# On Bounded Functional Interpretations

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**Abstract**

Bounded functional interpretations are variants of functional interpretations where bounds (rather than precise witnesses) are extracted from proofs. These have been particularly useful in computationally interpreting non-computational principles such as weak König's lemma. This paper presents a family of bounded functional interpretations – in the form of a parametrised interpretation – of both intuitionistic logic and (a fragment of) intuitionistic linear logic. We show how three different instantiations of the parameters give rise to three recently developed bounded interpretations: the bounded functional interpretation, bounded modified realizability and confined modified realizability.

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## 1. Introduction

Functional interpretations are proof interpretations where formulas $A$ are viewed as *sets $S_A$ of functionals* of finite type

$$A \quad \mapsto \quad S_A :\equiv \{f \ : \ f \in S_A\}$$

with provable formulas being interpreted as non-empty sets. They are called "proof interpretations" because any given proof of $A$ can be effectively turned into a witness to the fact that $S_A$ is not empty. The sets $S_A$ are normally described by specifying the input-output behavior of $f$, i.e.

$$A \quad \mapsto \quad \{f \ : \ \forall x \, T_A(x; fx)\}.$$

As such, we can think of functional interpretations as mapping formulas $A$ to *finite type specifications $T_A(x; y)$*. A proof in this case gives a concrete "implementation" of a functional satisfying this specification. For instance, the statement that a given set $P$ is infinite $A \equiv \forall n \exists m \geq n P(m)$ is interpreted as the specification

$$T_A(n; m) \ :\equiv \ (m \geq n) \wedge P(m)$$

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so that any proof of $A$ gives rise to a program computing arbitrarily large numbers in the set $P$. The most well-known functional interpretations are Gödel's Dialectica interpretation [8], Kreisel's modified realizability [10] and the Diller-Nahm interpretation [2].

Bounded functional interpretations, on the other hand, can be thought of as modifications of the usual functional interpretations where bounds (rather than precise witnesses) are extracted from proofs. For instance, in the example given above, rather than specifying a function that produces arbitrarily large elements in the set $P$, a bounded functional interpretation of $A$ specifies a function that, for any given $n$, computes a *bound on the value* of an element $m \in P$ greater than $n$.

Bounded interpretations have been particularly useful for the computational interpretation of non-computational principles such as weak König's lemma. Intuitively, non-computable functions might still have computable bounds (cf. [5]). The main challenge in producing a bounded interpretation is to show how one can work with bounds in a modular way, producing bounds for conclusions given bounds for the premises. Moreover, one needs a notion of "bound" which behaves well in all finite types. This is normally taken to be Howard/Bezem's notion of strong majorizability [1, 9], which says that a functional $f$ majorizes $g$ (written $g \leq^* f$) if

$$\forall y \forall x \leq^* y (gx \leq^* fy \wedge fx \leq^* fy).$$

Once such notion is in place, we can say that a "bounding witness" to an existential statement $\exists x A(x)$ is a functional $b$ such that $\exists x \leq^* b\, A(x)$. Although this is less information than a precise witness, it might come to our benefit since we then only require bounding witnesses from our axioms.

In this paper we shall present a family of bounded functional interpretations (in the form of a parametrised interpretation) of both intuitionistic logic and intuitionistic linear logic. In the context of intuitionistic logic we show how the bounded functional interpretation [5], bounded modified realizability [3], and confined modified realizability [7] all arise as particular instances of our bounded interpretation. The purpose of the bounded interpretation of intuitionistic *linear logic* is twofold. First, the finer setting of linear logic allows us to better understand the role of each of the parameters introduced in the intuitionistic context. But also, we carry out the proof of soundness for the linear logic interpretation, since in there logical contractions happen in a very controlled and localised manner. We then show how the soundness for intuitionistic logic can be obtained via the standard embedding of $\mathsf{IL}$ into $\mathsf{ILL}$.

It should be noted that a similar approach has been taken in previous papers of the authors (cf. [6, 12, 13], in particular [11]) for the standard functional interpretations such as Dialectica and modified realizability. The challenge in here is to provide an analogous unification in the realm of bounded variants. The subtlety of these bounded interpretation will be evident in the few restrictions we must impose to the linear logic setting, hinting that the bounded interpretations are almost intrinsically a feature of intuitionistic logic.

The paper is structured as follows. In the following section we describe the formal systems used in the paper. Then, in Section 3 the parametrised bounded interpretation of intuitionistic logic is presented. In Section 4 we give three instances of the parametrised interpretation and show they correspond to the bounded functional interpretation, bounded modified realizability and the confined modified realizability. Finally, in Section 5 we analyse the parametrised interpretation via intuitionistic linear logic, showing that most of the required properties must deal with contrac-

$$\frac{}{A \vdash A}\ (\mathrm{id}) \qquad \frac{}{\Gamma, 0 \vdash A}$$

$$\frac{\Gamma \vdash A}{\pi\{\Gamma\} \vdash A}\ (\mathrm{per}) \qquad \frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\ (\mathrm{cut})$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ (\otimes \mathrm{L}) \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ (\otimes \mathrm{R})$$

$$\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\ (\multimap \mathrm{L}) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ (\multimap \mathrm{R})$$

$$\frac{\Gamma, A \vdash B}{\Gamma, A \,\&\, C \vdash B}\ (\&\mathrm{L}) \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \,\&\, B \vdash C}\ (\&\mathrm{L}) \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}\ (\&\mathrm{R})$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}\ (\oplus \mathrm{L}) \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}\ (\oplus \mathrm{R}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\ (\oplus \mathrm{R})$$

Table 1: Intuitionistic Linear Logic (connectives)

tion and weakening. Also, the soundness of the parametrised interpretation for intuitionistic logic $\mathsf{IL}^\omega$ is proved via the soundness of the parametrised interpretation for intuitionistic linear logic in Section 5.3.

## 2. Preliminaries

In the following we will work with two formal systems $\mathsf{IL}^\omega$ and $\mathsf{ILL}^\omega_r$. By $\mathsf{IL}^\omega$ we denote intuitionistic logic in all finite types, where the types are inductively define in the usual way: there is a base type and if $\rho$ and $\sigma$ are finite types then $\rho \to \sigma$ is a finite type. By $\mathsf{ILL}^\omega_r$ we refer to the subsystem of $\mathsf{ILL}^\omega$ (intuitionistic linear logic in all finite types, whose formulation is shown in Tables 1 and 2) with the following restrictions: the &R-rule and the $\oplus$L-rule are permitted only when the context $\Gamma$ is of the form $!\Delta$ and all the implications occurring in the consequent formula $C$ of the rule $\oplus$L are of the form $!A \multimap B$. If $\Delta$ is a sequence of formulas $A_1, \ldots, A_n$ then by $!\Delta$ we denote the sequence $!A_1, \ldots, !A_n$. The necessity of some technical restrictions is discussed in Remark 5.6. In particular, as observed by an anonymous referee, the restrictions above determine the failure of basic properties such as $!(A \multimapboth B) \vdash C(A) \multimapboth C(B)$. Note, however, that $\mathsf{ILL}^\omega_r$ is strong enough to capture intuitionistic logic $\mathsf{IL}^\omega$ into the linear context, as precised in the following proposition.

**Proposition 2.1.** *Consider Girard's translation of $\mathsf{IL}^\omega$ into $\mathsf{ILL}^\omega$ defined inductively as follows:*

$$\frac{\Gamma, A[t/x] \vdash B}{\Gamma, \forall x A \vdash B}\ (\forall \mathrm{L}) \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \forall x A}\ (\forall \mathrm{R})$$

$$\frac{\Gamma, A \vdash B}{\Gamma, \exists x A \vdash B}\ (\exists \mathrm{L}) \qquad \frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x A}\ (\exists \mathrm{R})$$

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ (\mathrm{con}) \qquad \frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ (\mathrm{wkn}) \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ (!\mathrm{L}) \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ (!\mathrm{R})$$

Table 2: Intuitionistic Linear Logic (quantifiers and modality)

$$\begin{array}{llll}
P^* & :\equiv P, \quad \textit{if } P \textit{ atomic, } P \not\equiv \bot & & \\
\bot^* & :\equiv 0 & (A \to B)^* & :\equiv !A^* \multimap B^* \\
(A \wedge B)^* & :\equiv A^* \,\&\, B^* & (\forall x A)^* & :\equiv \forall x A^* \\
(A \vee B)^* & :\equiv !A^* \oplus !B^* & (\exists x A)^* & :\equiv \exists x !A^*
\end{array}$$

*The translation is such that,* $\Gamma \vdash A$ *is derivable in* $\mathsf{IL}^\omega$ *if and only if* $!\Gamma^* \vdash A^*$ *is derivable in* $\mathsf{ILL}^\omega_r$*.*

**Proof**. It is already known that $\Gamma \vdash_{\mathsf{IL}^\omega} A$ if and only if $!\Gamma^* \vdash_{\mathsf{ILL}^\omega} A^*$. The direct implication with $\mathsf{ILL}^\omega$ replaced by $\mathsf{ILL}^\omega_r$ just requires our attention in the rules &R and $\oplus$L. We can easily check that these rules are only applied with contexts of the form $!\Gamma$ and since the consequents of $\oplus$L are of the form $C^*$, by induction on the complexity of $C$, it is immediate to check that all implications there are of the form $!C_1 \multimap C_2$. The inverse implication with $\mathsf{ILL}^\omega$ replaced by $\mathsf{ILL}^\omega_r$ is a particular case of the original result. □

We will also consider a translation that undoes the previous one:

**Definition 2.2.** *Let A be a linear logic formula in the image of the translation* $(\cdot)^*$*. The intuitionistic translation of A, denoted by* $A^i$*, is defined inductively by:*

$$\begin{array}{llll}
P^i & :\equiv P, \quad \textit{if } P \textit{ atomic, } P \not\equiv 0 & (!A)^i & :\equiv A^i \\
0^i & :\equiv \bot & (A \multimap B)^i & :\equiv A^i \to B^i \\
(A \,\&\, B)^i & :\equiv A^i \wedge B^i & (\forall x A)^i & :\equiv \forall x A^i \\
(A \oplus B)^i & :\equiv A^i \vee B^i & (\exists x A)^i & :\equiv \exists x A^i.
\end{array}$$

The forgetful translation $(\cdot)^i$ reverses the translation $(\cdot)^*$ in the following sense:

**Proposition 2.3.** $A \equiv (A^*)^i$*, where* $\equiv$ *denotes syntactic equality.*

**Proof**. The proof follows immediately by induction on the complexity of the formula $A$. □

## 3. Parametrised Bounded Interpretation of Intuitionistic Logic

In this section we present a parametrised functional interpretation of $\mathsf{IL}^\omega$ that intends to capture and generalise the constructions behind the known bounded interpretations. The parameters of the interpretation will be assumed to satisfy certain conditions in order to ensure the soundness of the interpretation. In Section 4 we show how some instantiations of those parameters gives rise to three different bounded functional interpretations.

### *3.1. Parametrised verifying system* $\mathsf{IL}^\omega_\in$

While in the system under interpretation $\mathsf{IL}^\omega$ we just consider one single type structure $\mathcal{X}^\omega$, in the verifying system (denoted by $\mathsf{IL}^\omega_\in$) we assume two finite type structures $\mathcal{X}^\omega$ and $\mathcal{S}^\omega$. Intuitively, objects in $\mathcal{S}^\omega$ can be thought as subsets of (or bounds on) objects in $\mathcal{X}^\omega$. As such, the term language of $\mathsf{IL}^\omega_\in$ has two sets of variables of type $\rho$, one ranging over $\mathcal{X}^\rho$ and the other ranging over $\mathcal{S}^\rho$. When we write $x^\rho, y^\rho, z^\rho, \ldots$ it will be clear from the context if the variables take values in $\mathcal{X}^\rho$ or $\mathcal{S}^\rho$. Similarly by $t^\rho, q^\rho, \ldots$ we denote terms in $\mathcal{X}^\rho$ or $\mathcal{S}^\rho$. When not clear from the context, we specify if the constants, the variables and more generally the terms of type $\rho$ are intended to be interpreted in one structure or another by shortly saying that they are in $\mathcal{X}^\rho$ or $\mathcal{S}^\rho$ (or with the same meaning, that they are of type $\rho$ in $\mathcal{X}^\omega$ or $\mathcal{S}^\omega$). The first letters of the alphabet $a, b, c, \ldots$ are usually reserved for (variables or terms in) the latter structure.

We assume that $\mathsf{IL}^\omega_\in$ has a constant in $\mathcal{X}^0$, has functionals $\mathsf{m}^{\rho\to\rho\to\rho}$, $\mathsf{n}^{\rho\to\rho}$, $\mathsf{u}^{\rho\to\rho\to\rho}$, $\mathsf{v}^{(\tau\to\rho)\to\tau\to\rho}$ (for each types $\rho$ and $\tau$) in $\mathcal{S}^\omega$ and relations $\mathsf{ne}_\rho$, $\in_\rho$ and $\subseteq_\rho$ being the first a unary relation in $\mathcal{S}^\rho$ and the second and the third binary relations infixing between a term of $\mathcal{X}^\rho$ and a term of $\mathcal{S}^\rho$ and between two terms of $\mathcal{S}^\rho$ respectively. For the sake of intuition, it will be useful to have the following reading of these functionals and relations

| Functionals | Intuition | Relations | Intuition |
|---|---|---|---|
| $\mathsf{n}^{\rho\to\rho}(a)$ | superset of $a$ | $\mathsf{ne}_\rho(a)$ | set $a$ is non-empty |
| $\mathsf{m}^{\rho\to\rho\to\rho}$ | pointwise maximum | $x \in_\rho a$ | $x$ belongs to $a$ |
| $\mathsf{u}^{\rho\to\rho\to\rho}$ | union of two sets | $a \subseteq_\rho b$ | $a$ is a subset of $b$ |
| $\mathsf{v}^{(\tau\to\rho)\to\tau\to\rho}$ | flattening of a set of sets | | |

although the axioms we shall add on the constants and relations will not enforce this intuitive meaning.

The terms of $\mathsf{IL}^\omega_\in$ consist of two families of terms $\mathsf{T}_{\mathcal{X}^\omega}$ and $\mathsf{T}_{\mathcal{S}^\omega}$. The terms in $\mathsf{T}_{\mathcal{X}^\omega}$ are: the constants (including the typed combinators $\Pi^{\sigma\to\tau\to\sigma}$ and $\Sigma^{(\rho\to\sigma\to\tau)\to(\rho\to\sigma)\to\rho\to\tau}$), the infinitely many variables of each type in $\mathcal{X}^\omega$ and if $t^{\sigma\to\tau}$ and $s^\sigma$ are terms then the application of $t$ to $s$, $t(s)$, is a term of type $\tau$. Sometimes we omit the brackets writing just $ts$. We assume that $A[\Pi xy/w] \leftrightarrow A[x/w]$ and $A[\Sigma xyz/w] \leftrightarrow A[xz(yz)/w]$. By *combinatorial completeness*, we know that we can associate to each term $t^\sigma$ and variable $x^\tau$ a term $\lambda x.t$ of type $\tau \to \sigma$ that satisfy $A[(\lambda x.t)(s)/w] \leftrightarrow A[t[s/x]/w]$. So, alternatively (when useful) we can use the usual $\lambda$-term notation. The terms in the structure $\mathcal{S}^\omega$, whose set we denote by $\mathsf{T}_{\mathcal{S}^\omega}$, are defined precisely in the same way starting with constants (including the typed combinators $\Pi$ and $\Sigma$) and variables in $\mathcal{S}^\omega$. The formulas of $\mathsf{IL}^\omega_\in$ are the atomic formulas $P(t_1, \ldots, t_n)$ with $t_i$ a term of $\mathsf{T}_{\mathcal{X}^\omega} \bigcup \mathsf{T}_{\mathcal{S}^\omega}$ ($\perp$ is an atomic formula) and if $A$ and $B$ are formulas then $A \wedge B$, $A \vee B$, $A \to B$, $\forall x A$ and $\exists x A$ are also formulas. The rules of $\mathsf{IL}^\omega_\in$ are the usual intuitionistic logic rules. In $\mathsf{IL}^\omega_\in$ we will use the following abbreviations

$$
\begin{array}{lcllcl}
\exists z \in a A(z) & \equiv & \exists z(z \in a \wedge A(z)) & \forall z \in a A(z) & \equiv & \forall z(z \in a \to A(z)) \\
\exists y \subseteq a A(y) & \equiv & \exists y(y \subseteq a \wedge A(y)) & \forall y \subseteq a A(y) & \equiv & \forall y(y \subseteq a \to A(y)) \\
\tilde{\exists} y A(y) & \equiv & \exists y(\mathsf{ne}(y) \wedge A(y)) & \tilde{\forall} y A(y) & \equiv & \forall y(\mathsf{ne}(y) \to A(y)).
\end{array}
$$

The abbreviations concerning tuples of variables are also the usual ones, e.g. for $\boldsymbol{y} \equiv y_0, \ldots, y_n$ and $\boldsymbol{a} \equiv a_0, \ldots, a_n$, $\forall \boldsymbol{y} \subseteq \boldsymbol{a} A(\boldsymbol{y})$ denotes $\forall y_0 \subseteq a_0 \ldots \forall y_n \subseteq a_n A(y_0, \ldots, y_n)$. Instead of writing $\mathsf{ne}(a)$, sometimes we say, with the same meaning, that the element $a$ in $\mathcal{S}^\omega$ is *non-empty*.

