Hybrid Functional Interpretations of Linear and Intuitionistic Logic

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Abstract

This article shows how different functional interpretations can be combined into what we term *hybrid functional interpretations*. These hybrid interpretations work on the setting of a multi-modal linear logic. Functional interpretations of intuitionistic logic can be combined via Girard's embedding of intuitionistic logic into linear logic. We first show how to combine the usual Kreisel's modified realizability, Gödel's Dialectica interpretation, and the Diller-Nahm interpretation into a basic hybrid interpretation. We then prove a *monotone soundness theorem* for the basic hybrid interpretation, in the style of Kohlenbach's monotone interpretations. Finally, we present a *hybrid bounded functional interpretation* which, except for the additives, corresponds to a combination of the recently developed bounded functional interpretation and bounded modified realizability.

1 Introduction

This article deals with the problem of combining several functional interpretations when "mining" mathematical proofs for hidden computational content or bounds. Usually, each interpretation has its distinct features and limitations. The goal here is to maximise the applicability of these techniques by combining the best of each. For instance, Kreisel's modified realizability is well suited to deal with proofs that make heavy use of extensionality,

$$x \overset{\rho}{=} y \;\to\; fx \overset{\tau}{=} fy, \tag{1}$$

where primitive equality for basic types (say $n = m$ for numbers $n, m \in \mathbb{N}$) is assumed, and higher-type equality is defined as

$$f \overset{\rho\to\tau}{=} g \;:\equiv\; \forall x^\rho (fx \overset{\tau}{=} gx).$$

The Dialectica interpretation, however, requires witnesses for the universal quantifiers within $x \overset{\rho}{=} y$ of (1), which cannot be majorised in general [13] and hence cannot be expressed inside Gödel's system T. On the other hand, the Dialectica interpretation is ideal to handle (via the negative translation) classical proofs of Π_2-theorems, since it interprets the Markov principle

$$\neg\forall x A_{\mathsf{qf}}(x) \;\to\; \exists x \neg A_{\mathsf{qf}}(x)\ . \tag{2}$$

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The question we address here is: Can we have any meta-theorem about the unwinding of proofs which involve both full extensionally and the Markov principle? At present no proof interpretation is able to answer this question positively[1].

We propose a solution to this question via the use of linear logic (as a refinement of intuitionistic logic). Recall that intuitionistic proofs can be embedded into linear logic ones, with intuitionistic implications $A \to B$ translated as linear implications $!A \multimap B$. The difficulty of Dialectica in dealing with full extensionality is that the "negative information" in the assumption $!A \equiv !(x \overset{\rho}{=} y)$ of (1) should not (and cannot) be witnessed, i.e. the modality "!" should be given a modified realizability interpretation. That can be formalised by rewriting the extensionality axiom (1) using a "Kreisel modality" ($!_k A$) expressing that the information in the premise of the axiom schema should not be witnessed[2]

$$!_k(x \overset{\rho}{=} y) \ \multimap \ fx \overset{\tau}{=} fy \,. \tag{3}$$

In the case of Markov principle, since the premise of (2) corresponds in linear logic to $?\exists x A_{\mathsf{qf}}^{\perp}(x)$, the modality "?" should rather be treated as in Gödel's Dialectica interpretation, i.e. axiom (2) should be rewritten as

$$!_g \, ?_g \, \exists x \, A_{\mathsf{qf}}(x) \ \multimap \ \exists x \, ?_g \, A_{\mathsf{qf}}(x) \,. \tag{4}$$

For proofs which use both extensionality (3) and Markov principle (4), constructive information will be extracted whenever such a labelling of the modalities is possible.

This distinguished treatment of the modalities is possible because, as pointed out by Girard (cf. [3] and [8], p84), the modalities are not canonical, thus different modalities can coexist into a single system. We make use a *multi-modal linear logic*, which includes distinct modalities corresponding to each of the various functional interpretations.

