrow$ induced on the equivalence classes by $\otimes$ and $\multimap$. As the only sentences valid in the term model are those provable in the logic, completeness follows. The difficult detail is showing that the term models for the continuous logics satisfy our definition of a coop: it is easy to see that for any $x = [A]$, one has that $y = [A/2]$ satisfies $y = y \rightarrow x$, but is this y unique? We shall answer this question in the next section.

3 Automated Proofs and Counterexamples

In our early attempts to understand the relationships represented in Figure 3, we spent some time attempting to devise finite pocrims with interesting properties. This can be a surprisingly difficult and error-prone task. Verifying associativity, in particular, is irksome. Having painstakingly accumulated a small stock of examples, a conversation with Alison Pease reminded us of the existence of Bill McCune's Mace4 tool [10] that automatically searches for finite counter-examples to conjectures in a finitely axiomatised first-order theory.

It was fascinating to see Mace4 recreate examples similar to those we had already constructed. The following input asks Mace4 to produce a counterexample to the conjecture that all bounded pocrims are hoops:

```
op(500, infix, "==>").
formulas(assumptions).
   (x + y) + z = x + (y + z).        % monoid law 1
   x + y = y + x.                    % monoid law 2
   x + 0 = x.                        % monoid law 3
   x >= x.                           % ordering law 1
   x >= y & y >= z -> x >= z.        % ordering law 2
   x >= y & y >= x -> x = y.         % ordering law 3
   x >= y -> x + z >= y + z.         % ordering law 4
   x >= 0.                           % boundedness law
   x + 1 = 1.                        % annihilator law
   x + y >= z <-> x >= y ==> z.      % residuation law
end_of_list.
formulas(goals).
   x + (x ==> y) = y + (y ==> x).    % can we derive cwc?
end_of_list.
```

Here we use '`==>`' and '`>=`' to represent '$\rightarrow$' and '$\leq$' in the pocrim and '`&`', '`->`' and '`<->`' are Mace4 syntax for conjunction, implication and bi-implication.

Given the above, Mace4 quickly prints out the diagram of a pocrim on the ordered set $0 < p < q < 1$ with $x + y = 1$ whenever $\{x, y\} \subseteq \{p, q, 1\}$, a counter-example which we had already come up with over the course of an afternoon. That led us to test Mace4 on yet other conjectures which we had already refuted with some small counter-examples. Mace4, again and again, came up with similar counter-models to the ones we had contrived.

Some weeks later we were faced with the question of whether the two axiom schemata [HLB] and [HUB] uniquely determine the halving operator over the logic $\mathbf{ŁL_i}$. That would give us a very nice intuitionistic counterpart ($\mathbf{CL_i}$) to continuous logic $\mathbf{CL_c}$. More precisely, we found ourselves trying to show that the following rule is derivable in the theory $\mathbf{ŁL_i}$:

$$\frac{A \multimap B \vdash A \qquad A, A \vdash B \qquad C, C \vdash B \qquad C \multimap B \vdash C}{A \vdash C}$$

Despite several attempts we failed to come up with a proof, and even started looking at the possibility that this was simply not true. That is when we thought of using Prover9 to look for a proof. We entered the laws for a hoop, which include the law [cwc], together with the assumptions $a \rightarrow b = a$ and $c \rightarrow b = c$. Our goal was to derive $a = c$.

```
op(500, infix, "==>").
formulas(assumptions).
   (x + y) + z = x + (y + z).       % monoid law 1
   x + y = y + x.                   % monoid law 2
   x + 0 = x.                       % monoid law 3
   x >= x.                          % ordering law 1
   x >= y & y >= z -> x >= z.       % ordering law 2
   x >= y & y >= x -> x = y.        % ordering law 3
   x >= y -> x + z >= y + z.        % ordering law 4
   x >= 0.                          % boundedness law
   x + y >= z <-> x >= y ==> z.     % residuation law
   x + (x ==> y) = y + (y ==> x).   % commut. of wk. conjunction
   a ==> b = a.                     % assumption 1
   c ==> b = c.                     % assumption 2
end_of_list.
formulas(goals).
   a = c.
end_of_list.
```

To our surprise Prover9 took just a few seconds to answer

```
<< output proof >>
...
THEOREM PROVED. Exiting with 1 proof.
```

Interestingly, the Prover9 proof, despite having around seventy (70) steps, could easily be translated into natural language. After doing such translation, and having understood the main steps in the proof we were quite perplexed to find the following intricate construction which we reproduce here. The following lemma singles out the nine main sub-lemmas which require non-trivial proofs, and which are then used to prove Theorem 3.

Lemma 2 *Let $\mathbf{M} = (M, 0, +, \to; \leq)$ be a hoop and let $a, b, c, x, y \in M$. If $a \to b = a$ and $c \to b = c$, then the following hold:*

(1) $b \geq a$ *and* $b \geq c$.