We will assume that $\mathsf{IL}^\omega_\in$ has the following properties, which allows for a sound interpretation of $\mathsf{IL}^\omega$ into $\mathsf{IL}^\omega_\in$. The first group of properties concerns the interpretation of the quantifiers, and the modularity of the interpretation, meaning that bounds should be composable:

**Properties of $x \in a$ and $\mathsf{ne}(a)$**

(A1) For every closed term $t$ in $\mathcal{X}^\rho$ there is a closed term $\tilde{t}$ in $\mathcal{S}^\rho$ such that $t \in \tilde{t}$

(A2) $f \in g \wedge x \in y \to fx \in gy$

(A3) $x \in a \to \mathsf{ne}(a)$

(A4) $\mathsf{ne}(\Pi) \wedge \mathsf{ne}(\Sigma)$

(A5) $\mathsf{ne}(f) \wedge \mathsf{ne}(x) \to \mathsf{ne}(fx)$.

The second block of properties stipulates that the functional $\mathsf{m}(a, b)$ is a kind of pointwise maximum between two bounds (sets). This operation must preserve both the membership relation and the subset relation. As we will see the functional $\mathsf{m}(f, g)$ is only required for the interpretation of disjunction, in order to deal with the implicit contraction that occurs in the rule $\oplus$L.

**Properties of $\mathsf{m}(a, b)$**

(B1) $A[\mathsf{m}(f, g)(x)/w] \leftrightarrow A[\mathsf{m}(fx, gx)/w]$

(B2) $\mathsf{ne}(g) \wedge y \subseteq f \to y \subseteq \mathsf{m}(f, g) \wedge y \subseteq \mathsf{m}(g, f)$

(B3) $\mathsf{ne}(g) \wedge x \in f \to x \in \mathsf{m}(f, g) \wedge x \in \mathsf{m}(g, f)$

(B4) $\mathsf{ne}(\mathsf{m})$.

Finally, the third block of properties deals with the non-linear part of the logic, namely, the rules for $!A$ (see proof of Theorem 5.5). Although in the soundness in intuitionistic logic these properties would be used in verifying several rules, in linear logic each constant deals precisely with a particular rule: (C) deals with the contraction rule, (D) is necessary for the !R-rule (promotion) and (E) is used in the !L-rule (dereliction). The weakening rule is dealt by the assumption that each type is inhabited.

**Properties of $a \subseteq b$, $\mathsf{u}$, $\mathsf{v}$ and $\mathsf{n}$**

(C) $\mathsf{ne}(y) \wedge z \subseteq x \to z \subseteq \mathsf{u}(x, y) \wedge z \subseteq \mathsf{u}(y, x)$

(D) $\mathsf{ne}(f) \wedge x \subseteq y \wedge z \subseteq f(x) \to z \subseteq \mathsf{v}(f, y)$

(E) $\mathsf{ne}(x) \to x \subseteq \mathsf{n}(x)$

(F) $\mathsf{ne}(\mathsf{n}) \wedge \mathsf{ne}(\mathsf{u}) \wedge \mathsf{ne}(\mathsf{v})$.

**Remark 3.1.** *Note that when unifying the more traditional interpretations, the ones that provide precise witnesses instead of bounds [6, 11], the terms* $\mathsf{u}$, $\mathsf{n}$ *and* $\mathsf{v}$ *may depend on the formula* $A$*, considering that they are bounds for* $\forall y \subseteq (\cdot)A(y)$*. Since we are going to deal with interpretations that disregard precise witnesses just caring for bounds we no longer need that dependency on* $A$*, being enough the properties over the relation* $\subseteq$ *presented above.*

*3.2. Parametrised bounded functional interpretation*

In the next definition, for every formula $A$ of $\mathsf{IL}^\omega$ we introduce its parametrised interpretation $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$. The interpretation is presented in a kind of (two-player one-move) sequential game notation, where the variables $\boldsymbol{x}$ are the *witnessing variables* and $\boldsymbol{y}$ the *challenge variables*. First Eloise makes a move $\boldsymbol{x}$, then Abelard plays $\boldsymbol{y}$. If $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ is true Eloise wins the game, whereas if it is false Abelard is the winner.

**Definition 3.2 (Parametrised bounded interpretation of $\mathsf{IL}^\omega$).** *The interpretation of atomic formulas are the atomic formulas themselves. We extend the interpretation to all formulas of* $\mathsf{IL}^\omega$ *as follows. Assuming we have already defined* $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ *and* $|B|^{\boldsymbol{v}}_{\boldsymbol{w}}$*, we define*

$$
\begin{array}{lcl}
|A \wedge B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \wedge |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
|A \vee B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} & :\equiv & \tilde{\forall} \boldsymbol{y}' \subseteq \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}'} \vee \tilde{\forall} \boldsymbol{w}' \subseteq \boldsymbol{w} |B|^{\boldsymbol{v}}_{\boldsymbol{w}'} \\
|A \to B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} & :\equiv & \tilde{\forall} \boldsymbol{y} \subseteq \boldsymbol{f}\boldsymbol{x}\boldsymbol{w} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to |B|^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{w}} \\
|\forall z A(z)|^{\boldsymbol{f}}_{\boldsymbol{y},a} & :\equiv & \forall z \in a \, |A(z)|^{\boldsymbol{f}a}_{\boldsymbol{y}} \\
|\exists z A(z)|^{\boldsymbol{x},a}_{\boldsymbol{b}} & :\equiv & \exists z \in a \, \tilde{\forall} \boldsymbol{y} \subseteq \boldsymbol{b} |A(z)|^{\boldsymbol{x}}_{\boldsymbol{y}}.
\end{array}
$$

As noticed by an anonymous referee, this parametrised bounded interpretation is a kind of "least common multiple" of the bounded modified realizability, bounded functional interpretation and confined modified realizability (see Section 4), having all the features that characterise these interpretations. This explains why in Section 4 we are able to unify these bounded interpretations via the paramerised interpretation above.

In that section, we consider some instantiations where the subset relation $x \subseteq y$ is independent of $y$. In such cases we assume that we systematically omit the variable $y$ from the interpretation. The same could apply to the relation $z \in a$.

Let us briefly comment on the game theoretic intuition behind the interpretation above. The game $A \wedge B$ simply consists of playing both games $A$ and $B$ simultaneously, where Eloise first makes a move in both games ($\boldsymbol{x}$ in game $A$ and $\boldsymbol{v}$ in game $B$), followed by Abelard's move in both games ($\boldsymbol{y}$ in game $A$ and $\boldsymbol{w}$ in game $B$). Eloise wins the combined game if she wins both individual games. In the case of the game $A \vee B$, again Eloise makes a move in both games, but now Abelard has the advantage of choosing a set of moves $\boldsymbol{y}$ for the game $A$, and a set of moves $\boldsymbol{w}$ for the games $B$. Abelard wins $A \vee B$ if he has a wining move for $A$ and a winning move for $B$ in the two sets he has chosen. Note that out of the sets Abelard chooses, only the "good" moves are considered, i.e. moves $\boldsymbol{a}$ which satisfy $\mathsf{ne}(\boldsymbol{a})$. Intuitively, this is required because Eloise's move might be a functional, which makes use of Abelard's move. With this restriction, Eloise can be sure that only good moves are played, and she can make use of this. The game $A \to B$ is a bit more subtle, as this consists of playing game $B$ relative to (the oracle) game $A$. The game essentially consists of four moves, which can be reduced to two by Skolemisation. More precisely, Abelard starts by playing $\boldsymbol{x}$ in game $A$. This is followed by Eloise's move $\boldsymbol{g_x}$ in game $B$, the index showing the dependency of Eloise's move on Abelard's first move. On the third round, Abelard play $\boldsymbol{w}$ in game $B$. Finally, Eloise plays a sets of moves $\boldsymbol{f_{x,w}}$ in the "oracle" game $A$. Eloise's goal is either to win game $B$, or come up with a set of moves $\boldsymbol{f_{x,w}}$ containing a move that beats Abelard's move $\boldsymbol{x}$ in $A$. The quantifier games $\exists z A(z)$ and $\forall z A(z)$ are generalisations of the connective games $A \vee B$ and $A \wedge B$. The main difference is that in $\exists z A(z)$ Eloise chooses a particular subsets $a$ of the possible games, and she must win one of the games she chose. Dually, in the game $\forall z A(z)$ Abelard first chooses a set of games $a$ he wants to play, then Eloise makes her uniform move $\boldsymbol{f}_a$, followed by Abelard's move $\boldsymbol{y}$. Again, by Skolemisation this can be reduced to two rounds. The interpretation above is sound, in the sense that if $A$ is provable in intuitionistic logic then Eloise has a winning move in the game $A$, as the following theorem states:

**Theorem 3.3 (Soundness).** *Let $A_0, \ldots, A_n, B$ be formulas of $\mathsf{IL}^\omega$, with $\boldsymbol{z}$ as the only free-variables. If*

$$A_0(\boldsymbol{z}), \ldots, A_n(\boldsymbol{z}) \vdash B(\boldsymbol{z})$$

*is provable in $\mathsf{IL}^\omega$ then there are non-empty closed terms $\boldsymbol{a}_0, \ldots, \boldsymbol{a}_n, \boldsymbol{b}$ in $\mathcal{S}^\omega$ such that*

$$\Delta, \tilde{\forall} \boldsymbol{y}_0 \subseteq \boldsymbol{a}_0 \boldsymbol{x}_0 \ldots \boldsymbol{x}_n \boldsymbol{w} \boldsymbol{a} |A_0(\boldsymbol{z})|^{\boldsymbol{x}_0}_{\boldsymbol{y}_0}, \ldots, \tilde{\forall} \boldsymbol{y}_n \subseteq \boldsymbol{a}_n \boldsymbol{x}_0 \ldots \boldsymbol{x}_n \boldsymbol{w} \boldsymbol{a} |A_n(\boldsymbol{z})|^{\boldsymbol{x}_n}_{\boldsymbol{y}_n} \vdash |B(\boldsymbol{z})|^{\boldsymbol{b} \boldsymbol{x}_0 \ldots \boldsymbol{x}_n \boldsymbol{a}}_{\boldsymbol{w}}$$

$$\text{with } \Delta :\equiv \mathsf{ne}(\boldsymbol{x}_0), \ldots, \mathsf{ne}(\boldsymbol{x}_n), \mathsf{ne}(\boldsymbol{w}), \boldsymbol{z} \in \boldsymbol{a}$$

*is provable in $\mathsf{IL}^\omega_\in$.*

**Proof**. We postpone the proof of this theorem to Section 5 where a more general proof in the context of intuitionistic linear logic is presented. □

## 4. Instantiations

In this section we show how the three bounded interpretations of $\mathsf{IL}^\omega$ (bounded functional interpretation [4, 5], bounded modified realizability [3], and confined modified realizability [7]) can be obtained by instantiating the parametrised bounded interpretation in a suitable way.

### *4.1. Bounded modified realizability*

Consider the general framework of [3] where the bounded modified realizability was introduced. More precisely, $\mathsf{T}^\omega$ is an extension of $\mathsf{IL}^\omega_\in$ with two binary relation symbols $=_0$ and $\leq_0$ (infixing between terms of type 0) and two constants, one of type 0 and another, denoted by $\mathtt{m}$, of type $0 \to 0 \to 0$. $\mathsf{T}^\omega$ has the usual axioms for equality on type 0, the usual axioms for the combinators and the four axioms below asserting the reflexivity and transitivity of $\leq_0$ and its interaction with $\mathtt{m}$:

(P1) $x \leq_0 x$

(P2) $x \leq_0 y \wedge y \leq_0 z \to x \leq_0 z$

(P3) $x \leq_0 \mathtt{m}(x, y) \wedge y \leq_0 \mathtt{m}(x, y)$

(P4) $x \leq_0 x' \wedge y \leq_0 y' \to \mathtt{m}(x, y) \leq_0 \mathtt{m}(x', y')$.

Following [3], define Bezem's majorizability relation $\leq^*_\rho$ inductively as:

$$
\begin{array}{lcl}
x \leq^*_0 y & :\equiv & x \leq_0 y \\
x \leq^*_{\rho\to\sigma} y & :\equiv & \forall u^\rho, v^\rho (u \leq^*_\rho v \to xu \leq^*_\sigma yv \wedge yu \leq^*_\sigma yv)
\end{array}
$$

and the functionals $\mathtt{m}_\rho$ of type $\rho \to \rho \to \rho$ as:

$$
\begin{array}{lcl}
\mathtt{m}_0(n, m) & :\equiv & \mathtt{m}(n, m) \\
\mathtt{m}_{\rho\to\sigma}(x, y) & :\equiv & \lambda u^\rho . \mathtt{m}_\sigma(xu, yu).
\end{array}
$$

**Lemma 4.1.** *$\leq^*_\rho$ and $\mathtt{m}_\rho$ satisfy:*

- $x \leq^* y \to y \leq^* y$
- $x \leq^* y \wedge y \leq^* z \to x \leq^* z$
- $x \leq^* x \wedge y \leq^* y \to x \leq^* \mathtt{m}(x, y) \wedge y \leq^* \mathtt{m}(x, y)$
- $\mathtt{m}_\rho \leq^* \mathtt{m}_\rho$, $\Pi \leq^* \Pi$ *and* $\Sigma \leq^* \Sigma$.

We also assume that $\mathsf{T}^\omega$ is such that for every closed term $t^\rho$ there is another closed term $q^\rho$ such that $\mathsf{T}^\omega \vdash t \leq^* q$. We instantiate the framework of the parametrised interpretation in the following way:

$$
\begin{array}{lcl}
\mathsf{m}(a, b) & :\equiv & \mathtt{m}(a, b) \\
\mathsf{ne}(a) & :\equiv & a \leq^* a \\
x \in a & :\equiv & x \leq^* a \\
a \subseteq b & :\equiv & true
\end{array}
$$

and $\mathsf{u}$, $\mathsf{n}$, $\mathsf{v}$ as non-empty arbitrary functionals. As $a \subseteq b$ is independent of $b$ we assume the variable $b$ will be systematically omitted. We are considering that the two structures in all finite types in $\mathsf{IL}^\omega_\in$ are in fact the same. From the above, very easily we can check that all the properties of $\mathsf{IL}^\omega_\in$ are satisfied. So this first instantiation is an example of a sound interpretation of $\mathsf{IL}^\omega$ into $\mathsf{IL}^\omega_\in$.

In the next result we prove that this first instantiation corresponds to bounded modified realizability. Remember that bounded modified realizability, introduced in [3], associates to each formula $A$ of $\mathsf{IL}^\omega$ a formula $A_{\mathsf{br}}(\boldsymbol{a})$ in the following way:

$$
P_{\mathsf{br}}() \quad :\equiv \quad P \qquad \text{(for } P \text{ atomic).}
$$

If we have already interpretations for $A$ and $B$ given by $A_{\mathsf{br}}(\boldsymbol{a})$ and $B_{\mathsf{br}}(\boldsymbol{b})$ respectively then, we define:

$$
\begin{array}{lcl}
(A \wedge B)_{\mathsf{br}}(\boldsymbol{a}, \boldsymbol{b}) & :\equiv & A_{\mathsf{br}}(\boldsymbol{a}) \wedge B_{\mathsf{br}}(\boldsymbol{b}) \\
(A \vee B)_{\mathsf{br}}(\boldsymbol{a}, \boldsymbol{b}) & :\equiv & A_{\mathsf{br}}(\boldsymbol{a}) \vee B_{\mathsf{br}}(\boldsymbol{b}) \\
(A \to B)_{\mathsf{br}}(\boldsymbol{f}) & :\equiv & \tilde{\forall}\boldsymbol{a}(A_{\mathsf{br}}(\boldsymbol{a}) \to B_{\mathsf{br}}(\boldsymbol{f}\boldsymbol{a})) \\
((\forall z A(z))_{\mathsf{br}}(\boldsymbol{f}) & :\equiv & \tilde{\forall} a \forall z \leq^* a (A(z))_{\mathsf{br}}(\boldsymbol{f} a) \\
(\exists z A(z))_{\mathsf{br}}(\boldsymbol{a}, b) & :\equiv & \exists z \leq^* b (A(z))_{\mathsf{br}}(\boldsymbol{a}).
\end{array}
$$

where $\tilde{\forall} a A(a)$ abbreviates $\forall a(a \leq^* a \to A(a))$ and $\forall z \leq^* a A(z)$ and $\exists z \leq^* a A(z)$ are equivalent to $\forall z(z \leq^* a \to A(z))$ and $\exists z(z \leq^* a \wedge A(z))$ respectively.

**Proposition 4.2.** $\vdash_{\mathsf{T}^\omega} A_{\mathsf{br}}(\boldsymbol{a}) \leftrightarrow \tilde{\forall}\boldsymbol{b}|A|^{\boldsymbol{a}}_{\boldsymbol{b}}$.