The paper is organised as follows. For the rest of this introduction we present the formal system of multi-modal linear logic. In Section 2 we introduce a basic hybrid functional interpretation of the multi-modal system. Section 2.3 contains a few illustrative applications of this basic hybrid interpretation. In Section 3 we present an algorithm for decorating a (linear translation of a) given intuitionistic proof with different modalities, allowing us to apply the techniques developed here to the context of intuitionistic logic (and hence classical logic, via the negative translation). Finally, in Section 4 we consider a *monotone* soundness theorem for the hybrid interpretation, and a *bounded* variant of the hybrid interpretation. Due to the absence of the monotonicity property, the bounded hybrid interpretation does not apply to the additives. This bounded variant incorporates into a single interpretation (the additive-free fragment of) both the bounded modified realizability [4] and (a variant of) the bounded functional interpretation [5].

1.1 Multi-modal linear logic (in all finite types)

The set of *finite types* $\mathcal{T}$ is inductively defined by:

[1] By placing restrictions on the types involved, however, one can often guarantee the elimination of extensionality via Luckhardt's elimination procedure [17].

[2] Note that this generalises Spector's quantifier-free rule of extensionality (see [16]) since it allows us to derive $rs \overset{\tau}{=} rt$ from $s \overset{\rho}{=} t$ in any context of the form $!_k\Delta$.

$A_{\mathsf{at}}, A_{\mathsf{at}}^{\perp}$ (id)	$\dfrac{\Gamma, A \quad \Delta, A^{\perp}}{\Gamma, \Delta}$ (cut)	$\dfrac{\Gamma}{\pi\{\Gamma\}}$ (per)
$\dfrac{\Gamma[\boldsymbol{\gamma}_0], A \quad \Gamma[\boldsymbol{\gamma}_1], B}{\Gamma[(z)(\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1)], A \diamond_z B}$ $(\diamond_z)$	$\dfrac{\Gamma, A}{\Gamma, A \diamond_{\mathrm{t}} B}$ $(\diamond_{\mathrm{t}})$	$\dfrac{\Gamma, B}{\Gamma, A \diamond_{\mathrm{f}} B}$ $(\diamond_{\mathrm{f}})$
$\dfrac{\Gamma, A, B}{\Gamma, A \parr B}$ $(\parr)$ $\quad$ $\dfrac{\Gamma, A \quad \Delta, B}{\Gamma, \Delta, A \otimes B}$ $(\otimes)$	$\dfrac{\Gamma, A}{\Gamma, \forall z A}$ $(\forall)$	$\dfrac{\Gamma, A[t/z]}{\Gamma, \exists z A}$ $(\exists)$

Table 1: Pure classical linear logic

- $\mathbb{N}, \mathbb{B} \in \mathcal{T}$,
- if $\rho, \sigma \in \mathcal{T}$ then $\rho \to \sigma \in \mathcal{T}$.

For simplicity, we deal with only two basic finite types $\mathbb{N}$ (integers) and $\mathbb{B}$ (booleans). The multi-modal classical linear logic $\mathsf{LL}^{\omega}_{\mathsf{h}}$ is defined as follows[3]. The terms of $\mathsf{LL}^{\omega}_{\mathsf{h}}$ contain all typed λ-terms, i.e. variables x^{ρ} for each finite type ρ, λ-abstractions $(\lambda x^{\rho}.t^{\sigma})^{\rho\to\sigma}$, applications $(t^{\rho\to\sigma}s^{\rho})^{\sigma}$, and conditionals $(s^{\mathbb{B}})(t^{\rho}, r^{\rho})$. The atomic formulas of $\mathsf{LL}^{\omega}_{\mathsf{h}}$ are $A_{\mathsf{at}}, B_{\mathsf{at}}, \ldots$ and $A_{\mathsf{at}}^{\perp}, B_{\mathsf{at}}^{\perp}, \ldots$. For simplicity, the standard propositional constants $0, 1, \perp, \top$ of linear logic have been omitted, since the interpretation of atomic formulas is trivial (see Definition 2.1). Formulas of $\mathsf{LL}^{\omega}_{\mathsf{h}}$ are built from atomic formulas via:

- connectives $A \parr B$ (par), $A \otimes B$ (tensor), $A \diamond_z B$ (if-then-else),
- quantifiers $\forall x A$ and $\exists x A$, and
- modalities described below.