(2) $a + a = b$.

(3) $a \to (a \to c) = 0$.

(4) $(x \to y) + z \geq x \to (y + (y \to x) + z)$.

(5) $c \to (a + a + x) \geq c$.

(6) $c \to a \geq a \to c$.

(7) $c \to a = a \to c$.

(8) $c + (c \to a) + ((a \to c) \to a) = b$.

(9) $a + c = b$.

Proof: (1): we have $b \geq a \to b = a$ and similarly for $b \geq c$.
(2): by (1) we have $b \to a = 0$. Therefore

$$a + a = a + (a \to b) \stackrel{[\mathsf{cwc}]}{=} b + (b \to a) = b.$$

(3): by (1) we have $a = a \to b \geq a \to c$ and hence $0 \geq a \to (a \to c)$, which implies (3).
(4): by [cwc]

$$x + (x \to y) + z = y + (y \to x) + z$$

and hence (4) follows.
(5): since $c \to (b + x) \geq c \to b = c$, this follows from (2).
(6): by (4) we have $(c \to a) + a \geq c \to (a + a + (a \to c))$. By (5) we have $(c \to a) + a \geq c$ and hence (6).
(7): this follows by symmetry from (6).
(8): we have

$$\begin{aligned} c + (c \to a) + ((a \to c) \to a) &\stackrel{[\mathsf{cwc}]}{=} a + (a \to c) + ((a \to c) \to a) \\ &\stackrel{[\mathsf{cwc}]}{=} a + a + (a \to (a \to c)) \\ &\stackrel{(2)}{=} b + (a \to (a \to c)) \\ &\stackrel{(3)}{=} b. \end{aligned}$$

(9): we have

$$\begin{aligned} b &\overset{(8)}{=} c + (c \to a) + ((a \to c) \to a) \\ &\overset{(7)}{=} c + (a \to c) + ((a \to c) \to a) \\ &\overset{[\mathsf{cwc}]}{=} c + a + (a \to (a \to c)) \\ &\overset{(3)}{=} c + a. \end{aligned}$$

This completes the proof of the lemma. It is interesting to note the complexity of the proof in terms of uses of [cwc] (used 6 times!) and the important sub-lemma (2) (used twice) as depicted in the following outline proof tree.

$$\dfrac{\dfrac{(1)}{(3)} \qquad \dfrac{\dfrac{\dfrac{}{(4)}\,[\mathsf{cwc}] \quad \dfrac{\dfrac{(1)}{(2)}\,[\mathsf{cwc}]}{(5)}}{(6)}}{(7)} \qquad \dfrac{\dfrac{(1)}{(2)}\,[\mathsf{cwc}] \quad \dfrac{(1)}{(3)}}{(8)}\,2 \times [\mathsf{cwc}]}{(9)}\,[\mathsf{cwc}]$$

Finally, from (9) of Lemma 2 we have the theorem that, given b, the equation $a \to b = a$ uniquely determines a:

Theorem 3 *In any hoop the following holds: if $a \to b = a$ and $c \to b = c$ then $a = c$.*

Proof: By symmetry it is enough to show $c \geq a$. By Lemma 2 (9) we have $c \geq a \to b$ and hence $c \geq a$.

We already have the part of Theorem 1 that gives soundness and completeness of $\mathbf{ŁL_i}$ for bounded hoops. Theorem 3 now gives us that the continuous logic axioms [HLB] and [HUB] uniquely determine halving, even over $\mathbf{ŁL_i}$, and that is exactly what we need to complete the proof of Theorem 1:

4 Final Remarks

We should note that Prover9 has also found some other intricate (although already known) proofs in this area. For example, it can prove a lemma on pocrims implying that the axiom schemata [CWC] + [DNE] is equivalent to [CSD] over intuitionistic affine logic $\mathbf{AL_i}$. This implies the aforementioned result that in the $\mathbf{AL_i}$-$\mathbf{ŁL_c}$ square of Figure 3, the north-east logic $\mathbf{ŁL_c}$ is the least extension of the south-west logic $\mathbf{AL_i}$ that contains the other two logics $\mathbf{AL_c}$ and $\mathbf{ŁL_i}$.

We also stress that Prover9's proof of Theorem 3 was surprisingly concise iand easily human-readable. The only effort we put was in separating the more trivial facts from the ones which required several steps to be derived, and grouping these into a proof-like (tree-like) structure. That is what led us to the nine sub-lemmas of Lemma 2, and the proof structure depicted above.

Our work on the algebraic semantics of continuous logic is in its early days. It will be interesting to see where a combination of automated proof and more conventional mathematical methods will eventually lead us.

Clearly our application is one to which technology such as Mace4 and Prover9 is well suited. It is nonetheless a tribute to the memory of the late Bill McCune that the accessibility and ease of use of these tools have enabled us to get useful results with very little effort.

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