**Proof**. The proof is done by induction on the complexity of the formula $A$. When dealing with quantifications of the form $\tilde{\forall}$ (for instance in $\tilde{\forall} x(A \wedge B) \to \tilde{\forall} x A \wedge B$, with $x$ not free in $B$) we may use the fact that every type in $\mathcal{S}^\omega$ is inhabited at least by a non-empty element. For $A$ an atomic formula the result is trivial. The composite cases are treated below:

*Conjunction*.

$$
\begin{array}{lcl}
(A \wedge B)_{\mathsf{br}}(\boldsymbol{a}, \boldsymbol{b}) & \equiv & A_{\mathsf{br}}(\boldsymbol{a}) \wedge B_{\mathsf{br}}(\boldsymbol{b}) \\
& \overset{\text{I.H.}}{\leftrightarrow} & \tilde{\forall}\boldsymbol{u}|A|^{\boldsymbol{a}}_{\boldsymbol{u}} \wedge \tilde{\forall}\boldsymbol{v}|B|^{\boldsymbol{b}}_{\boldsymbol{v}} \\
& \leftrightarrow & \tilde{\forall}\boldsymbol{u}\tilde{\forall}\boldsymbol{v}(|A|^{\boldsymbol{a}}_{\boldsymbol{u}} \wedge |B|^{\boldsymbol{b}}_{\boldsymbol{v}}) \quad \equiv \quad \tilde{\forall}\boldsymbol{u}\tilde{\forall}\boldsymbol{v}|A \wedge B|^{\boldsymbol{a},\boldsymbol{b}}_{\boldsymbol{u},\boldsymbol{v}}.
\end{array}
$$

*Disjunction*.

$$
\begin{array}{lcl}
(A \vee B)_{\mathsf{br}}(\boldsymbol{a}, \boldsymbol{b}) & \equiv & A_{\mathsf{br}}(\boldsymbol{a}) \vee B_{\mathsf{br}}(\boldsymbol{b}) \\
& \overset{\text{I.H.}}{\leftrightarrow} & \tilde{\forall}\boldsymbol{u}|A|^{\boldsymbol{a}}_{\boldsymbol{u}} \vee \tilde{\forall}\boldsymbol{v}|B|^{\boldsymbol{b}}_{\boldsymbol{v}} \quad \leftrightarrow \quad |A \vee B|^{\boldsymbol{a},\boldsymbol{b}}.
\end{array}
$$

*Implication*.

$$
\begin{array}{lcl}
(A \to B)_{\mathsf{br}}(\boldsymbol{f}) & \equiv & \tilde{\forall}\boldsymbol{a}(A_{\mathsf{br}}(\boldsymbol{a}) \to B_{\mathsf{br}}(\boldsymbol{f}\boldsymbol{a})) \\
& \overset{\text{I.H.}}{\leftrightarrow} & \tilde{\forall}\boldsymbol{a}(\tilde{\forall}\boldsymbol{u}|A|^{\boldsymbol{a}}_{\boldsymbol{u}} \to \tilde{\forall}\boldsymbol{v}|B|^{\boldsymbol{f}\boldsymbol{a}}_{\boldsymbol{v}}) \\
& \leftrightarrow & \tilde{\forall}\boldsymbol{a}\tilde{\forall}\boldsymbol{v}(\tilde{\forall}\boldsymbol{u}|A|^{\boldsymbol{a}}_{\boldsymbol{u}} \to |B|^{\boldsymbol{f}\boldsymbol{a}}_{\boldsymbol{v}}) \quad \leftrightarrow \quad \tilde{\forall}\boldsymbol{a}\tilde{\forall}\boldsymbol{v}|A \to B|^{\boldsymbol{f}}_{\boldsymbol{a},\boldsymbol{v}}.
\end{array}
$$

*Universal quantification*.

$$
\begin{array}{lcl}
(\forall z A(z))_{\mathsf{br}}(\boldsymbol{f}) & \equiv & \tilde{\forall} a \forall z \leq^* a (A(z))_{\mathsf{br}}(\boldsymbol{f} a) \\
& \overset{\text{I.H.}}{\leftrightarrow} & \tilde{\forall} a \forall z \leq^* a \tilde{\forall}\boldsymbol{u}|A(z)|^{\boldsymbol{f} a}_{\boldsymbol{u}} \\
& \leftrightarrow & \tilde{\forall}\boldsymbol{u}\tilde{\forall} a \forall z \leq^* a|A(z)|^{\boldsymbol{f} a}_{\boldsymbol{u}} \quad \leftrightarrow \quad \tilde{\forall}\boldsymbol{u}\tilde{\forall} a|\forall z A(z)|^{\boldsymbol{f}}_{\boldsymbol{u},a}.
\end{array}
$$

*Existential quantification*.

$$
\begin{aligned}
(\exists z A(z))_{\mathsf{br}}(\boldsymbol{a}, b) \quad &\equiv \quad \exists z \leq^* b (A(z))_{\mathsf{br}}(\boldsymbol{a}) \\
&\overset{\text{I.H.}}{\leftrightarrow} \quad \exists z \leq^* b \tilde{\forall} \boldsymbol{u} |A(z)|^{\boldsymbol{a}}_{\boldsymbol{u}} \quad \leftrightarrow \quad |\exists z A(z)|^{\boldsymbol{a}, b}.
\end{aligned}
$$

That concludes the proof. □

The notation $\leq_0$ and $\mathtt{m}$ is inspired by the "less than or equal to" relation and the "maximum of two numbers" when we consider $\mathbb{N}$ as the base type. Note however that the previous way of obtaining a sound interpretation of $\mathsf{IL}^\omega$ into $\mathsf{IL}^\omega_\in$ is quite general. Any pair of a binary relation $\mathsf{r} \subseteq 0 \times 0$ on type 0 and a functional of type $\mathsf{f} \colon 0 \to 0 \to 0$ satisfying the four axioms (P1), ..., (P4) in the beginning of the section and extended to all finite types in the same inductive way:

$$
\begin{aligned}
x \,\mathsf{r}_0\, y \quad &:\equiv \quad x \,\mathsf{r}\, y \\
x \,\mathsf{r}_{\rho \to \sigma}\, y \quad &:\equiv \quad \forall u^\rho, v^\rho (u \,\mathsf{r}_\rho\, v \to xu \,\mathsf{r}_\sigma\, yv \wedge yu \,\mathsf{r}_\sigma\, yv)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathsf{f}_0(n, m) \quad &:\equiv \quad \mathsf{f}(n, m) \\
\mathsf{f}_{\rho \to \sigma}(x, y) \quad &:\equiv \quad \lambda u^\rho . \mathsf{f}_\sigma(xu, yu)
\end{aligned}
$$

gives rise, in a similar manner, to a sound interpretation. Moreover, the assumption that for every closed term $t$ there exists a closed term $q$ such that $\mathsf{T}^\omega \vdash t \,\mathsf{r}\, q$ is just needed if the theory has constants other than the one of type 0, $\Pi$, $\Sigma$ and $\mathsf{f}$. The result for $c^0$ follows immediately by the reflexivity on the base type and we also have $\Pi \,\mathsf{r}\, \Pi$, $\Sigma \,\mathsf{r}\, \Sigma$ and $\mathsf{f} \,\mathsf{r}\, \mathsf{f}$ for free by the defining axiomatization of these constants.

The concrete example presented next, in the framework of Heyting arithmetic in all finite types, illustrates precisely a situation in which the relation and the constant used in the instantiation of the parametrised interpretation are not "less than or equal to" nor the "maximum of two numbers".

#### *4.1.1. Bounding witnesses as multiples*

In the framework of the natural numbers, where we take the basic type 0 to be $\mathbb{N}$, we define the binary relation "$n$ divides $m$" by:

$$
n \mid m :\equiv \exists z (m = n \cdot z)
$$

where the symbol $\cdot$ (usually omitted) stands for multiplication of natural numbers in infixed notation and can be seen in all finite types as a constant of type $\mathbb{N} \to \mathbb{N} \to \mathbb{N}$.

The following properties, that correspond to the previous (P1), (P2), (P3) and (P4) when considering the relation "divides" and the constant "multiplication", can trivially be checked:

$$
\begin{aligned}
&x \mid x \\
&x \mid y \wedge y \mid z \to x \mid z \\
&x \mid xy \wedge y \mid xy \\
&x \mid x' \wedge y \mid y' \to xy \mid x'y'.
\end{aligned}
$$

So, from what we pointed before, extending the relation $x \,|\, y$ and the function $n \cdot m$ to all finite types in the standard hereditary way [1]:

$$\begin{array}{lcl} x \,|_0\, y & :\equiv & x \,|\, y \\ x \,|_{\rho\to\sigma}\, y & :\equiv & \forall u^\rho, v^\rho (u \,|_\rho\, v \to xu \,|_\sigma\, yv \wedge yu \,|_\sigma\, yv) \end{array}$$

and

$$\begin{array}{lcl} n \cdot_0 m & :\equiv & nm \\ x \cdot_{\rho\to\sigma} y & :\equiv & \lambda u^\rho.(xu \cdot_\sigma yu), \end{array}$$

and proving that the constants successor $S$ and recursor $R$ verify $S \,|\, t$ and $R \,|\, q$ for certain closed terms $t$ and $q$, automatically we get a sound interpretation via the instantiation:

$$\begin{array}{lcl} \mathsf{m}(x, y) & :\equiv & x \cdot y \\ \mathsf{ne}(a) & :\equiv & a \,|\, a \\ x \in a & :\equiv & x \,|\, a \\ a \subseteq b & :\equiv & \mathit{true} \\ \mathsf{u}, \mathsf{n}, \mathsf{v} & :\equiv & \text{arbitrary non-empty functionals.} \end{array}$$

Note that instead of upper bounds for the precise witnesses, in this interpretation the "bounding witnesses" are multiples of the precise witnesses. It just remains to prove that successor and recursor can be "bounded" under the relation $x \,|\, y$. Although 0, with $0^{\mathbb{N}} :\equiv 0$ and $0^{\rho\to\sigma} :\equiv \lambda u^\rho.0^\sigma$, is always a bound, the existence of non-trivial multiples for successor and recursor can be obtained adapting Bezem's argument in [1] when dealing with strong majorizability, to the case of divisibility in all finite types. First we establish some notation and auxiliary results. Given $\alpha$ a functional of type $\mathbb{N} \to \sigma$, we denote by $\alpha^*$ the functional of the same type defined by:

$$\alpha^*(n) :\equiv \begin{cases} 0^\sigma & \text{if} \quad n = 0 \\ \alpha(n)\alpha(n-1)\cdots\alpha(0) & \text{if} \quad n \neq 0, \end{cases}$$

where $\alpha(n)\alpha(n-1)\cdots\alpha(0)$ is an abbreviation for several applications of multiplication.

**Lemma 4.3.** *Being $\alpha$ and $\beta$ functionals of type $\mathbb{N} \to \sigma$, we have the following:*

(*a*) $\forall n(\alpha(n) \,|\, \beta(n)) \Rightarrow \alpha \,|\, \beta^*$

(*b*) $\forall n(\alpha(n) \,|\, \beta(n)) \Rightarrow \alpha^* \,|\, \beta^*$.

**Proof**. (*a*) Without loss of generality assume that $\sigma$ is the type $\sigma_1 \to \cdots \to \sigma_k \to \mathbb{N}$. Take $n \,|\, n'$, $h_1 \,|\, h'_1, \ldots, h_k \,|\, h'_k$. We want to prove that

$$\alpha n h_1 \ldots h_k \,|\, \beta^* n' h'_1 \ldots h'_k \text{ and } \beta^* n h_1 \ldots h_k \,|\, \beta^* n' h'_1 \ldots h'_k.$$

[1]This was first suggested by Ulrich Kohlenbach (personal communication).

Note that we are using the following fact: $t \,|_\rho\, q \leftrightarrow \forall \boldsymbol{u} \forall \boldsymbol{v} (\boldsymbol{u} \,|\, \boldsymbol{v} \to t\boldsymbol{u} \,|_0\, q\boldsymbol{v} \wedge q\boldsymbol{u} \,|_0\, q\boldsymbol{v})$, easily derived by unfolding the recursive definition of $|_\rho$.

If $n' = 0$, both assertions are trivial because, by the definition of $\beta^*$, we have $\beta^* n' h'_1 \ldots h'_k = 0^{\mathbb{N}}$. If $n' \neq 0$, and since $n \,|\, n'$, we know that $n \leq n'$ and $n \neq 0$. By hypothesis $\alpha n \,|\, \beta n$ and since $h_i \,|\, h'_i$, we have $\alpha n h_1 \ldots h_k \,|\, \beta n h'_1 \ldots h'_k$. Applying (P2) and (P3) several times, we can deduce that

$$\beta n h'_1 \ldots h'_k \,|\, (\beta n' h'_1 \ldots h'_k) \cdots (\beta n h'_1 \ldots h'_k) \cdots (\beta 0 h'_1 \ldots h'_k).$$

Therefore, by transitivity we get

$$\alpha n h_1 \ldots h_k \,|\, (\beta n' h'_1 \ldots h'_k) \cdots (\beta n h'_1 \ldots h'_k) \cdots (\beta 0 h'_1 \ldots h'_k).$$

Since, by definition, $\beta^* n' h'_1 \ldots h'_k = (\beta n' h'_1 \ldots h'_k) \cdots (\beta n h'_1 \ldots h'_k) \cdots (\beta 0 h'_1 \ldots h'_k)$, we proved the first assertion. The second assertion follows in a similar way from the observation that, by definition, $\beta^* n h'_1 \ldots h'_k = (\beta n h'_1 \ldots h'_k) \cdots (\beta 0 h'_1 \ldots h'_k)$.

($b$) Following the proof of ($a$) we just have to replace the assertion $\alpha n h_1 \ldots h_k \,|\, \beta^* n' h'_1 \ldots h'_k$ by a proof of $\alpha^* n h_1 \ldots h_k \,|\, \beta^* n' h'_1 \ldots h'_k$. The case $n' = 0$, as we saw before, is trivial. For $n' \neq 0$, again using the fact that $n \leq n'$ and $n \neq 0$, we have

$$\alpha^* n h_1 \ldots h_k = (\alpha n h_1 \ldots h_k) \cdots (\alpha 0 h_1 \ldots h_k) \text{ and}$$

$$\beta^* n' h'_1 \ldots h'_k = (\beta n' h'_1 \ldots h'_k) \cdots (\beta n h'_1 \ldots h'_k) \cdots (\beta 0 h'_1 \ldots h'_k).$$

From our hypothesis, we know that for all $0 \leq i \leq n$, $\alpha i h_1 \ldots h_k \,|\, \beta i h'_1 \ldots h'_k$. Therefore, applying (P4) several times we get

$$(\alpha n h_1 \ldots h_k) \cdots (\alpha 0 h_1 \ldots h_k) \,|\, (\beta n h'_1 \ldots h'_k) \cdots (\beta 0 h'_1 \ldots h'_k).$$

Finally, by (P2) and (P3), we conclude that $\alpha^* n h_1 \ldots h_k \,|\, \beta^* n' h'_1 \ldots h'_k$. □

We are now able to present non-trivial "bounds" for the constants successor and recursor.

**Proposition 4.4.** *Following the notation above, we have:*

($a$) $S \,|\, S^*$

($b$) $R \,|\, \lambda f, g.(Rfg)^*$.

**Proof**. ($a$) By reflexivity of the relation $n \,|\, m$ on the base type, we know that for every integer $n$, $Sn \,|\, Sn$. Thus, by Lemma 4.3 ($a$), we conclude that $S \,|\, S^*$.

($b$) First we prove, by induction on $n$, that

$$(+)\ \forall n, f, g (f \,|\, f' \wedge g \,|\, g' \to Rfgn \,|\, Rf'g'n).$$

The case $n = 0$ follows immediately from $g \,|\, g'$. For the induction step, assuming the result valid for $n$, we have to prove that

$$\forall f, g (f \,|\, f' \wedge g \,|\, g' \to Rfg(n+1) \,|\, Rf'g'(n+1)),$$

which, by the way the recursor is defined, has conclusion $fn(Rfgn) \,|\, f'n(Rf'g'n)$. This conclusion can be obtained by applying the induction hypothesis and the fact that $f \,|\, f'$ and $n \,|\, n$. From the assertion (+), we deduce, using Lemma 4.3 that

$$\forall f, g(f \,|\, f' \wedge g \,|\, g' \to Rfg \,|\, (Rf'g')^* \wedge (Rfg)^* \,|\, (Rf'g')^*).$$

Therefore, $R \,|\, \lambda f, g.(Rfg)^*$. □

Instead of considering the integers as our base type, we can also work with a general commutative monoid $(\mathcal{A}, \cdot)$. In this context, the binary relation defined by:

$$x \,|\, y :\equiv \exists z \in \mathcal{A}\, (y = x \cdot z)$$

still satisfies the four properties (P1), ..., (P4). Therefore, extending $x \,|\, y$ and $x \cdot y$ to all finite types in the usual hereditary way, we get a sound interpretation via the instantiation already presented. This allow us to work with, for instance, the framework of the $m \times n$ matrices in $\mathbb{N}$ with the sum operation, or bounded semilattices. In such cases, one would have to verify whether the usual operators in each particular framework could be majorized in the general setting.

*4.2. Bounded functional interpretation*

Consider now the framework of [4], which we also denote by $\mathsf{T}^\omega$, that is pretty similar to the previous one except for the use of the intensional relation $\trianglelefteq_\rho$, for each type $\rho$. The relation $\trianglelefteq_\rho$ is defined inductively by two axioms and a rule:

M1: $x \trianglelefteq_0 y \leftrightarrow x \leq_0 y$

M2: $x \trianglelefteq_{\rho\to\sigma} y \to \forall u \trianglelefteq_\rho v (xu \trianglelefteq_\sigma yv \wedge yu \trianglelefteq_\sigma yv)$

$$\frac{A_b \wedge u \trianglelefteq v \to su \trianglelefteq tv \wedge tu \trianglelefteq tv}{A_b \to s \trianglelefteq t}$$

where $\forall u \trianglelefteq vA$ and $\exists u \trianglelefteq vA$ are equivalent to $\forall u(u \trianglelefteq v \to A)$ and $\exists u(u \trianglelefteq v \wedge A)$ respectively and are called bounded quantifiers, $A_b$ is a bounded formula (i.e. all quantifiers are bounded) and $u$ and $v$ are variables that do not occur free in the conclusion of the rule. We also assume that $\mathsf{T}^\omega$ is such that for every closed term $t^\rho$ there is another closed term $q^\rho$ such that $\mathsf{T}^\omega \vdash t \trianglelefteq q$.

**Lemma 4.5.** *From [4], we already know that:*

- $x \trianglelefteq y \to y \trianglelefteq y$
- $x \trianglelefteq y \wedge y \trianglelefteq z \to x \trianglelefteq z$
- $x \trianglelefteq x \wedge y \trianglelefteq y \to x \trianglelefteq \mathtt{m}(x, y) \wedge y \trianglelefteq \mathtt{m}(x, y)$
- $\mathtt{m}_\rho \trianglelefteq \mathtt{m}_\rho$, $\Pi \trianglelefteq \Pi$ *and* $\Sigma \trianglelefteq \Sigma$.