The *linear implication* $A \multimap B$ abbreviates $A^{\perp} \parr B$, where the *linear negation* $(\cdot)^{\perp}$ is an abbreviation such that $(A^{\perp})^{\perp}$ is syntactically equal to A (see [7, 19]). Note that (following [20]) we have deviated from the standard formulation of linear logic and use the if-then-else logical constructor $A \diamond_z B$ instead of standard additive conjunction and disjunction[4]. In terms of quantification over booleans, the standard additives can be defined as

$$A \wedge B \;:\equiv\; \forall z^{\mathbb{B}}(A \diamond_z B) \qquad\qquad A \vee B \;:\equiv\; \exists z^{\mathbb{B}}(A \diamond_z B)$$

The logical rules of $\mathsf{LL}^{\omega}_{\mathsf{h}}$ are shown in Table 1 (see also [7, 19]).

[3] We will use LL^{ω} to denote the standard system of linear logic in all finite types.

[4] See Girard's comments in [7] (p13) and [8] (p73) on the relation between the additive connectives and the if-then-else construct.

$$\frac{?_Y\Gamma, A}{?_Y\Gamma,\ !_X A}\,(!_X) \qquad \frac{\Gamma, A}{\Gamma, ?_X A}\,(?_X)$$

$$\frac{\Gamma, ?_{Z_0}A, ?_{Z_1}A}{\Gamma, ?_X A}\,(\mathrm{con}_X, \star) \qquad \frac{\Gamma}{\Gamma, ?_X A}\,(\mathrm{wkn}_X)$$

Table 2: Rules for the exponentials (where $X, Y \in \{k < d < g\}$ and $Y \leq X \leq Z_i$)

The author [19, 20] has recently studied possible different interpretations for the exponentials ! and ?, and how these correspond to well-known functional interpretations of intuitionistic logic. We here introduce syntactically distinct exponentials (see Table 2) and show how these different interpretations can coexist (whence the "hybrid" denomination). We consider here the "Kreisel", "Diller-Nahm" and "Gödel" modalities, denoted $!_k, !_d$, and $!_g$, respectively (together with their duals $?_k, ?_d$ and $?_g$). This will correspond to a combination of Kreisel's modified realizability, Diller-Nahm interpretation and Gödel's Dialectica interpretation into a single functional interpretation which supersedes all of them.

The rules for all three exponentials are presented in Table 2, where $?_Y\Gamma \equiv ?_Y B_0, \ldots, ?_Y B_n$. Note that an "information ordering" is assumed on the distinct modalities, and this ordering allows for some information to be lost in the promotion and contraction rules. This is because, as will be reflected in the hybrid interpretation given below, the Gödel "whynot" is meant to carry a finer information than $?_d$, and the $?_d$ a finer information than $?_k$ (symmetrically for the !).

Definition 1.1 (Computation/refutation relevant, and fixed formulas) *Let* CR *(computation relevant) denote the smallest classes of formulas satisfying:*

- $\exists x A \in \mathsf{CR}$,
- *if* $A \in \mathsf{CR}$ *then* $\{\forall x A, ?_d A, ?_g A\} \subseteq \mathsf{CR}$,
- *if* $A \in \mathsf{CR}$ *then* $\{!_k A, !_d A, !_g A\} \subseteq \mathsf{CR}$,
- *if* $A_i \in \mathsf{CR}$ *then* $A_0 \,\square\, A_1 \in \mathsf{CR} \quad (\square \in \{\, \parr, \otimes, \diamond_z \,\})$.

Also, let RR *(refutation relevant) denote the class of formulas A such that $A^\perp \in$* CR*. We call a formula A computation (resp. refutation) irrelevant if it is not computation (resp. refutation) relevant. A formula which is both computation and refutation irrelevant will be called a* fixed formula.

The computation irrelevant formulas correspond to the intuitionistic notion of Harrop formulas[5]. Refutation relevant formulas are the dual notion. In mixing the three interpretations, we must add also the following restriction on the "Gödel" contraction rule con_g:

[5]Recall that a formula A is called *Harrop* if it does not contain a strictly positive sub-formula of the kind $\exists x B$ (cf. [21]).

($\star$) if the contraction formula A in con_g is *computation relevant*, then it must not contain any Kreisel whynot $?_k$ in front of a *computation relevant* subformula, nor any Kreisel bang $!_k$ in front of a *refutation relevant* subformula.

As we will see, condition ($\star$) ensures that the interpretation of a contraction formula A is decidable (assuming that bounded formulas are decidable).