Our second instantiation of the parametrised functional interpretation is the following one:

$$\begin{array}{lllllll} \mathsf{ne}(a) & :\equiv & a \trianglelefteq a & \quad \mathsf{u} & :\equiv & \mathsf{m} & :\equiv \ \mathtt{m} \\ x \in a & :\equiv & x \trianglelefteq a & \quad \mathsf{n} & :\equiv & \lambda x.x & \\ a \subseteq b & :\equiv & a \trianglelefteq b & \quad \mathsf{v} & :\equiv & \lambda x, y.xy. & \end{array}$$

We are again considering that the two structures in all finite types in $\mathsf{IL}^{\omega}_{\in}$ are the same. From the above properties proved in [4], we can see very easily that all the conditions imposed over the parameters of our bounded functional interpretation are verified, thus we obtain our second sound interpretation of $\mathsf{IL}^{\omega}$ into $\mathsf{IL}^{\omega}_{\in}$.

In order to prove that this interpretation corresponds exactly to the well-known bounded functional interpretation (cf. [4]) we start by remembering its assignment of formulas. The bounded functional interpretation associates to each formula $A$ of $\mathsf{IL}^{\omega}$ a formula $A_{\mathsf{b}}(\boldsymbol{a}; \boldsymbol{b})$ in the following way:

$$P_{\mathsf{b}}(;) \quad :\equiv \quad P \qquad \text{(for } P \text{ atomic).}$$

If we have already interpretations for $A$ and $B$ given by $A_{\mathsf{b}}(\boldsymbol{b}; \boldsymbol{c})$ and $B_{\mathsf{b}}(\boldsymbol{d}; \boldsymbol{e})$ respectively then, we define:

$$\begin{array}{lll} (A \wedge B)_{\mathsf{b}}(\boldsymbol{b}, \boldsymbol{d}; \boldsymbol{c}, \boldsymbol{e}) & :\equiv & A_{\mathsf{b}}(\boldsymbol{b}; \boldsymbol{c}) \wedge B_{\mathsf{b}}(\boldsymbol{d}; \boldsymbol{e}) \\ (A \vee B)_{\mathsf{b}}(\boldsymbol{b}, \boldsymbol{d}; \boldsymbol{c}, \boldsymbol{e}) & :\equiv & \tilde{\forall} \boldsymbol{c}' \trianglelefteq \boldsymbol{c} A_{\mathsf{b}}(\boldsymbol{b}; \boldsymbol{c}') \vee \tilde{\forall} \boldsymbol{e}' \trianglelefteq \boldsymbol{e} B_{\mathsf{b}}(\boldsymbol{d}; \boldsymbol{e}') \\ (A \to B)_{\mathsf{b}}(\boldsymbol{f}, \boldsymbol{g}; \boldsymbol{b}, \boldsymbol{e}) & :\equiv & \tilde{\forall} \boldsymbol{c} \trianglelefteq \boldsymbol{f}\boldsymbol{b}\boldsymbol{e} A_{\mathsf{b}}(\boldsymbol{b}; \boldsymbol{c}) \to B_{\mathsf{b}}(\boldsymbol{g}\boldsymbol{b}; \boldsymbol{e}) \\ (\forall z A)_{\mathsf{b}}(\boldsymbol{f}; \boldsymbol{c}, a) & :\equiv & \forall z \trianglelefteq a A_{\mathsf{b}}(\boldsymbol{f} a; \boldsymbol{c}) \\ (\exists z A)_{\mathsf{b}}(\boldsymbol{b}, a; \boldsymbol{c}) & :\equiv & \exists z \trianglelefteq a \tilde{\forall} \boldsymbol{c}' \trianglelefteq \boldsymbol{c} A_{\mathsf{b}}(\boldsymbol{b}; \boldsymbol{c}'). \end{array}$$

where $\tilde{\forall} a A$ abbreviates $\forall a (a \trianglelefteq a \to A)$.

**Proposition 4.6.** $\vdash_{\mathsf{T}^{\omega}} A_{\mathsf{b}}(\boldsymbol{a}; \boldsymbol{b}) \leftrightarrow |A|^{\boldsymbol{a}}_{\boldsymbol{b}}$.

**Proof**. The proof is done by induction on the logic structure of $A$, being all the cases completely straightforward. □

A remark similar to the one concerning the generality of the instantiation presented in the previous section, can also be made concerning this second instantiation. Starting with a binary relation of type 0 and a constant of type $0 \to 0 \to 0$ that satisfy the axioms (P1), …, (P4) and defining their extensions to all finite types in the same hereditary way, the corresponding instantiation gives rise to a sound interpretation.

As noticed by an anonymous referee, the result still holds when, instead of the axiom and the rule in M2, we have the axiom with $\to$ replaced by $\leftrightarrow$. In this case, instead of the intensional $\trianglelefteq_{\rho}$ we would produce Bezem's majorizability relation $\leq^{*}_{\rho}$ already described. The reason why in the bounded functional interpretation one works with the rule instead of the corresponding axiom is because the majorizability relation occurs in the interpreted system and the axiom can not be interpreted. In our case we are free to choose the rule or the axiom because the majorizability relation only occurs in the verifying system.

*4.3. Confined modified realizability*

To present our third instantiation, consider a theory that incorporates the framework where confined modified realizability was introduced (cf. [7]). In particular, we have a constant $c^0$ of base type, relation symbols $=_0$, $\leq_0$ defined as in the first example and functionals mi and ma of type $0 \to (0 \to 0)$ satisfying:

$$x \leq_0 \mathrm{ma}(x, y) \wedge y \leq_0 \mathrm{ma}(x, y)$$
$$x \leq_0 x' \wedge y \leq_0 y' \to \mathrm{ma}(x, y) \leq_0 \mathrm{ma}(x', y')$$
$$\mathrm{mi}(x, y) \leq_0 x \wedge \mathrm{mi}(x, y) \leq_0 y$$
$$x \leq_0 x' \wedge y \leq_0 y' \to \mathrm{mi}(x, y) \leq_0 \mathrm{mi}(x', y').$$

In this environment we can define, by induction on the types, the functionals $\mathrm{mi}_\rho$ and $\mathrm{ma}_\rho$ of type $\rho \to (\rho \to \rho)$ as follows:

$$\begin{array}{lllllll} \mathrm{mi}_0(n, m) & :\equiv & \mathrm{mi}(n, m) & \qquad & \mathrm{ma}_0(n, m) & :\equiv & \mathrm{ma}(n, m) \\ \mathrm{mi}_{\rho\to\sigma}(x, y) & :\equiv & \lambda u^\rho.\mathrm{mi}_\sigma(xu, yu) & & \mathrm{ma}_{\rho\to\sigma}(x, y) & :\equiv & \lambda u^\rho.\mathrm{ma}_\sigma(xu, yu). \end{array}$$

Consider now a theory $\mathsf{T}^\omega_{\in^*}$ with a richer language (intended to extend $\mathsf{IL}^\omega_\in$). In addition to the language in all finite types already described, it has for each type $\rho$ a second type, we also call $\rho$, of functionals $[\cdot;\cdot]^\rho$ of elements of the first type $\rho$. We define the terms in the second structure by:

$$\Pi \equiv [\Pi; \Pi]$$

$$\Sigma \equiv [\Sigma; \Sigma]$$

$$[s; t]^{\rho\to\sigma}[x; y]^\rho \equiv [sx; ty]^\sigma.$$

From the above we can derive $\lambda[x; y]^\rho.[t; q]^\sigma \equiv [\lambda x.t; \lambda y.q]^{\rho\to\sigma}$.

The quantifications over the second structure have the form $\forall[x; y]$ and $\exists[x; y]$ and we can define, by induction on the types, the relation $\in^*_\rho$ between a term of type $\rho$ of the first structure and a term of type $\rho$ of the second by:

$$\begin{array}{lll} x \in^*_0 [a; b] & :\equiv & a \leq_0 x \wedge x \leq_0 b \\ x \in^*_{\rho\to\sigma} [a; b] & :\equiv & \forall[c; d]^\rho \forall y^\rho (y \in^*_\rho [c; d] \to xy \in^*_\sigma [ac; bd] \wedge ay \in^*_\sigma [ac; bd] \wedge by \in^*_\sigma [ac; bd]). \end{array}$$

Consider that $\mathsf{T}^\omega_{\in^*}$ is a confined theory, i.e. for every constant $c^\rho$ of the first structure there is a closed term $[t; q]^\rho$ of the second such that $\mathsf{T}^\omega_{\in^*} \vdash c \in^* [t; q]$. $\mathsf{T}^\omega_{\in^*}$ can be seen as $\mathsf{IL}^\omega_\in$ when we define $\mathsf{ne}$, $\in$, $\subseteq$ and $\mathsf{m}$ as

$$\begin{array}{lll} \mathsf{ne}([x; y]) & :\equiv & x \in^* [x; y] \wedge y \in^* [x; y] \\ z \in [x; y] & :\equiv & z \in^* [x; y] \\ [x; y] \subseteq [w; v] & :\equiv & \mathit{true} \\ \mathsf{m} & :\equiv & [\mathrm{mi}; \mathrm{ma}] \end{array}$$

and $\mathsf{u}$, $\mathsf{n}$, $\mathsf{v}$ as non-empty arbitrary functionals.

**Lemma 4.7.** *The following is provable in* $\mathsf{T}^{\omega}_{\in^*}$

- $x \in^* [a;b] \to (a \in^* [a;b] \wedge b \in^* [a;b])$
- $\Pi \in^* [\Pi;\Pi] \wedge \Sigma \in^* [\Sigma;\Sigma]$
- $x \in^* [a;b] \wedge a \in^* [c;d] \wedge b \in^* [c;d] \to x \in^* [c;d]$
- $a, b \in^* [a;b] \wedge c, d \in^* [c;d] \to a, b, c, d \in^* [\mathrm{mi}(a,c); \mathrm{ma}(b,d)]$
- $\mathrm{mi} \in^* [\mathrm{mi};\mathrm{ma}] \wedge \mathrm{ma} \in^* [\mathrm{mi};\mathrm{ma}]$.

*Moreover, for every closed term* $t^{\rho}$ *of the first structure there is a closed term* $[q;r]^{\rho}$ *of the second such that* $\mathsf{T}^{\omega}_{\in^*} \vdash t \in^* [q;r]$.

**Proof**. All assertions follow immediately from the work done in [7]. The ones not explicitly there can immediately be derived from the corresponding properties concerning $\subseteq$ and using the fact (also proved in [7]) that $[x;y] \subseteq [a;b] \leftrightarrow x \in^* [a;b] \wedge y \in^* [a;b]$. By $\subseteq$ in this proof we refer to the relation introduced in [7] and not to the parametrised relation with the same symbol used throughout this paper. $\square$

Thus, it can easily be checked that all the properties of $\mathsf{IL}^{\omega}_{\in}$ are satisfied and we have a third example of a bounded functional interpretation of $\mathsf{IL}^{\omega}$ into $\mathsf{IL}^{\omega}_{\in}$. Next we will see that this instantiation corresponds to confined modified realizability. The confined modified realizability assigns to each formula $A$ of $\mathsf{IL}^{\omega}$ a formula $A_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}]$ according to the following clauses:

$$P_{\mathsf{cr}}[;] \quad :\equiv \quad P \qquad \text{(for } P \text{ atomic).}$$

If we have already interpretations for $A$ and $B$ given by $A_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}]$ and $B_{\mathsf{cr}}[\boldsymbol{c};\boldsymbol{d}]$ respectively then, we define:

$$\begin{array}{lcl}
(A \wedge B)_{\mathsf{cr}}[\boldsymbol{a},\boldsymbol{c};\boldsymbol{b},\boldsymbol{d}] & :\equiv & A_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}] \wedge B_{\mathsf{cr}}[\boldsymbol{c};\boldsymbol{d}] \\
(A \vee B)_{\mathsf{cr}}[\boldsymbol{a},\boldsymbol{c};\boldsymbol{b},\boldsymbol{d}] & :\equiv & A_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}] \vee B_{\mathsf{cr}}[\boldsymbol{c};\boldsymbol{d}] \\
(A \to B)_{\mathsf{cr}}[\boldsymbol{f};\boldsymbol{g}] & :\equiv & \tilde{\forall}[\boldsymbol{a};\boldsymbol{b}](A_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}] \to B_{\mathsf{cr}}[\boldsymbol{f}\boldsymbol{a};\boldsymbol{g}\boldsymbol{b}]) \\
(\forall z A)_{\mathsf{cr}}[\boldsymbol{f};\boldsymbol{g}] & :\equiv & \tilde{\forall}[a;b]\forall z \in^* [a;b] A_{\mathsf{cr}}[\boldsymbol{f}a;\boldsymbol{g}b] \\
(\exists z A)_{\mathsf{cr}}[\boldsymbol{a},c;\boldsymbol{b},d] & :\equiv & \exists z \in^* [c;d] A_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}]
\end{array}$$

where $\tilde{\forall}[a;b]A$ abbreviates $\forall[a;b](a \in^* [a;b] \wedge b \in^* [a;b] \to A)$ and $\forall z \in^* [a;b]A$ and $\exists z \in^* [a;b]A$ are equivalent to $\forall z(z \in^* [a;b] \to A)$ and $\exists z(z \in^* [a;b] \wedge A)$ respectively. The tuples $[\boldsymbol{a};\boldsymbol{b}]$ should be seen as $[a_1;b_1], \ldots, [a_n;b_n]$.

**Proposition 4.8.** $\vdash_{\mathsf{T}^{\omega}_{\in^*}} A_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}] \leftrightarrow \tilde{\forall}[\boldsymbol{u};\boldsymbol{v}]|A|^{[\boldsymbol{a};\boldsymbol{b}]}_{[\boldsymbol{u};\boldsymbol{v}]}$.

**Proof**. By induction on the complexity of the formula $A$. Since the result is immediate for atomic formulas we study below some of the other cases, assuming the result valid for $A$ and $B$:

*Implication*.

$$
\begin{aligned}
(A \to B)_{\mathsf{cr}}[\boldsymbol{f};\boldsymbol{g}] \quad &\equiv \quad \tilde{\forall}[\boldsymbol{a};\boldsymbol{b}](A_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}] \to B_{\mathsf{cr}}[\boldsymbol{f}\boldsymbol{a};\boldsymbol{g}\boldsymbol{b}]) \\
&\overset{\text{I.H.}}{\leftrightarrow} \quad \tilde{\forall}[\boldsymbol{a};\boldsymbol{b}](\tilde{\forall}[\boldsymbol{u};\boldsymbol{v}]|A|^{[\boldsymbol{a};\boldsymbol{b}]}_{[\boldsymbol{u};\boldsymbol{v}]} \to \tilde{\forall}[\boldsymbol{r};\boldsymbol{s}]|B|^{[\boldsymbol{f}\boldsymbol{a};\boldsymbol{g}\boldsymbol{b}]}_{[\boldsymbol{r};\boldsymbol{s}]}) \\
&\leftrightarrow \quad \tilde{\forall}[\boldsymbol{a};\boldsymbol{b}]\tilde{\forall}[\boldsymbol{r};\boldsymbol{s}](\tilde{\forall}[\boldsymbol{u};\boldsymbol{v}]|A|^{[\boldsymbol{a};\boldsymbol{b}]}_{[\boldsymbol{u};\boldsymbol{v}]} \to |B|^{[\boldsymbol{f}\boldsymbol{a};\boldsymbol{g}\boldsymbol{b}]}_{[\boldsymbol{r};\boldsymbol{s}]}) \\
&\leftrightarrow \quad \tilde{\forall}[\boldsymbol{a};\boldsymbol{b}]\tilde{\forall}[\boldsymbol{r};\boldsymbol{s}]|A \to B|^{[\boldsymbol{f};\boldsymbol{g}]}_{[\boldsymbol{a};\boldsymbol{b}],[\boldsymbol{r};\boldsymbol{s}]}.
\end{aligned}
$$

*Existential quantification.*

$$
\begin{aligned}
(\exists z A(z))_{\mathsf{cr}}[\boldsymbol{a},c;\boldsymbol{b},d] \quad &\equiv \quad \exists z \in^* [c;d](A(z))_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}] \\
&\overset{\text{I.H.}}{\leftrightarrow} \quad \exists z \in^* [c;d]\tilde{\forall}[\boldsymbol{u};\boldsymbol{v}]|A(z)|^{[\boldsymbol{a};\boldsymbol{b}]}_{[\boldsymbol{u};\boldsymbol{v}]} \quad \leftrightarrow \quad |\exists z A(z)|^{[\boldsymbol{a};\boldsymbol{b}],[c;d]}.
\end{aligned}
$$

The other cases are treated similarly. □

The instantiation presented in this section also works with relations and constants hereditarily derived (in the same way) from any binary relation of type 0 and any two constants of type $0 \to 0 \to 0$ verifying the initial conditions.

## 5. Parametrised Bounded Interpretation of Linear Logic

In the previous section we presented an unified view of the bounded interpretations through a parametrised interpretation of $\mathsf{IL}^\omega$. All the study was carried on in intuitionistic logic. In this section we try to capture the notion of bounded interpretation in the linear logic context. Our goal is to show how the finer setting of linear logic clarifies the role played by each of the parameters introduced. The move to a more refined calculus is by no means a novel approach. In [6] and [14] the same strategy was used to get a better understanding of the differences and similarities of intuitionistic interpretations such as Gödel's Dialectica interpretation, Kreisel's modified realizability and the Diller-Nahm interpretation.

### *5.1. Parametrised verifying system* $\mathsf{ILL}^\omega_\in$

Since our goal is to obtain a parametrised bounded functional interpretation of $\mathsf{ILL}^\omega_r$ we are going to use a structure similar to the one previously presented, but this time in linear logic: in the system under interpretation $\mathsf{ILL}^\omega_r$ we assume a single structure $\mathcal{X}^\omega$ in all finite types, whereas in the verifying system (denoted by $\mathsf{ILL}^\omega_\in$) we have two structures $\mathcal{X}^\omega$ and $\mathcal{S}^\omega$ as described in the beginning of Section 3.