Finally, we assume that $\mathsf{LL}_{\mathsf{h}}^{\omega}$ contains equality (together with defining axioms) for the basic types $\mathbb{B}, \mathbb{N}$. Higher order equality is defined as

$$f \overset{\rho\to\tau}{=} g \;:\equiv\; \forall x^{\rho}(fx \overset{\tau}{=} gx).$$

We then assume the (Kreisel) extensionality schema

$$!_k(x \overset{\rho}{=} y) \multimap fx \overset{\tau}{=} fy. \tag{5}$$

2 Hybrid Interpretation of Linear Logic

To each formula A of $\mathsf{LL}_{\mathsf{h}}^{\omega}$ (multi-modal linear logic) we associate a formula $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ of LL^{ω} (standard linear logic), where $\boldsymbol{x}, \boldsymbol{y}$ are fresh variables not appearing in A. The length and types of $\boldsymbol{x}, \boldsymbol{y}$ are inductively determined by the logical structure of the formula A. Therefore, for the sake of readability we will avoid writing types explicitly. The variables $\boldsymbol{x}$ in the superscript are called the *witnessing variables*, while the subscript variables $\boldsymbol{y}$ are called the *challenge variables*. Intuitively, the interpretation of A is a two-player (Eloise and Abelard) one-move game, where $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ is the adjudication relation. We want that Eloise has a winning move whenever A is provable. Moreover, the proof of A will provide Eloise's winning move $\boldsymbol{a}$, i.e., $\forall \boldsymbol{y} |A|^{\boldsymbol{a}}_{\boldsymbol{y}}$ will hold, where $\boldsymbol{a}$ is a tuple of terms of the corresponding types.

Definition 2.1 (Hybrid Interpretation) *The interpretation of atomic formulas are the atomic formulas themselves, with empty sets of witnessing and challenge variables, i.e.* $|A_{\mathsf{at}}| :\equiv A_{\mathsf{at}}$ *and* $|A_{\mathsf{at}}^{\perp}| :\equiv A_{\mathsf{at}}^{\perp}$. *Assuming* $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ *and* $|B|^{\boldsymbol{v}}_{\boldsymbol{w}}$ *already defined, we define*

$$\begin{array}{lcl}
|A \parr B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{y},\boldsymbol{w}} & :\equiv & |A|^{\boldsymbol{f}\boldsymbol{w}}_{\boldsymbol{y}} \parr |B|^{\boldsymbol{g}\boldsymbol{y}}_{\boldsymbol{w}} \\
|A \otimes B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{f},\boldsymbol{g}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{v}} \otimes |B|^{\boldsymbol{v}}_{\boldsymbol{g}\boldsymbol{x}} \\
|A \diamond_z B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \diamond_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
|\exists z A(z)|^{\boldsymbol{x},z}_{\boldsymbol{f}} & :\equiv & |A(z)|^{\boldsymbol{x}}_{\boldsymbol{f}z} \\
|\forall z A(z)|^{\boldsymbol{f}}_{\boldsymbol{y},z} & :\equiv & |A(z)|^{\boldsymbol{f}z}_{\boldsymbol{y}}.
\end{array}$$

The three sets of modalities are given different interpretations as[6]

[6]We are assuming a language extended with finite-multiset types τ^* for every type τ, together with primitive constructs such as singleton sets $\{\cdot\} : \tau \to \tau^*$ and union $(\cdot) \cup (\cdot) : \tau^* \times \tau^* \to \tau^*$. We assume also that a family of primitive binary relations $x^{\tau} \in a^{\tau^*}$ (with simple universal axioms) is also available, and that $\exists x^{\tau} \in a^{\tau^*} A$ is an abbreviation for $\exists x(x \in a \wedge A)$ (similarly for universal quantifiers).