Once more, we assume that $\mathsf{ILL}^\omega_\in$ has a constant in $\mathcal{X}^0$, functionals $\mathsf{m}^{\rho\to\rho\to\rho}$, $\mathsf{n}^{\rho\to\rho}$, $\mathsf{u}^{\rho\to\rho\to\rho}$, $\mathsf{v}^{(\tau\to\rho)\to\tau\to\rho}$ (for each types $\rho$ and $\tau$) in $\mathcal{S}^\omega$ and relations $\mathsf{ne}_\rho$, $\in_\rho$ and $\subseteq_\rho$ with the arity and interaction on the structures already described. The terms in each structure $\mathcal{X}^\omega$ and $\mathcal{S}^\omega$ obey the usual construction we previously mentioned. Remember that the rules in $\mathsf{ILL}^\omega_r$ have some restrictions but $\mathsf{ILL}^\omega_\in$ has the usual intuitionistic linear logic rules. Also, in $\mathsf{ILL}^\omega_\in$ we will use the following abbreviations

$$
\begin{array}{llllll}
\exists z \in a A(z) & \equiv & \exists z(!(z \in a) \otimes A(z)) & \forall z \in a A(z) & \equiv & \forall z(!(z \in a) \multimap A(z)) \\
\exists y \subseteq a A(y) & \equiv & \exists y(!(y \subseteq a) \otimes A(y)) & \forall y \subseteq a A(y) & \equiv & \forall y(!(y \subseteq a) \multimap A(y)) \\
\tilde{\forall} y A(y) & \equiv & \forall y(!\mathsf{ne}(y) \multimap A(y)) & \tilde{\exists} y A(y) & \equiv & \exists y(!\mathsf{ne}(y) \otimes A(y)).
\end{array}
$$

Instead of writing $!\mathsf{ne}(a)$, sometimes we say, with the same meaning, that the element $a$ in $\mathcal{S}^\omega$ is *non-empty*. Having in view the construction of a sound interpretation, we assume that $\mathsf{ILL}^\omega_\in$ has a linear logic version of the properties described in Section 3.1, namely:

**Properties of $x \in a$ and $\mathsf{ne}(a)$**

(A1) For every closed term $t$ in $\mathcal{X}^\rho$ there is a closed term $\tilde{t}$ in $\mathcal{S}^\rho$ such that $!(t \in \tilde{t})$

(A2) $!(f \in g) \otimes !(x \in y) \multimap !(fx \in gy)$

(A3) $!(x \in a) \multimap !(\mathsf{ne}(a))$

(A4) $!\mathsf{ne}(\Pi) \otimes !\mathsf{ne}(\Sigma)$

(A5) $!\mathsf{ne}(f) \otimes !\mathsf{ne}(x) \multimap !\mathsf{ne}(fx)$

**Properties of $\mathsf{m}(a, b)$**

(B1) $!A[\mathsf{m}(f, g)x/w] \;\circ\!\!-\!\!\circ\; !A[\mathsf{m}(fx, gx)/w]$

(B2) $!\mathsf{ne}(g) \otimes !(y \subseteq f) \multimap !(y \subseteq \mathsf{m}(f, g)) \otimes !(y \subseteq \mathsf{m}(g, f))$

(B3) $!\mathsf{ne}(g) \otimes !(x \in f) \multimap !(x \in \mathsf{m}(f, g)) \otimes !(x \in \mathsf{m}(g, f))$

(B4) $\mathsf{ne}(\mathsf{m})$.

**Properties of $a \subseteq b$, $\mathsf{u}$, $\mathsf{v}$ and $\mathsf{n}$**

(C) $!\mathsf{ne}(y) \otimes !(z \subseteq x) \multimap !(z \subseteq \mathsf{u}(x, y)) \otimes !(z \subseteq \mathsf{u}(y, x))$

(D) $!\mathsf{ne}(f) \otimes !(x \subseteq y) \otimes !(z \subseteq f(x)) \multimap !(z \subseteq \mathsf{v}(f, y))$

(E) $!\mathsf{ne}(x) \multimap !(x \subseteq \mathsf{n}(x))$

(F) $!\mathsf{ne}(\mathsf{n}) \otimes !\mathsf{ne}(\mathsf{u}) \otimes !\mathsf{ne}(\mathsf{v})$.

Note that (A1) together with the fact that there is a constant of type 0 in $\mathcal{X}^\omega$ ensure that any type in $\mathcal{X}^\omega$ and $\mathcal{S}^\omega$ is inhabited at least by a closed term. Moreover, by (A3), we know that any type in $\mathcal{S}^\omega$ has a non-empty closed term.

Instead of considering $\mathsf{u}$, $\mathsf{v}$ and $\mathsf{n}$ as functionals in our theory, we could have adopted a more general approach. The interpretation would still be sound if we had simply required that for given terms $t$, $q$ and $s$ in $\mathcal{S}^\omega$, there were terms $\mathsf{u}(t, q)$, $\mathsf{n}(t)$ and $\mathsf{v}(s, q)$ in $\mathcal{S}^\omega$ satisfying the universal closure of conditions ($C$), ($D$), ($E$) and ($F$). Note that in this way the construction of the new terms does not need to be uniform in $t$, $q$ and $s$. Identical observation can be done concerning the properties in the intuitionistic setting presented before. We opted in both cases to present a simpler (not so general) approach since all our three examples fit in this simpler version.

**Lemma 5.1.** *We have the following results:*

($a$) $!\mathsf{ne}(\lambda x.x)$

(b) $!\mathsf{ne}(t) \otimes !\mathsf{ne}(s) \multimap\ !\mathsf{ne}(E[t,s])$, *where E is a term constructed from t, s and the combinators* $\Pi$ *and* $\Sigma$ *via application.*

(c) $!\mathsf{ne}(\lambda y.t[y]) \multimap\ !\mathsf{ne}(\lambda x, y.t[y])$.

**Proof.** (a) Since $\lambda x.x \equiv \Sigma\Pi\Pi$, by Properties (A4) and (A5) we know that $\lambda x.x$ is non-empty. Let us prove (b) by induction on the complexity of the term $E$. If $E$ is the term $t$ or $s$, it is non-empty by hypothesis. If $E$ is the term $\Pi$ or $\Sigma$, applying Property (A4) we also know it is non-empty. Suppose now that $E[t,s]$ is of the form $E'[t,s](E''[t,s])$. Immediately by induction hypothesis and Property (A5) we have that $E$ is non-empty. (c) follows from (b) noticing that $\lambda x, y.t[y] \equiv \Pi(\lambda y.t[y])$. □

Note that in assertion (b) we can have an arbitrary number of non-empty terms in the premise and assertion (c) is still valid if instead of single variables we allow tuples of variables. We illustrate the latter with a tuple of two variables.

$$\begin{aligned} \lambda x_1, x_2, y.t[y] &\equiv \lambda x_1.(\Pi(\lambda y.t[y])) \\ &\equiv \Sigma(\lambda x_1.\Pi)(\lambda x_1.(\lambda y.t[y])) \\ &\equiv \Sigma(\Pi\Pi)(\Pi(\lambda y.t[y])). \end{aligned}$$

**Lemma 5.2.** *If* $t[x]$ *is a term in* $\mathcal{X}^\omega$ *then there is a closed term* $\tilde{t}$ *in* $\mathcal{S}^\omega$ *such that* $!(\lambda x.t[x] \in \tilde{t})$. *Moreover, if* $!(x \in a)$ *then* $!(t[x] \in \tilde{t}(a))$.

**Proof.** From $t[x]$ a term in $\mathcal{X}^\omega$, we can construct the closed term $\lambda x.t[x]$. By (A1), there exists a closed term $\tilde{t}$ such that $!(\lambda x.t[x] \in \tilde{t})$. Assuming $!(x \in a)$ it follows by (A2) that $!(t[x] \in \tilde{t}(a))$. □

### *5.2. Parametrised bounded functional interpretation*

In order to distinguish from the parametrised bounded interpretation in $\mathsf{IL}^\omega$, in the next definition we use $[A]^{\boldsymbol{x}}_{\boldsymbol{y}}$ instead of $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ to denote the interpretation of the intuitionistic linear formula $A$.

**Definition 5.3 (Parametrised Bounded Functional Interpretation of $\mathsf{ILL}^\omega_r$).** *The interpretation of atomic formulas are the atomic formulas themselves. We extend the interpretation to all formulas of* $\mathsf{ILL}^\omega_r$ *as follows. Assuming we have already defined* $[A]^{\boldsymbol{x}}_{\boldsymbol{y}}$ *and* $[B]^{\boldsymbol{v}}_{\boldsymbol{w}}$, *we define*

$$\begin{aligned} [A \multimap B]^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} &:\equiv [A]^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{x}\boldsymbol{w}} \multimap [B]^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{w}} \\ [A \otimes B]^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} &:\equiv [A]^{\boldsymbol{x}}_{\boldsymbol{y}} \otimes [B]^{\boldsymbol{v}}_{\boldsymbol{w}} \\ [A \,\&\, B]^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} &:\equiv [A]^{\boldsymbol{x}}_{\boldsymbol{y}} \,\&\, [B]^{\boldsymbol{v}}_{\boldsymbol{w}} \\ [A \oplus B]^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} &:\equiv [A]^{\boldsymbol{x}}_{\boldsymbol{y}} \oplus [B]^{\boldsymbol{v}}_{\boldsymbol{w}} \\ [\exists z A(z)]^{\boldsymbol{x},a}_{\boldsymbol{y}} &:\equiv \exists z \in a\, [A(z)]^{\boldsymbol{x}}_{\boldsymbol{y}} \\ [\forall z A(z)]^{\boldsymbol{f}}_{\boldsymbol{y},a} &:\equiv \forall z \in a\, [A(z)]^{\boldsymbol{f}a}_{\boldsymbol{y}} \\ [!A]^{\boldsymbol{x}}_{\boldsymbol{a}} &:\equiv !\tilde{\forall} \boldsymbol{y} \subseteq \boldsymbol{a} [A]^{\boldsymbol{x}}_{\boldsymbol{y}}. \end{aligned}$$

Note how the parameter $\boldsymbol{y} \subseteq \boldsymbol{a}$ is only used for the interpretation of $!A$, and, as such, it captures precisely the interpretation of contraction and weakening. In particular, for the bounded functional interpretation of *pure* multiplicative linear logic (with quantifiers) this parameter is not necessary.

**Lemma 5.4 (Monotonicity).** *Assume that all the implications in $A$ are of the form $!B \multimap C$. Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{y}$ be non-empty. $\mathsf{ILL}^{\omega}_{\in}$ proves both $[A]^{\boldsymbol{a}}_{\boldsymbol{y}} \multimap [A]^{\mathsf{m}(\boldsymbol{a},\boldsymbol{b})}_{\boldsymbol{y}}$ and $[A]^{\boldsymbol{b}}_{\boldsymbol{y}} \multimap [A]^{\mathsf{m}(\boldsymbol{a},\boldsymbol{b})}_{\boldsymbol{y}}$.*

**Proof.** We present a proof of the first assertion. The second is entirely similar. The proof is done by induction on the complexity of the formula $A$. When $A$ is of the form $B \otimes C$, $B \,\&\, C$ and $B \oplus C$ the result follows trivially from the induction hypothesis and the inference rules of $\mathsf{ILL}^{\omega}_{\in}$.

$A \equiv \exists z B(z)$. We want to prove that $[\exists z B(z)]^{\boldsymbol{x},a}_{\boldsymbol{y}} \multimap [\exists z B(z)]^{\mathsf{m}(\boldsymbol{x},\boldsymbol{x}'),\mathsf{m}(a,a')}_{\boldsymbol{y}}$, knowing that $\boldsymbol{x}, a, \boldsymbol{x}', a'$ and $\boldsymbol{y}$ are non-empty. But

$$\begin{array}{rcl}
[\exists z B(z)]^{\boldsymbol{x},a}_{\boldsymbol{y}} & :\equiv & \exists z \in a [B(z)]^{\boldsymbol{x}}_{\boldsymbol{y}} \overset{\text{I.H.}}{\multimap} \exists z \in a [B(z)]^{\mathsf{m}(\boldsymbol{x},\boldsymbol{x}')}_{\boldsymbol{y}} \\
& \equiv & \exists z (!(z \in a) \otimes [B(z)]^{\mathsf{m}(\boldsymbol{x},\boldsymbol{x}')}_{\boldsymbol{y}}) \\
& \overset{\text{(B3)}}{\multimap} & \exists z (!(z \in \mathsf{m}(a,a')) \otimes [B(z)]^{\mathsf{m}(\boldsymbol{x},\boldsymbol{x}')}_{\boldsymbol{y}}) \\
& \equiv & \exists z \in \mathsf{m}(a,a')[B(z)]^{\mathsf{m}(\boldsymbol{x},\boldsymbol{x}')}_{\boldsymbol{y}} \quad \equiv \quad [\exists z B(z)]^{\mathsf{m}(\boldsymbol{x},\boldsymbol{x}'),\mathsf{m}(a,a')}_{\boldsymbol{y}}.
\end{array}$$

$A \equiv \forall z B(z)$. We want to prove that $[\forall z B(z)]^{\boldsymbol{f}}_{\boldsymbol{y},a} \multimap [\forall z B(z)]^{\mathsf{m}(\boldsymbol{f},\boldsymbol{f}')}_{\boldsymbol{y},a}$, knowing that $\boldsymbol{f}, \boldsymbol{f}', \boldsymbol{y}$ and $a$ are non-empty.

$$\begin{array}{rcl}
[\forall z B(z)]^{\boldsymbol{f}}_{\boldsymbol{y},a} & :\equiv & \forall z \in a [B(z)]^{\boldsymbol{f}a}_{\boldsymbol{y}} \overset{\text{(A5)/I.H.}}{\multimap} \forall z \in a [B(z)]^{\mathsf{m}(\boldsymbol{f}a,\boldsymbol{f}'a)}_{\boldsymbol{y}} \\
& \overset{\text{(B1)}}{\equiv} & \forall z \in a [B(z)]^{\mathsf{m}(\boldsymbol{f},\boldsymbol{f}')a}_{\boldsymbol{y}} \quad :\equiv \quad [\forall z B(z)]^{\mathsf{m}(\boldsymbol{f},\boldsymbol{f}')}_{\boldsymbol{y},a}.
\end{array}$$

$A \equiv {!B}$. Let us prove that $[!B]^{\boldsymbol{x}}_{\boldsymbol{a}} \multimap [!B]^{\mathsf{m}(\boldsymbol{x},\boldsymbol{x}')}_{\boldsymbol{a}}$, when $\boldsymbol{x}, \boldsymbol{x}'$ and $\boldsymbol{a}$ are non-empty.

$$[!B]^{\boldsymbol{x}}_{\boldsymbol{a}} \;:\equiv\; !\tilde{\forall} \boldsymbol{y} \subseteq \boldsymbol{a} [B]^{\boldsymbol{x}}_{\boldsymbol{y}} \overset{\text{I.H.}}{\multimap} !\tilde{\forall} \boldsymbol{y} \subseteq \boldsymbol{a} [B]^{\mathsf{m}(\boldsymbol{x},\boldsymbol{x}')}_{\boldsymbol{y}} \;:\equiv\; [!B]^{\mathsf{m}(\boldsymbol{x},\boldsymbol{x}')}_{\boldsymbol{a}}.$$

$A \equiv {!B} \multimap C$. In this case note that we are imposing a restriction that all implications in $A$ are of the form $!D \multimap E$. So, we need to prove that $[!B \multimap C]^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} \multimap [!B \multimap C]^{\mathsf{m}(\boldsymbol{f},\boldsymbol{f}'),\mathsf{m}(\boldsymbol{g},\boldsymbol{g}')}_{\boldsymbol{x},\boldsymbol{w}}$, when $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{x}, \boldsymbol{w}, \boldsymbol{f}'$ and $\boldsymbol{g}'$ are non-empty. By the way we interpret linear implication, we have to prove that

$$\vdash_{\mathsf{ILL}^{\omega}_{\in}} ([!B]^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{x}\boldsymbol{w}} \multimap [C]^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{w}}) \multimap ([!B]^{\boldsymbol{x}}_{\mathsf{m}(\boldsymbol{f},\boldsymbol{f}')\boldsymbol{x}\boldsymbol{w}} \multimap [C]^{\mathsf{m}(\boldsymbol{g},\boldsymbol{g}')\boldsymbol{x}}_{\boldsymbol{w}}).$$

Applying the interpretation of bang and (B1), it is the same as proving that

$$\vdash_{\mathsf{ILL}^{\omega}_{\in}} (!\tilde{\forall} \boldsymbol{y} \subseteq \boldsymbol{f}\boldsymbol{x}\boldsymbol{w}[B]^{\boldsymbol{x}}_{\boldsymbol{y}} \multimap [C]^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{w}}) \multimap (!\tilde{\forall} \boldsymbol{y} \subseteq \mathsf{m}(\boldsymbol{f}\boldsymbol{x}\boldsymbol{w}, \boldsymbol{f}'\boldsymbol{x}\boldsymbol{w})[B]^{\boldsymbol{x}}_{\boldsymbol{y}} \multimap [C]^{\mathsf{m}(\boldsymbol{g}\boldsymbol{x},\boldsymbol{g}'\boldsymbol{x})}_{\boldsymbol{w}}).$$

Starting with (B2), and since $\boldsymbol{f}'\boldsymbol{x}\boldsymbol{w}$ is non-empty, we know that

$$(\star) \quad !(\boldsymbol{y} \subseteq \boldsymbol{f}\boldsymbol{x}\boldsymbol{w}) \vdash \; !(\boldsymbol{y} \subseteq \mathsf{m}(\boldsymbol{f}\boldsymbol{x}\boldsymbol{w}, \boldsymbol{f}'\boldsymbol{x}\boldsymbol{w})).$$

Using the straightforward facts that from $A \vdash_{\mathsf{ILL}^{\omega}_{\in}} B$ we can deduce $B \multimap C \vdash_{\mathsf{ILL}^{\omega}_{\in}} A \multimap C$ and from $A \vdash_{\mathsf{ILL}^{\omega}_{\in}} B$ we can derive $!\tilde{\forall} y A \vdash_{\mathsf{ILL}^{\omega}_{\in}} !\tilde{\forall} y B$, from $(\star)$ we obtain

$$!\tilde{\forall}\boldsymbol{y} \subseteq \mathsf{m}(\boldsymbol{f}\boldsymbol{x}\boldsymbol{w}, \boldsymbol{f}'\boldsymbol{x}\boldsymbol{w})[B]^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash_{\mathsf{ILL}^{\omega}_{\in}} !\tilde{\forall}\boldsymbol{y} \subseteq \boldsymbol{f}\boldsymbol{x}\boldsymbol{w}[B]^{\boldsymbol{x}}_{\boldsymbol{y}}.$$

Applying once more one of the previous facts, we have

$$!\tilde{\forall}\boldsymbol{y} \subseteq \boldsymbol{f}\boldsymbol{x}\boldsymbol{w}[B]^{\boldsymbol{x}}_{\boldsymbol{y}} \multimap [C]^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{w}} \vdash_{\mathsf{ILL}^{\omega}_{\in}} !\tilde{\forall}\boldsymbol{y} \subseteq \mathsf{m}(\boldsymbol{f}\boldsymbol{x}\boldsymbol{w}, \boldsymbol{f}'\boldsymbol{x}\boldsymbol{w})[B]^{\boldsymbol{x}}_{\boldsymbol{y}} \multimap [C]^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{w}}.$$

By induction hypothesis we also have $[C]^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{w}} \vdash_{\mathsf{ILL}^{\omega}_{\in}} [C]^{\mathsf{m}(\boldsymbol{g}\boldsymbol{x}, \boldsymbol{g}'\boldsymbol{x})}_{\boldsymbol{w}}$. □

We are now able to prove the following soundness result:

**Theorem 5.5 (Soundness).** *Let $A_0, \dots, A_n, B$ be formulas of $\mathsf{ILL}^{\omega}_{r}$, with $z$ as the only free-variables. If*

$$A_0(z), \dots, A_n(z) \vdash B(z)$$

*is provable in $\mathsf{ILL}^{\omega}_{r}$ then there are non-empty closed terms $\boldsymbol{a}_0, \dots, \boldsymbol{a}_n, \boldsymbol{b}$ in $\mathcal{S}^{\omega}$ such that*

$$!\mathsf{ne}(\boldsymbol{x}_0), \dots, !\mathsf{ne}(\boldsymbol{x}_n), !\mathsf{ne}(\boldsymbol{w}), !(z \in \boldsymbol{a}), [A_0(z)]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\dots\boldsymbol{x}_n\boldsymbol{w}\boldsymbol{a}}, \dots, [A_n(z)]^{\boldsymbol{x}_n}_{\boldsymbol{a}_n\boldsymbol{x}_0\dots\boldsymbol{x}_n\boldsymbol{w}\boldsymbol{a}} \vdash [B(z)]^{\boldsymbol{b}\boldsymbol{x}_0\dots\boldsymbol{x}_n\boldsymbol{a}}_{\boldsymbol{w}}$$

*is provable in $\mathsf{ILL}^{\omega}_{\in}$.*

**Proof**. The proof is done by induction on the derivation of $A_0(z), \dots, A_n(z) \vdash B(z)$. To simplify notation, when not essential, we omit the formulas $!\mathsf{ne}(\cdot)$ from the sequent, we consider the contexts of the rules with just one formula or no formulas at all, and when not relevant the variables $z$ will usually be omitted. The axioms are easily checked. Note that for $A(z) \vdash A(z)$ the terms $\boldsymbol{a}_0 :\equiv \lambda\boldsymbol{x}_0, \boldsymbol{w}, \boldsymbol{a}.\boldsymbol{w}$ and $\boldsymbol{b} :\equiv \lambda\boldsymbol{x}_0, \boldsymbol{a}.\boldsymbol{x}_0$ are closed and non-empty (see Lemma 5.1) and we have $!\mathsf{ne}(\boldsymbol{x}_0), !\mathsf{ne}(\boldsymbol{w}), !(z \in \boldsymbol{a}), [A(z)]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}\boldsymbol{a}} \vdash_{\mathsf{ILL}^{\omega}_{\in}} [A(z)]^{\boldsymbol{b}\boldsymbol{x}_0\boldsymbol{a}}_{\boldsymbol{w}}$. The axioms for $\Sigma$ and $\Pi$ can be studied in a similar way. For $\Gamma, 0 \vdash A$ note that the interpretation of atomic formulas are the atomic formulas themselves and every type is inhabited by a non-empty closed term. Let us consider some of the non-trivial rules:

*Cut*. By induction hypothesis there are non-empty closed terms $\boldsymbol{a}_0, \boldsymbol{b}$ such that for all non-empty $\boldsymbol{x}_0, \boldsymbol{w}$ we have $[A]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}} \vdash [B]^{\boldsymbol{b}\boldsymbol{x}_0}_{\boldsymbol{w}}$ and there are non-empty closed terms $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{c}$ such that for all non-empty $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{v}$ we have

$$[C]^{\boldsymbol{x}_1}_{\boldsymbol{a}_1\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{v}}, [B]^{\boldsymbol{x}_2}_{\boldsymbol{a}_2\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{v}} \vdash [D]^{\boldsymbol{c}\boldsymbol{x}_1\boldsymbol{x}_2}_{\boldsymbol{v}}.$$

But then,

$$\dfrac{\dfrac{[A]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}} \vdash [B]^{\boldsymbol{b}\boldsymbol{x}_0}_{\boldsymbol{w}}}{[A]^{\boldsymbol{x}_0}_{\boldsymbol{\alpha}_0\boldsymbol{x}_0\boldsymbol{x}_1\boldsymbol{v}} \vdash [B]^{\boldsymbol{b}\boldsymbol{x}_0}_{\boldsymbol{\alpha}_2\boldsymbol{x}_0\boldsymbol{x}_1\boldsymbol{v}}}\,[\tfrac{\boldsymbol{\alpha}_2\boldsymbol{x}_0\boldsymbol{x}_1\boldsymbol{v}}{\boldsymbol{w}}] \qquad \dfrac{[C]^{\boldsymbol{x}_1}_{\boldsymbol{a}_1\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{v}}, [B]^{\boldsymbol{x}_2}_{\boldsymbol{a}_2\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{v}} \vdash [D]^{\boldsymbol{c}\boldsymbol{x}_1\boldsymbol{x}_2}_{\boldsymbol{v}}}{[C]^{\boldsymbol{x}_1}_{\boldsymbol{\alpha}_1\boldsymbol{x}_0\boldsymbol{x}_1\boldsymbol{v}}, [B]^{\boldsymbol{b}\boldsymbol{x}_0}_{\boldsymbol{\alpha}_2\boldsymbol{x}_0\boldsymbol{x}_1\boldsymbol{v}} \vdash [D]^{\boldsymbol{\beta}\boldsymbol{x}_0\boldsymbol{x}_1}_{\boldsymbol{v}}}\,[\tfrac{\boldsymbol{b}\boldsymbol{x}_0}{\boldsymbol{x}_2}]}{[A]^{\boldsymbol{x}_0}_{\boldsymbol{\alpha}_0\boldsymbol{x}_0\boldsymbol{x}_1\boldsymbol{v}}, [C]^{\boldsymbol{x}_1}_{\boldsymbol{\alpha}_1\boldsymbol{x}_0\boldsymbol{x}_1\boldsymbol{v}} \vdash [D]^{\boldsymbol{\beta}\boldsymbol{x}_0\boldsymbol{x}_1}_{\boldsymbol{v}}}\,(\text{cut})$$

with

$$\begin{aligned}
\boldsymbol{\alpha}_0 &:\equiv \lambda\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{v}.\boldsymbol{a}_0\boldsymbol{x}_0(\boldsymbol{a}_2\boldsymbol{x}_1(\boldsymbol{b}\boldsymbol{x}_0)\boldsymbol{v}) \\
\boldsymbol{\alpha}_1 &:\equiv \lambda\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{v}.\boldsymbol{a}_1\boldsymbol{x}_1(\boldsymbol{b}\boldsymbol{x}_0)\boldsymbol{v} \\
\boldsymbol{\alpha}_2 &:\equiv \lambda\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{v}.\boldsymbol{a}_2\boldsymbol{x}_1(\boldsymbol{b}\boldsymbol{x}_0)\boldsymbol{v} \\
\boldsymbol{\beta} &:\equiv \lambda\boldsymbol{x}_0, \boldsymbol{x}_1.\boldsymbol{c}\boldsymbol{x}_1(\boldsymbol{b}\boldsymbol{x}_0).
\end{aligned}$$

To see that the terms above are non-empty apply Lemma 5.1. Thus we proved that there are non-empty closed terms $\boldsymbol{\alpha}_0$, $\boldsymbol{\alpha}_1$ and $\boldsymbol{\beta}$ such that for all non-empty $\boldsymbol{x}_0$, $\boldsymbol{x}_1$ and $\boldsymbol{v}$ we have

$$[A]^{\boldsymbol{x}_0}_{\boldsymbol{\alpha_0 x_0 x_1 v}}, [C]^{\boldsymbol{x}_1}_{\boldsymbol{\alpha_1 x_0 x_1 v}} \vdash [D]^{\boldsymbol{\beta x_0 x_1}}_{\boldsymbol{v}}.$$

Note that the precise formulation of the proof with $!\mathsf{ne}(\cdot)$ inside the sequents brings no problem, since in the antecedent of the conclusion we get $!\mathsf{ne}(\boldsymbol{x}_0)$, $!\mathsf{ne}(\boldsymbol{x}_1)$ and $!\mathsf{ne}(\boldsymbol{v})$ and the others can be removed by cut. E.g. $!\mathsf{ne}(\boldsymbol{bx}_0)$ can be cut with $!\mathsf{ne}(\boldsymbol{x}_0) \vdash\ !\mathsf{ne}(\boldsymbol{bx}_0)$. We are using Property (A5). Note also that during the derivation we have to get ride of the conditions $!(z \in \boldsymbol{a})$ for variables $z$ free in $B$ but not in $A$, $C$ and $D$. That can be done with a convenient substitution followed by cut since in each type we have closed terms $\boldsymbol{s}$ and $\boldsymbol{t}$ such that $!(\boldsymbol{s} \in \boldsymbol{t})$.

*Linear implication - right introduction*

$$\dfrac{\dfrac{[A]^{\boldsymbol{x}_0}_{\boldsymbol{a_0x_0x_1w}}, [B]^{\boldsymbol{x}_1}_{\boldsymbol{a_1x_0x_1w}} \vdash [C]^{\boldsymbol{bx_0x_1}}_{\boldsymbol{w}}}{[A]^{\boldsymbol{x}_0}_{\boldsymbol{a_0x_0x_1w}} \vdash [B]^{\boldsymbol{x}_1}_{\boldsymbol{a_1x_0x_1w}} \multimap [C]^{\boldsymbol{bx_0x_1}}_{\boldsymbol{w}}}\ (\multimap\mathrm{R})}{[A]^{\boldsymbol{x}_0}_{\boldsymbol{a_0x_0x_1w}} \vdash [B \multimap C]^{\boldsymbol{a_1x_0},\boldsymbol{bx_0}}_{\boldsymbol{x_1},\boldsymbol{w}}}\ (\mathrm{D.}\ 5.3)$$

*Linear implication - left introduction*

$$\dfrac{\dfrac{\dfrac{[A]^{\boldsymbol{x}_0}_{\boldsymbol{a_0x_0w}} \vdash [B]^{\boldsymbol{bx_0}}_{\boldsymbol{w}}}{[A]^{\boldsymbol{x}_0}_{\boldsymbol{a_0x_0(fba_2)}} \vdash [B]^{\boldsymbol{bx_0}}_{\boldsymbol{fba_2}}}\ [\tfrac{\boldsymbol{fba_2}}{\boldsymbol{w}}] \qquad \dfrac{[C]^{\boldsymbol{x}_1}_{\boldsymbol{a_1x_1x_2v}}, [D]^{\boldsymbol{x}_2}_{\boldsymbol{a_2x_1x_2v}} \vdash [E]^{\boldsymbol{cx_1x_2}}_{\boldsymbol{v}}}{[C]^{\boldsymbol{x}_1}_{\boldsymbol{a_1x_1(g(bx_0))v}}, [D]^{\boldsymbol{g(bx_0)}}_{\boldsymbol{a_2x_1(g(bx_0))v}} \vdash [E]^{\boldsymbol{cx_1(g(bx_0))}}_{\boldsymbol{v}}}\ [\tfrac{\boldsymbol{g(bx_0)}}{\boldsymbol{x_2}}]}{[A]^{\boldsymbol{x}_0}_{\boldsymbol{a_0x_0(fba_2)}}, [C]^{\boldsymbol{x}_1}_{\boldsymbol{a_1x_1(g(bx_0))v}}, [B]^{\boldsymbol{bx_0}}_{\boldsymbol{fba_2}} \multimap [D]^{\boldsymbol{g(bx_0)}}_{\boldsymbol{a_2x_1(g(bx_0))v}} \vdash [E]^{\boldsymbol{cx_1(g(bx_0))}}_{\boldsymbol{v}}}\ (\multimap\mathrm{L})}{[A]^{\boldsymbol{x}_0}_{\boldsymbol{\alpha_0x_0x_1fgv}}, [C]^{\boldsymbol{x}_1}_{\boldsymbol{\alpha_1x_0x_1fgv}}, [B \multimap D]^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{\alpha_2x_0x_1fgv},\boldsymbol{\alpha_3x_0x_1fgv}} \vdash [E]^{\boldsymbol{\beta x_0x_1},\boldsymbol{fg}}_{\boldsymbol{v}}}$$

where $\boldsymbol{fba}_2$ stands for $\boldsymbol{f}(\boldsymbol{bx}_0)(\boldsymbol{a}_2\boldsymbol{x}_1(\boldsymbol{g}(\boldsymbol{bx}_0))\boldsymbol{v})$ and

$$\begin{aligned}
\boldsymbol{\alpha}_0 &:\equiv \lambda \boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{v}.\boldsymbol{a}_0\boldsymbol{x}_0(\boldsymbol{fba}_2)\\
\boldsymbol{\alpha}_1 &:\equiv \lambda \boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{v}.\boldsymbol{a}_1\boldsymbol{x}_1(\boldsymbol{g}(\boldsymbol{bx}_0))\boldsymbol{v}\\
\boldsymbol{\alpha}_2 &:\equiv \lambda \boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{v}.\boldsymbol{bx}_0\\
\boldsymbol{\alpha}_3 &:\equiv \lambda \boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{v}.\boldsymbol{a}_2\boldsymbol{x}_1(\boldsymbol{g}(\boldsymbol{bx}_0))\boldsymbol{v}\\
\boldsymbol{\beta} &:\equiv \lambda \boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{f}, \boldsymbol{g}.\boldsymbol{cx}_1(\boldsymbol{g}(\boldsymbol{bx}_0)).
\end{aligned}$$

Note that the $!\mathsf{ne}(\boldsymbol{fba}_2)$ we omitted in antecedent of the conclusion can be replaced by $!\mathsf{ne}(\boldsymbol{f})$ since we know that $!\mathsf{ne}(\boldsymbol{f}) \vdash\ !\mathsf{ne}(\boldsymbol{fba}_2)$. A similar observation can be done concerning $!\mathsf{ne}(\boldsymbol{g}(\boldsymbol{bx}_0))$. The closed terms $\boldsymbol{\alpha}_0$, $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, $\boldsymbol{\alpha}_3$ and $\boldsymbol{\beta}$ being non-empty follows from Lemma 5.1.

*With - left introduction*

$$\dfrac{\dfrac{[A]^{\boldsymbol{x}_0}_{\boldsymbol{a_0x_0w}} \vdash [B]^{\boldsymbol{bx_0}}_{\boldsymbol{w}}}{[A]^{\boldsymbol{x}_0}_{\boldsymbol{a_0x_0w}} \mathbin{\&} [C]^{\boldsymbol{x}_1}_{\boldsymbol{a_1x_0x_1w}} \vdash [B]^{\boldsymbol{bx_0}}_{\boldsymbol{w}}}\ (\&\mathrm{L})}{[A \mathbin{\&} C]^{\boldsymbol{x}_0,\boldsymbol{x}_1}_{(\lambda \boldsymbol{x}_0,\boldsymbol{x}_1,\boldsymbol{w}.\boldsymbol{a_0x_0w})\boldsymbol{x_0x_1w},\boldsymbol{a_1x_0x_1w}} \vdash [B]^{(\lambda \boldsymbol{x}_0,\boldsymbol{x}_1.\boldsymbol{bx}_0)\boldsymbol{x_0x_1}}_{\boldsymbol{w}}}\ (\mathrm{D.}\ 5.3)$$

Note that the existence of $\boldsymbol{a}_1$ is guaranteed since each type has a non-empty closed term. The second rule of $\&$ L introduction is entirely similar.