$$
\begin{array}{rclcrcl}
|!_k A|^{\boldsymbol{x}} & :\equiv & !\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} & \qquad & |?_k A|_{\boldsymbol{y}} & :\equiv & ?\exists \boldsymbol{x} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \\
|!_d A|^{\boldsymbol{x}}_{\boldsymbol{f}} & :\equiv & !\forall \boldsymbol{y} \in \boldsymbol{f}\boldsymbol{x}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} & & |?_d A|^{\boldsymbol{f}}_{\boldsymbol{y}} & :\equiv & ?\exists \boldsymbol{x} \in \boldsymbol{f}\boldsymbol{y}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \\
|!_g A|^{\boldsymbol{x}}_{\boldsymbol{f}} & :\equiv & !|A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{x}} & & |?_g A|^{\boldsymbol{f}}_{\boldsymbol{y}} & :\equiv & ?|A|^{\boldsymbol{f}\boldsymbol{y}}_{\boldsymbol{y}}.
\end{array}
$$

It is easy to check that $|A^{\perp}|^{\boldsymbol{y}}_{\boldsymbol{x}} \equiv (|A|^{\boldsymbol{x}}_{\boldsymbol{y}})^{\perp}$ *and thus* $|A \multimap B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} \equiv |A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{w}} \multimap |B|^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{w}}$.

Note that the games $|?_k A|_{\boldsymbol{y}}$, $|?_d A|^{\boldsymbol{f}}_{\boldsymbol{y}}$ and $|?_g A|^{\boldsymbol{f}}_{\boldsymbol{y}}$ correspond to a break of symmetry in the game $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$, where Eloise has access to Abelard's move, i.e. Abelard plays first, and Eloise observes Abelard's move. Moreover, Eloise is then allowed to play a *set of moves*, and wins if any move in the set is winning. The nature of the set, however, changes with the different interpretations: it can be whole set of possible moves ($?_k$), a finite set ($?_d$), or a singleton set ($?_g$). Dually for the games $|!_k A|^{\boldsymbol{x}}$, $|!_d A|^{\boldsymbol{x}}_{\boldsymbol{f}}$ and $|!_g A|^{\boldsymbol{x}}_{\boldsymbol{f}}$.

Proposition 2.2 *Let* $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ *be the hybrid interpretation on* A. *Then the following holds*

(i) $A \in \mathsf{CR}$ *iff the tuple* $\boldsymbol{x}$ *is not empty.*

(ii) $A \in \mathsf{RR}$ *iff the tuple* $\boldsymbol{y}$ *is not empty.*

Theorem 2.3 (Soundness of hybrid interpretation) *Let* $A_0, \ldots, A_n$ *be a sequence of formulas of* $\mathsf{LL}^{\omega}_{\mathsf{h}}$, *with* $\boldsymbol{z}$ *as the only free-variables. If*

$$\vdash_{\mathsf{LL}^{\omega}_{\mathsf{h}}} A_0, \ldots, A_n$$

then terms $\boldsymbol{a}_0, \ldots, \boldsymbol{a}_n$ *can be synthesised from its formal proof, such that*

$$\vdash_{\mathsf{LL}^{\omega}} |A_0|^{\boldsymbol{a}_0}_{\boldsymbol{y}_0}, \ldots, |A_n|^{\boldsymbol{a}_n}_{\boldsymbol{y}_n},$$

where $\mathsf{FV}(\boldsymbol{a}_i) \in \{\boldsymbol{z}, \boldsymbol{y}_0, \ldots, \boldsymbol{y}_n\} \backslash \{\boldsymbol{y}_i\}$.

Proof. The soundness proof where only the Kreisel modality is considered is given in [20]. The interpretation of the Gödel and Diller-Nahm modalities are shown to be (independently) sound in [19]. In order to obtain the soundness of the hybrid interpretation we just need to observe that these three different modalities only interfere with each other in the promotion and contraction rules, where "loss of information" is allowed. Consider, for instance, the promotion rule (first rule on Table 2) where $Y = d$ and $X = g$. We have:

$$
\cfrac{\cfrac{\cfrac{|?_d \Gamma|^{\boldsymbol{\gamma}[\boldsymbol{y}]}_{\boldsymbol{w}}, |A|^{\boldsymbol{a}[\boldsymbol{w}]}_{\boldsymbol{y}}}{?\exists \boldsymbol{v} \in \boldsymbol{\gamma}[\boldsymbol{y}]\boldsymbol{w}\, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}}, |A|^{\boldsymbol{a}[\boldsymbol{w}]}_{\boldsymbol{y}}}}{?\exists \boldsymbol{v} \in \boldsymbol{\gamma}[\boldsymbol{f}(\boldsymbol{a}[\boldsymbol{w}])]\boldsymbol{w}\, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}}, !|A|^{\boldsymbol{a}[\boldsymbol{w}]}_{\boldsymbol{f}(\boldsymbol{a}[\boldsymbol{w}])}} \left[\frac{\boldsymbol{f}(\boldsymbol{a}[\boldsymbol{w}])}{\boldsymbol{y}}\right]}{|?_d \Gamma|^{\lambda \boldsymbol{w}.\boldsymbol{\gamma}[\boldsymbol{f}(\boldsymbol{a}[\boldsymbol{w}])]\boldsymbol{w}}_{\boldsymbol{w}}, |!_g A|^{\boldsymbol{a}[\boldsymbol{w}]}_{\boldsymbol{f}}}
$$

Similarly, for the other combinations, and the contraction rule. One has also to be careful with the Gödel modalities, due to its side condition that the interpretation of the contraction formula has to be decidable. That is, however, guaranteed by our restriction ($\star$). $\Box$

2.1 Interpretable principles

We call a principle P *interpretable*, for a given fixed interpretation $|\cdot|$, if there exists a term $\boldsymbol{t}$ such that $\vdash_{\mathsf{LL}^\omega} |P|^{\boldsymbol{t}}_{\boldsymbol{y}}$. We have shown in Theorem 2.3 that every theorem A of multi-modal linear logic is interpretable. It turns out some other principles, not provable in LL^ω, are also interpretable.

Before we proceed to discuss the interpretable principles of the hybrid interpretation, let us extend linear logic with a variant of Henkin's branching quantifier, which we call *simultaneous quantifier*. The logical rule for the simultaneous quantifier is as follows:

$$\frac{A_0(\boldsymbol{a}_0, \boldsymbol{y}_0), \ldots, A_n(\boldsymbol{a}_n, \boldsymbol{y}_n)}{\exists\!\!\!\forall^{\boldsymbol{x}_0}_{\boldsymbol{y}_0} A_0(\boldsymbol{x}_0, \boldsymbol{y}_0), \ldots, \exists\!\!\!\forall^{\boldsymbol{x}_n}_{\boldsymbol{y}_n} A_n(\boldsymbol{x}_n, \boldsymbol{y}_n)}\ (\exists\!\!\!\forall)$$

with the side-condition that $\boldsymbol{y}_i$ may only appear free in the terms $\boldsymbol{a}_j$, for $j \neq i$. In particular, we will have that each $\boldsymbol{y}_i$ will not be free in the conclusion of the rule. We assume that when $\boldsymbol{x}_i$ and $\boldsymbol{y}_i$ are empty tuples, the quantifier is omitted. Therefore, the simultaneous quantifiers generalise both the universal and existential quantifiers. For instance, when $\boldsymbol{a}_i$, for $0 \leq i \leq n$, are empty and $\boldsymbol{y}_j$, for $0 \leq j < n$, are empty, the rule above becomes the universal introduction rule (with side formulas $A_0, \ldots, A_{n-1}$).

The hybrid interpretation can be extended to deal with the simultaneous quantifiers as (cf. [20])

$$|\exists\!\!\!\forall^{\boldsymbol{v}}_{\boldsymbol{w}} A(\boldsymbol{v}, \boldsymbol{w})|^{\boldsymbol{f},\boldsymbol{v}}_{\boldsymbol{g},\boldsymbol{w}} \quad :\equiv \quad |A(\boldsymbol{v}, \boldsymbol{w})|^{\boldsymbol{f}\boldsymbol{w}}_{\boldsymbol{g}\boldsymbol{v}}.$$

Proposition 2.4 (Characterisation) *Let A_{ri} denote an arbitrary refutation irrelevant formula, and $A_{\mathsf{fix}}, B_{\mathsf{fix}}$ denote fixed formulas. The following principles are interpretable by the hybrid functional interpretation described above:*