*With - right introduction*

$$\dfrac{\dfrac{\dfrac{[!A]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}} \vdash [B]^{\boldsymbol{b}\boldsymbol{x}_0}_{\boldsymbol{w}}}{!\tilde{\forall}\boldsymbol{y} \subseteq \boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}[A]^{\boldsymbol{x}_0}_{\boldsymbol{y}} \vdash [B]^{\boldsymbol{b}\boldsymbol{x}_0}_{\boldsymbol{w}}}}{!\tilde{\forall}\boldsymbol{y} \subseteq \mathsf{u}(\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}, \boldsymbol{a}_1\boldsymbol{x}_0\boldsymbol{v})[A]^{\boldsymbol{x}_0}_{\boldsymbol{y}} \vdash [B]^{\boldsymbol{b}\boldsymbol{x}_0}_{\boldsymbol{w}}}\,(\mathrm{C}) \qquad \dfrac{\dfrac{\dfrac{[!A]^{\boldsymbol{x}_1}_{\boldsymbol{a}_1\boldsymbol{x}_1\boldsymbol{v}} \vdash [C]^{\boldsymbol{c}\boldsymbol{x}_1}_{\boldsymbol{v}}}{!\tilde{\forall}\boldsymbol{y} \subseteq \boldsymbol{a}_1\boldsymbol{x}_1\boldsymbol{v}[A]^{\boldsymbol{x}_1}_{\boldsymbol{y}} \vdash [C]^{\boldsymbol{c}\boldsymbol{x}_1}_{\boldsymbol{v}}}}{!\tilde{\forall}\boldsymbol{y} \subseteq \boldsymbol{a}_1\boldsymbol{x}_0\boldsymbol{v}[A]^{\boldsymbol{x}_0}_{\boldsymbol{y}} \vdash [C]^{\boldsymbol{c}\boldsymbol{x}_0}_{\boldsymbol{v}}}\,[\tfrac{\boldsymbol{x}_0}{\boldsymbol{x}_1}]}{!\tilde{\forall}\boldsymbol{y} \subseteq \mathsf{u}(\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}, \boldsymbol{a}_1\boldsymbol{x}_0\boldsymbol{v})[A]^{\boldsymbol{x}_0}_{\boldsymbol{y}} \vdash [C]^{\boldsymbol{c}\boldsymbol{x}_0}_{\boldsymbol{v}}}\,(\mathrm{C})}{\dfrac{!\tilde{\forall}\boldsymbol{y} \subseteq \mathsf{u}(\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}, \boldsymbol{a}_1\boldsymbol{x}_0\boldsymbol{v})[A]^{\boldsymbol{x}_0}_{\boldsymbol{y}} \vdash [B]^{\boldsymbol{b}\boldsymbol{x}_0}_{\boldsymbol{w}} \,\&\, [C]^{\boldsymbol{c}\boldsymbol{x}_0}_{\boldsymbol{v}}}{[!A]^{\boldsymbol{x}_0}_{\mathsf{u}(\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}, \boldsymbol{a}_1\boldsymbol{x}_0\boldsymbol{v})} \vdash [B \,\&\, C]^{\boldsymbol{b}\boldsymbol{x}_0, \boldsymbol{c}\boldsymbol{x}_0}_{\boldsymbol{w},\boldsymbol{v}}}}\,(\&\mathrm{R})$$

Note that by weakening we can write $!\mathsf{ne}(\boldsymbol{x}_0)$, $!\mathsf{ne}(\boldsymbol{w})$, $!\mathsf{ne}(\boldsymbol{v})$ in both branches of the proof. In step (C), we must first show that $\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}$ and $\boldsymbol{a}_1\boldsymbol{x}_0\boldsymbol{v}$ are non-empty, but this follows since $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$ are non-empty and their arguments are also assumed to be non-empty.

*Plus - left introduction*

$$\dfrac{\dfrac{[A]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}} \vdash [C]^{\boldsymbol{b}\boldsymbol{x}_0}_{\boldsymbol{w}}}{[A]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}} \vdash [C]^{\mathsf{m}(\boldsymbol{b}\boldsymbol{x}_0, \boldsymbol{c}\boldsymbol{x}_1)}_{\boldsymbol{w}}}\,(\mathrm{L.5.4}) \qquad \dfrac{\dfrac{[B]^{\boldsymbol{x}_1}_{\boldsymbol{a}_1\boldsymbol{x}_1\boldsymbol{v}} \vdash [C]^{\boldsymbol{c}\boldsymbol{x}_1}_{\boldsymbol{v}}}{[B]^{\boldsymbol{x}_1}_{\boldsymbol{a}_1\boldsymbol{x}_1\boldsymbol{w}} \vdash [C]^{\boldsymbol{c}\boldsymbol{x}_1}_{\boldsymbol{w}}}\,[\tfrac{\boldsymbol{w}}{\boldsymbol{v}}]}{[B]^{\boldsymbol{x}_1}_{\boldsymbol{a}_1\boldsymbol{x}_1\boldsymbol{w}} \vdash [C]^{\mathsf{m}(\boldsymbol{b}\boldsymbol{x}_0, \boldsymbol{c}\boldsymbol{x}_1)}_{\boldsymbol{w}}}\,(\mathrm{L.5.4})}{\dfrac{[A]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}} \oplus [B]^{\boldsymbol{x}_1}_{\boldsymbol{a}_1\boldsymbol{x}_1\boldsymbol{w}} \vdash [C]^{\mathsf{m}(\boldsymbol{b}\boldsymbol{x}_0, \boldsymbol{c}\boldsymbol{x}_1)}_{\boldsymbol{w}}}{[A \oplus B]^{\boldsymbol{x}_0, \boldsymbol{x}_1}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}, \boldsymbol{a}_1\boldsymbol{x}_1\boldsymbol{w}} \vdash [C]^{\mathsf{m}(\boldsymbol{b}\boldsymbol{x}_0, \boldsymbol{c}\boldsymbol{x}_1)}_{\boldsymbol{w}}}}\,(\oplus\mathrm{L})$$

Again we use weakening to deal with the non-empty elements. Because of the restriction on the rule ($\oplus$L), all implications on $C$ are of the form $!D \multimap E$ and hence we can apply Lemma 5.4.

*Existential quantifier - right introduction*

$$\dfrac{\dfrac{!(z \in \boldsymbol{a}), [A(z)]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}\boldsymbol{a}} \vdash [B(t[z])]^{\boldsymbol{b}\boldsymbol{x}_0\boldsymbol{a}}_{\boldsymbol{w}} \qquad !(z \in \boldsymbol{a}) \overset{\mathrm{L.\,5.2}}{\vdash} !(t[z] \in \tilde{t}(\boldsymbol{a}))}{!(z \in \boldsymbol{a}), [A(z)]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}\boldsymbol{a}} \vdash !(t[z] \in \tilde{t}(\boldsymbol{a})) \otimes [B(t[z])]^{\boldsymbol{b}\boldsymbol{x}_0\boldsymbol{a}}_{\boldsymbol{w}}}}{\dfrac{!(z \in \boldsymbol{a}), [A(z)]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}\boldsymbol{a}} \vdash \exists x \in \tilde{t}(\boldsymbol{a})[B(x)]^{\boldsymbol{b}\boldsymbol{x}_0\boldsymbol{a}}_{\boldsymbol{w}}}{!(z \in \boldsymbol{a}), [A(z)]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}\boldsymbol{a}} \vdash [\exists x B(x)]^{\boldsymbol{b}\boldsymbol{x}_0\boldsymbol{a}, \tilde{t}(\boldsymbol{a})}_{\boldsymbol{w}}}}\,(\exists\mathrm{R})$$

Note that the existence of a closed term $\tilde{t}$ such that $!\lambda z.t[z] \in \tilde{t}$ is ensured by Lemma 5.2. Immediately by Property (A3) the term $\tilde{t}$ is non-empty and so also $\lambda\boldsymbol{x}_0, \boldsymbol{a}.\tilde{t}(\boldsymbol{a})$ is closed and non-empty (see Lemma 5.1). If the variables $z$ do not occur free in $A$ or $\exists x B(x)$, we can remove $!(z \in \boldsymbol{a})$ from the conclusion, instantiating these variables in a way that $\vdash\ !(z \in \boldsymbol{a})$ and applying the cut rule. Note that appropriate instantiations verifying the assertion above are always possible.

*Existential quantifier - left introduction*

$$\dfrac{\dfrac{\dfrac{!(z \in a), [A(z)]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}a} \vdash [B]^{\boldsymbol{b}\boldsymbol{x}_0 a}_{\boldsymbol{w}}}{!(z \in a) \otimes [A(z)]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}a} \vdash [B]^{\boldsymbol{b}\boldsymbol{x}_0 a}_{\boldsymbol{w}}}}{\exists z \in a[A(z)]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}a} \vdash [B]^{\boldsymbol{b}\boldsymbol{x}_0 a}_{\boldsymbol{w}}} \ (\exists \mathrm{L})}{[\exists z A(z)]^{\boldsymbol{x}_0, a}_{(\lambda \boldsymbol{x}_0, a, \boldsymbol{w}.\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}a)\boldsymbol{x}_0 a\boldsymbol{w}} \vdash [B]^{\boldsymbol{b}\boldsymbol{x}_0 a}_{\boldsymbol{w}}}$$

*Universal quantifier - right introduction*

$$\dfrac{\dfrac{\dfrac{!(z \in a), [A]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}a} \vdash [B(z)]^{\boldsymbol{b}\boldsymbol{x}_0 a}_{\boldsymbol{w}}}{[A]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}a} \vdash \ !(z \in a) \multimap [B(z)]^{\boldsymbol{b}\boldsymbol{x}_0 a}_{\boldsymbol{w}}}}{[A]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}a} \vdash \forall z \in a[B(z)]^{\boldsymbol{b}\boldsymbol{x}_0 a}_{\boldsymbol{w}}} \ (\forall \mathrm{R})}{[A]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}a} \vdash [\forall z B(z)]^{\boldsymbol{b}\boldsymbol{x}_0}_{\boldsymbol{w}, a}}$$

*Universal quantifier - left introduction*

$$\dfrac{\dfrac{\dfrac{\dfrac{!(z \in \boldsymbol{a}) \overset{\text{L. 5.2}}{\vdash} \ !(t[z] \in \tilde{t}(\boldsymbol{a})) \qquad !(z \in \boldsymbol{a}), [A(t[z])]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}\boldsymbol{a}} \vdash [B(z)]^{\boldsymbol{b}\boldsymbol{x}_0\boldsymbol{a}}_{\boldsymbol{w}}}{!(z \in \boldsymbol{a}), !(t[z] \in \tilde{t}(\boldsymbol{a})) \multimap [A(t[z])]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}\boldsymbol{a}} \vdash [B(z)]^{\boldsymbol{b}\boldsymbol{x}_0\boldsymbol{a}}_{\boldsymbol{w}}}}{!(z \in \boldsymbol{a}), \forall x \in \tilde{t}(\boldsymbol{a})[A(x)]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}\boldsymbol{a}} \vdash [B(z)]^{\boldsymbol{b}\boldsymbol{x}_0\boldsymbol{a}}_{\boldsymbol{w}}} \ (\forall \mathrm{L})}{!(z \in \boldsymbol{a}), \forall x \in \tilde{t}(\boldsymbol{a})[A(x)]^{\boldsymbol{f}(\tilde{t}\boldsymbol{a})}_{\boldsymbol{a}_0(\boldsymbol{f}(\tilde{t}\boldsymbol{a}))\boldsymbol{w}\boldsymbol{a}} \vdash [B(z)]^{\boldsymbol{b}(\boldsymbol{f}(\tilde{t}\boldsymbol{a}))\boldsymbol{a}}_{\boldsymbol{w}}} \ [\tfrac{\boldsymbol{f}(\tilde{t}\boldsymbol{a})}{\boldsymbol{x}_0}]}{!(z \in \boldsymbol{a}), [\forall x A(x)]^{\boldsymbol{f}}_{\boldsymbol{a}_0(\boldsymbol{f}(\tilde{t}\boldsymbol{a}))\boldsymbol{w}\boldsymbol{a}, \tilde{t}\boldsymbol{a}} \vdash [B(z)]^{\boldsymbol{b}(\boldsymbol{f}(\tilde{t}\boldsymbol{a}))\boldsymbol{a}}_{\boldsymbol{w}}}$$

*Contraction*

$$\dfrac{\dfrac{\dfrac{\dfrac{[!A]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{x}_1\boldsymbol{w}}, [!A]^{\boldsymbol{x}_1}_{\boldsymbol{a}_1\boldsymbol{x}_0\boldsymbol{x}_1\boldsymbol{w}} \vdash [B]^{\boldsymbol{b}\boldsymbol{x}_0\boldsymbol{x}_1}_{\boldsymbol{w}}}{[!A]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{x}_0\boldsymbol{w}}, [!A]^{\boldsymbol{x}_0}_{\boldsymbol{a}_1\boldsymbol{x}_0\boldsymbol{x}_0\boldsymbol{w}} \vdash [B]^{\boldsymbol{b}\boldsymbol{x}_0\boldsymbol{x}_0}_{\boldsymbol{w}}} \ [\tfrac{\boldsymbol{x}_0}{\boldsymbol{x}_1}]}{!\tilde{\forall}\boldsymbol{y} \subseteq \boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{x}_0\boldsymbol{w}[A]^{\boldsymbol{x}_0}_{\boldsymbol{y}}, !\tilde{\forall}\boldsymbol{z} \subseteq \boldsymbol{a}_1\boldsymbol{x}_0\boldsymbol{x}_0\boldsymbol{w}[A]^{\boldsymbol{x}_0}_{\boldsymbol{z}} \vdash [B]^{\boldsymbol{b}\boldsymbol{x}_0\boldsymbol{x}_0}_{\boldsymbol{w}}}}{!\tilde{\forall}\boldsymbol{y} \subseteq \mathsf{u}(\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{x}_0\boldsymbol{w}, \boldsymbol{a}_1\boldsymbol{x}_0\boldsymbol{x}_0\boldsymbol{w})[A]^{\boldsymbol{x}_0}_{\boldsymbol{y}}, !\tilde{\forall}\boldsymbol{y} \subseteq \mathsf{u}(\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{x}_0\boldsymbol{w}, \boldsymbol{a}_1\boldsymbol{x}_0\boldsymbol{x}_0\boldsymbol{w})[A]^{\boldsymbol{x}_0}_{\boldsymbol{y}} \vdash [B]^{\boldsymbol{b}\boldsymbol{x}_0\boldsymbol{x}_0}_{\boldsymbol{w}}} \ (\mathrm{C})}{\dfrac{!\tilde{\forall}\boldsymbol{y} \subseteq \mathsf{u}(\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{x}_0\boldsymbol{w}, \boldsymbol{a}_1\boldsymbol{x}_0\boldsymbol{x}_0\boldsymbol{w})[A]^{\boldsymbol{x}_0}_{\boldsymbol{y}} \vdash [B]^{\boldsymbol{b}\boldsymbol{x}_0\boldsymbol{x}_0}_{\boldsymbol{w}}}{[!A]^{\boldsymbol{x}_0}_{\mathsf{u}(\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{x}_0\boldsymbol{w}, \boldsymbol{a}_1\boldsymbol{x}_0\boldsymbol{x}_0\boldsymbol{w})} \vdash [B]^{\boldsymbol{b}\boldsymbol{x}_0\boldsymbol{x}_0}_{\boldsymbol{w}}}} \ (\mathrm{con})$$

*Weakening*

$$\dfrac{\dfrac{[A]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}} \vdash [C]^{\boldsymbol{b}\boldsymbol{x}_0}_{\boldsymbol{w}}}{[A]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}}, !\tilde{\forall}\boldsymbol{y} \subseteq \boldsymbol{c}_1[B]^{\boldsymbol{x}_1}_{\boldsymbol{y}} \vdash [C]^{\boldsymbol{b}\boldsymbol{x}_0}_{\boldsymbol{w}}} \ (\mathrm{wkn})}{[A]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\boldsymbol{w}}, [!B]^{\boldsymbol{x}_1}_{\boldsymbol{c}_1} \vdash [C]^{\boldsymbol{b}\boldsymbol{x}_0}_{\boldsymbol{w}}}$$

Again we are using the fact that any type is inhabited by a non-empty closed term.

*Bang - right introduction*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[!A]^{x_0}_{a_0x_0w} \vdash [B]^{bx_0}_w}{!\tilde{\forall} y \subseteq a_0x_0w[A]^{x_0}_y \vdash [B]^{bx_0}_w}}{!\tilde{\forall} w \subseteq v !\tilde{\forall} y \subseteq a_0x_0w[A]^{x_0}_y \vdash \ !\tilde{\forall} w \subseteq v[B]^{bx_0}_w} \text{(!R)}}{!\tilde{\forall} y \subseteq \mathsf{v}(a_0x_0, v)[A]^{x_0}_y \vdash \ !\tilde{\forall} w \subseteq v[B]^{bx_0}_w} \text{(D)}}{[!A]^{x_0}_{\mathsf{v}(a_0x_0,v)} \vdash [!B]^{bx_0}_v}}{!\mathsf{ne}(v), [!A]^{x_0}_{\mathsf{v}(a_0x_0,v)} \vdash [!B]^{bx_0}_v}$$

*Bang - left introduction*

$$\dfrac{\dfrac{[A]^{x_0}_{a_0x_0w} \vdash [B]^{bx_0}_w}{!\tilde{\forall} y \subseteq \mathsf{n}(a_0x_0w)[A]^{x_0}_y \vdash [B]^{bx_0}_w} \text{(E)}}{[!A]^{x_0}_{\mathsf{n}(a_0x_0w)} \vdash [B]^{bx_0}_w}$$

That concludes the proof. □

**Remark 5.6 (Necessity of restrictions).** *In order to see why we imposed restrictions on the rules* &R *and* ⊕L*, consider an instance of* &R *where* $[\Delta]^x_y$ *is not monotone on* $y$*, i.e.* $[\Delta]^x_{\mathsf{u}(a,b)}$ *does not necessarily imply* $[\Delta]^x_a$*. For instance, take* $\Delta \equiv \forall x \exists y P(x, y)$ *with* $P(x, y)$ *an atomic formula, so that*

$$[\Delta]^f_a \equiv \forall x \in a\, \exists y \in fa\, P(x, y).$$

*In such cases, given two witnesses* $a$ *and* $b$ *such that*

$$[\Delta]^x_{axy} \vdash [A]_y \qquad \text{and} \qquad [\Delta]^x_{bxw} \vdash [B]_w$$

*we must produce a single witness* $c$ *satisfying* $[\Delta]^x_{cxyw} \vdash [A]_y \,\&\, [B]_w$*. This would be possible if we could find a* $c$ *such that both* $[\Delta]^x_{cxyw} \multimap [\Delta]^x_{axy}$ *and* $[\Delta]^x_{cxyw} \multimap [\Delta]^x_{bxw}$*. Such* $c$ *always exists given the restrictions we added in the rule* &R*. The restriction in* ⊕L *can be argued in a similar way, in this case by constructing a* $C$ *which does not have the necessary monotonicity property derived in Lemma 5.4.*

### *5.3. Proof of Theorem 3.3*

As pointed before, the proof of the soundness theorem for the parametrised bounded interpretation of $\mathsf{IL}^\omega$ follows from the soundness of the parametrised interpretation of $\mathsf{ILL}^\omega_r$. Next we present some results that help us establishing the above relation.

Denote by $\mathsf{P}_i$ the properties listed in Section 3.1 (for $\mathsf{IL}^\omega_\in$). Let us associate each of the relations and functionals $\mathsf{ne}$, $\in$, $\subseteq$, $\mathsf{m}$, $\mathsf{n}$, $\mathsf{u}$ and $\mathsf{v}$ of $\mathsf{IL}^\omega_\in$ with their linear logic counterparts $\mathsf{ne}^*$, $\in^*$, $\subseteq^*$, $\mathsf{m}$, $\mathsf{n}$, $\mathsf{u}$ and $\mathsf{v}$. Let $\mathsf{ILL}^\omega_{\in^*}$ be the system whose axioms for the definition of $\mathsf{m}$ (respectively $\mathsf{n}$, $\mathsf{u}$ and $\mathsf{v}$) are the Girard's $(\cdot)^*$ translations of the axioms in $\mathsf{IL}^\omega_\in$ to define $\mathsf{m}$ (respectively $\mathsf{n}$, $\mathsf{u}$ and $\mathsf{v}$). The same happens with the axioms involved in $\mathsf{ne}$, $\in$, $\subseteq$ and we just opted to distinguish the corresponding relations in linear logic by $\mathsf{ne}^*$, $\in^*$, $\subseteq^*$ since, if the formers are not primitive symbols in the language the translations via $(\cdot)^*$ of their definitions are the definitions of the relations in linear logic, e.g. $(x \in a)^* \equiv x \in^* a$. By Girard's $(\cdot)^*$ translation we have if $\mathsf{IL}^\omega_\in$ satisfies the properties $\mathsf{P}_i$ then $\mathsf{ILL}^\omega_{\in^*}$ satisfies the properties $(\mathsf{P}_i)^*$.

Denote by $\mathsf{P}_l$ the properties initially imposed over the linear setting (cf. Section 5.1). The two sets of conditions can be related through the following result:

**Proposition 5.7.** *If $\mathsf{ILL}^\omega_{\in^*}$ satisfies the properties $(\mathsf{P}_i)^*$ then $\mathsf{ILL}^\omega_{\in^*}$ satisfies the properties $\mathsf{P}_l$.*

**Proof**. All the properties are easily checked since we have the following:

If $\vdash_{\mathsf{ILL}^\omega_{\in^*}} A$ then $\vdash_{\mathsf{ILL}^\omega_{\in^*}} !A$

If $\vdash_{\mathsf{ILL}^\omega_{\in^*}} !A \multimap C$ then $\vdash_{\mathsf{ILL}^\omega_{\in^*}} !A \multimap\ !C$

and $!(A \,\&\, B) \multimap\!\!\circ\ !A \otimes\ !B$. □

**Lemma 5.8.** *From the above we can easily check the following correspondences concerning the translations of the bounded quantifications:*

($a$) $(\tilde{\forall} y \subseteq aA)^* \equiv \tilde{\forall} y \subseteq^* a\, A^*$

($b$) $(\forall z {\in} a\, A)^* \equiv \forall z {\in^*} a\, A^*$

($c$) $(\exists z {\in} a\, A)^* \equiv \exists z {\in^*} a\ !A^*$.

**Proof**. We illustrate the idea with the proof of the first assertion:

$$\begin{aligned}(\tilde{\forall} y \subseteq aA)^* &\equiv (\forall y(\mathsf{ne}(y) \to (y \subseteq a \to A)))^* \\ &\equiv \forall y(!(\mathsf{ne}^*(y)) \multimap (!(y \subseteq^* a) \multimap A^*)) \\ &\equiv \tilde{\forall} y(!(y \subseteq^* a) \multimap A^*) \equiv \tilde{\forall} y \subseteq^* a\, A^*.\end{aligned}$$

The other cases are treated similarly. □

So, the interpretations $|\cdot|^{\boldsymbol{x}}_{\boldsymbol{y}}$ and $[\cdot]^{\boldsymbol{x}}_{\boldsymbol{y}}$ from $\mathsf{IL}^\omega$ to $\mathsf{IL}^\omega_\in$ and $\mathsf{ILL}^\omega_r$ to $\mathsf{ILL}^\omega_{\in^*}$ respectively, are related in the following way.

**Proposition 5.9.** $[A^*]^{\boldsymbol{x}}_{\boldsymbol{y}} \equiv (|A|^{\boldsymbol{x}}_{\boldsymbol{y}})^*$.

**Proof**. The proof is done by induction on the complexity of the formula $A$. For $A$ atomic the result is immediately. The other cases are studied below.

*Conjunction*.

$$\begin{aligned}[(A \wedge B)^*]^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} &\equiv [A^* \,\&\, B^*]^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} \equiv [A^*]^{\boldsymbol{x}}_{\boldsymbol{y}} \,\&\, [B^*]^{\boldsymbol{v}}_{\boldsymbol{w}} \stackrel{\text{I.H.}}{\equiv} (|A|^{\boldsymbol{x}}_{\boldsymbol{y}})^* \,\&\, (|B|^{\boldsymbol{v}}_{\boldsymbol{w}})^* \\ &\equiv (|A|^{\boldsymbol{x}}_{\boldsymbol{y}} \wedge |B|^{\boldsymbol{v}}_{\boldsymbol{w}})^* \equiv (|A \wedge B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}})^*.\end{aligned}$$

*Disjunction*.

$$
\begin{aligned}
[(A \vee B)^*]^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{a},\boldsymbol{b}} \quad &\equiv \quad [!A^* \oplus\ !B^*]^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{a},\boldsymbol{b}} \quad \equiv \quad [!A^*]^{\boldsymbol{x}}_{\boldsymbol{a}} \oplus [!B^*]^{\boldsymbol{v}}_{\boldsymbol{b}} \\
&\equiv \quad !\tilde{\forall} \boldsymbol{y} \subseteq^* \boldsymbol{a}[A^*]^{\boldsymbol{x}}_{\boldsymbol{y}} \oplus\ !\tilde{\forall} \boldsymbol{w} \subseteq^* \boldsymbol{b}[B^*]^{\boldsymbol{v}}_{\boldsymbol{w}} \\
&\overset{\text{I.H.}}{\equiv} \quad !\tilde{\forall} \boldsymbol{y} \subseteq^* \boldsymbol{a}(|A|^{\boldsymbol{x}}_{\boldsymbol{y}})^* \oplus\ !\tilde{\forall} \boldsymbol{w} \subseteq^* \boldsymbol{b}(|B|^{\boldsymbol{v}}_{\boldsymbol{w}})^* \\
&\overset{\text{L.5.8}}{\equiv} \quad !(\tilde{\forall} \boldsymbol{y} \subseteq \boldsymbol{a}|A|^{\boldsymbol{x}}_{\boldsymbol{y}})^* \oplus\ !(\tilde{\forall} \boldsymbol{w} \subseteq \boldsymbol{b}|B|^{\boldsymbol{v}}_{\boldsymbol{w}})^* \\
&\equiv \quad (\tilde{\forall} \boldsymbol{y} \subseteq \boldsymbol{a}|A|^{\boldsymbol{x}}_{\boldsymbol{y}} \vee \tilde{\forall} \boldsymbol{w} \subseteq \boldsymbol{b}|B|^{\boldsymbol{v}}_{\boldsymbol{w}})^* \quad \equiv \quad (|A \vee B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{a},\boldsymbol{b}})^*.
\end{aligned}
$$

*Implication.*

$$
\begin{aligned}
[(A \to B)^*]^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} \quad &\equiv \quad [!A^* \multimap B^*]^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} \quad \equiv \quad [!A^*]^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{x}\boldsymbol{w}} \multimap [B^*]^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{w}} \\
&\equiv \quad !\tilde{\forall} \boldsymbol{y} \subseteq^* \boldsymbol{f}\boldsymbol{x}\boldsymbol{w}[A^*]^{\boldsymbol{x}}_{\boldsymbol{y}} \multimap [B^*]^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{w}} \\
&\overset{\text{I.H.}}{\equiv} \quad !\tilde{\forall} \boldsymbol{y} \subseteq^* \boldsymbol{f}\boldsymbol{x}\boldsymbol{w}(|A|^{\boldsymbol{x}}_{\boldsymbol{y}})^* \multimap (|B|^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{w}})^* \\
&\overset{\text{L.5.8}}{\equiv} \quad !(\tilde{\forall} \boldsymbol{y} \subseteq \boldsymbol{f}\boldsymbol{x}\boldsymbol{w}|A|^{\boldsymbol{x}}_{\boldsymbol{y}})^* \multimap (|B|^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{w}})^* \\
&\equiv \quad (\tilde{\forall} \boldsymbol{y} \subseteq \boldsymbol{f}\boldsymbol{x}\boldsymbol{w}|A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to |B|^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{w}})^* \quad \equiv \quad (|A \to B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}})^*.
\end{aligned}
$$

*Universal quantification.*

$$
\begin{aligned}
[(\forall z A(z))^*]^{\boldsymbol{f}}_{\boldsymbol{y},a} \quad &\equiv \quad [\forall z (A(z))^*]^{\boldsymbol{f}}_{\boldsymbol{y},a} \quad \equiv \quad \forall z \in^* a[(A(z))^*]^{\boldsymbol{f}a}_{\boldsymbol{y}} \\
&\overset{\text{I.H.}}{\equiv} \quad \forall z \in^* a(|A(z)|^{\boldsymbol{f}a}_{\boldsymbol{y}})^* \\
&\overset{\text{L.5.8}}{\equiv} \quad (\forall z \in a|A(z)|^{\boldsymbol{f}a}_{\boldsymbol{y}})^* \quad \equiv \quad (|\forall z A(z)|^{\boldsymbol{f}}_{\boldsymbol{y},a})^*.
\end{aligned}
$$

*Existential quantification.*

$$
\begin{aligned}
[(\exists z A(z))^*]^{\boldsymbol{x},a}_{\boldsymbol{b}} \quad &\equiv \quad [\exists z !(A(z))^*]^{\boldsymbol{x},a}_{\boldsymbol{b}} \quad \equiv \quad \exists z \in^* a[!(A(z))^*]^{\boldsymbol{x}}_{\boldsymbol{b}} \\
&\equiv \quad \exists z \in^* a!\tilde{\forall} \boldsymbol{y} \subseteq^* \boldsymbol{b}[(A(z))^*]^{\boldsymbol{x}}_{\boldsymbol{y}} \\
&\overset{\text{I.H.}}{\equiv} \quad \exists z \in^* a!\tilde{\forall} \boldsymbol{y} \subseteq^* \boldsymbol{b}(|A(z)|^{\boldsymbol{x}}_{\boldsymbol{y}})^* \\
&\overset{\text{L.5.8}}{\equiv} \quad (\exists z \in a\tilde{\forall} \boldsymbol{y} \subseteq \boldsymbol{b}|A(z)|^{\boldsymbol{x}}_{\boldsymbol{y}})^* \quad \equiv \quad (|\exists z A(z)|^{\boldsymbol{x},a}_{\boldsymbol{b}})^*.
\end{aligned}
$$

That concludes the proof. □

From the above, we immediately derive the following result:

**Lemma 5.10.** $([A^*]^{\boldsymbol{x}}_{\boldsymbol{y}})^i \equiv |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$.

The soundness in $\mathsf{IL}^\omega$ can now be deduced as follows: If

$A_0(z), \ldots, A_n(z) \vdash_{\mathsf{IL}^\omega} B(z)$ and $\mathsf{IL}^\omega_\in$ satisfies the properties $\mathsf{P}_i$

then through the translation $(\cdot)^*$ we know that

$!(A_0(z))^*, \ldots, !(A_n(z))^* \vdash_{\mathsf{ILL}^\omega_r} (B(z))^*$ and $\mathsf{ILL}^\omega_{\in^*}$ satisfies the properties $(\mathsf{P}_i)^*$.

Applying Proposition 5.7, we know that the properties $\mathsf{P}_l$ are valid for $\mathsf{ne}^*$, $\in^*$, $\subseteq^*$, $\mathsf{m}$, $\mathsf{n}$, $\mathsf{u}$ and $\mathsf{v}$, so we can apply the soundness theorem in $\mathsf{ILL}^\omega_r$ and deduce that there exist non-empty closed terms $\boldsymbol{a}_0, \ldots, \boldsymbol{a}_n, \boldsymbol{b}$ such that

$$!\mathsf{ne}^*(\boldsymbol{x}_0), \ldots, !\mathsf{ne}^*(\boldsymbol{x}_n), !\mathsf{ne}^*(\boldsymbol{w}), !(z \in^* \boldsymbol{a}), [!(A_0(z))^*]^{\boldsymbol{x}_0}_{\boldsymbol{a}_0\boldsymbol{x}_0\ldots\boldsymbol{x}_n\boldsymbol{w}\boldsymbol{a}}, \ldots, [!(A_n(z))^*]^{\boldsymbol{x}_n}_{\boldsymbol{a}_n\boldsymbol{x}_0\ldots\boldsymbol{x}_n\boldsymbol{w}\boldsymbol{a}} \vdash [(B(z))^*]^{\boldsymbol{b}\boldsymbol{x}_0\ldots\boldsymbol{x}_n\boldsymbol{a}}_{\boldsymbol{w}}.$$

I.e., assuming $!\mathsf{ne}^*(\boldsymbol{x}_0), \ldots, !\mathsf{ne}^*(\boldsymbol{x}_n), !\mathsf{ne}^*(\boldsymbol{w}), !(z \in^* \boldsymbol{a})$ we have

$$!\tilde{\forall}\boldsymbol{y}_0 \subseteq^* \boldsymbol{a}_0\boldsymbol{x}_0\ldots\boldsymbol{x}_n\boldsymbol{w}\boldsymbol{a}[(A_0(z))^*]^{\boldsymbol{x}_0}_{\boldsymbol{y}_0}, \ldots, !\tilde{\forall}\boldsymbol{y}_n \subseteq^* \boldsymbol{a}_n\boldsymbol{x}_0\ldots\boldsymbol{x}_n\boldsymbol{w}\boldsymbol{a}[(A_n(z))^*]^{\boldsymbol{x}_n}_{\boldsymbol{y}_n} \vdash [(B(z))^*]^{\boldsymbol{b}\boldsymbol{x}_0\ldots\boldsymbol{x}_n\boldsymbol{a}}_{\boldsymbol{w}}.$$

Applying Lemma 5.8 and Proposition 5.9 we can easily see that the assertion above is of the form $!\Gamma^* \vdash C^*$. Since from Propositions 2.1 and 2.3 we know that $!\Gamma^* \vdash_{\mathsf{ILL}^\omega_{\in^*}} C^*$ iff $(\Gamma^*)^i \vdash_{\mathsf{IL}^\omega_{\in}} (C^*)^i$, we conclude that there exist non-empty closed terms $\boldsymbol{a}_0, \ldots, \boldsymbol{a}_n, \boldsymbol{b}$ such that

$$\Delta, \tilde{\forall}\boldsymbol{y}_0 \subseteq \boldsymbol{a}_0\boldsymbol{x}_0\ldots\boldsymbol{x}_n\boldsymbol{w}\boldsymbol{a}|A_0(z)|^{\boldsymbol{x}_0}_{\boldsymbol{y}_0}, \ldots, \tilde{\forall}\boldsymbol{y}_n \subseteq \boldsymbol{a}_n\boldsymbol{x}_0\ldots\boldsymbol{x}_n\boldsymbol{w}\boldsymbol{a}|A_n(z)|^{\boldsymbol{x}_n}_{\boldsymbol{y}_n} \vdash |B(z)|^{\boldsymbol{b}\boldsymbol{x}_0\ldots\boldsymbol{x}_n\boldsymbol{a}}_{\boldsymbol{w}}$$

$$\text{with } \Delta :\equiv \mathsf{ne}(\boldsymbol{x}_0), \ldots, \mathsf{ne}(\boldsymbol{x}_n), \mathsf{ne}(\boldsymbol{w}), z \in \boldsymbol{a},$$

which is precisely the conclusion of the Soundness Theorem 3.3.

## 6. Final Remarks

It should be observed that usually bounded interpretations of $\mathsf{IL}^\omega$ (e.g. the bounded modified realizability) have primitive bounded quantifiers in the interpreted system. The interpretations are then designed so that these quantifiers are considered as empty of computational information. In our abstract setting, however, we do not consider that possibility since the bounded quantifier depends on the relation $x \in a$ which is a parameter for our general interpretation. In order to show that the bounded quantifier could be interpreted we would need to have an exact definition for the relation $\in$, or at least we would have to assume that it behaved well with respect to the interpretation that we were trying to define. Note, however, that in each of the three instances considered one can check that the bounded quantifiers are indeed interpretable. We refrained from stating general properties guaranteeing when this is the case.

Moreover, another implication of working in a parametrised setting is that we cannot precisely state the characterisation principles of the interpretation as such principles would have to be stated in a parametrised form. But, since we have not defined the interpretation of formulas containing parameters, it would be impossible to show that the principles were interpretable.


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