$$\begin{array}{lcl}
\mathsf{AC}_s & : & \forall z \exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A_{\mathsf{ri}}(\boldsymbol{x}, \boldsymbol{y}, z) \multimap \exists\!\!\!\forall^{\boldsymbol{f}}_{\boldsymbol{y},z} A_{\mathsf{ri}}(\boldsymbol{f}z, \boldsymbol{y}, z), \\
\mathsf{AC}_p & : & (\exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A_{\mathsf{fix}}(\boldsymbol{y}) \multimap \exists\!\!\!\forall^{\boldsymbol{v}}_{\boldsymbol{w}} B_{\mathsf{fix}}(\boldsymbol{v})) \multimap \exists\!\!\!\forall^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} (A_{\mathsf{fix}}(\boldsymbol{f}\boldsymbol{w}) \multimap B_{\mathsf{fix}}(\boldsymbol{g}\boldsymbol{x})), \\
\mathsf{TA} & : & !_* \exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A \multimap \exists \boldsymbol{x} !_* \forall \boldsymbol{y} A, \quad (* \in \{k, d, g\}) \\
\mathsf{MP}^g_l & : & ?_g \exists \boldsymbol{x} A_{\mathsf{ri}} \multimap \exists \boldsymbol{x} ?_g A_{\mathsf{ri}}, \\
\mathsf{MP}^d_l & : & ?_d \exists \boldsymbol{x} A_{\mathsf{ri}} \multimap \exists \boldsymbol{s} ?_d \exists \boldsymbol{x} \in \boldsymbol{s}\, A_{\mathsf{ri}}.
\end{array}$$

These principles are also sufficient to characterise the hybrid interpretation, meaning that they prove the equivalence between A and its interpretation[7] $\exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$.

The abbreviations above stand for: sequential and parallel choice, trump advantage, Gödel Markov principle, and Diller-Nahm Markov principle, respectively.

It is well known that both the Dialectica interpretation and modified realizability, for instance, interpret the axiom of choice

$$\mathsf{AC} \quad : \quad !\forall z \exists \boldsymbol{x} A(\boldsymbol{x}, z) \multimap \exists \boldsymbol{f} \forall z A(\boldsymbol{f}z, z)$$

[7]Note that A is a formula of $\mathsf{LL}^\omega_{\mathsf{h}}$, whereas $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ is a formula of the standard LL^ω. For the equivalence above we are assuming that $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ is translated back into $\mathsf{LL}^\omega_{\mathsf{h}}$, by labelling the modalities in $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ following the structure of A.

for arbitrary matrices $A(\boldsymbol{x}, z)$. Note, however, that AC is weaker than its purely linear variant (without the ! in the premise), since the bang (!) makes the premise stronger, and hence the whole principle weaker. As expected, AC as above is also interpretable by the hybrid interpretation (for any choice of !), and, as such, it is provable from the principles above.

One might consider splitting the principle AC_p into an initial prenexation following by an application of (a suitable form of) the axiom of choice, as done in the characterisation of the Dialectica interpretation. In our context, however, this initial prenexation would need a proper Henkin quantifier, going from $\exists\!\!\!/\,^{\boldsymbol{x}}_{\boldsymbol{y}} A_{\mathsf{fix}}(\boldsymbol{y}) \multimap \exists\!\!\!/\,^{\boldsymbol{v}}_{\boldsymbol{w}} B_{\mathsf{fix}}(\boldsymbol{v})$ to

$$\left(\begin{array}{c}\forall \boldsymbol{x} \exists \boldsymbol{v} \\ \forall \boldsymbol{w} \exists \boldsymbol{y}\end{array}\right)(A_{\mathsf{fix}}(\boldsymbol{y}) \multimap B_{\mathsf{fix}}(\boldsymbol{v})),$$

which can only be expressed with our (simpler) simultaneous quantifier once a "choice step" is performed.

The fact that TA is valid for all three interpretations, for arbitrary formulas A, suggests that this should probably be a valid principle of linear logic. In particular (even in LL^ω, without simultaneous quantifiers), the commuting property $!\exists x A \multimap \exists x !A$ should be derivable in linear logic. The intuitive justification in terms of games is as follows: Although the game $!\exists x A$ consists of several copies of the game $\exists x A$, Eloise must make a uniform move for all copies of the game. Hence, it is actually as if she is playing the game $\exists x !A$. It would be interesting to investigate if other interpretations of linear logic (other than game interpretations) also validate this principle.

2.2 Self-interpretable principles

We call a principle P *self-interpretable*, for a fixed given interpretation $|\cdot|$, if there exists a term $\boldsymbol{t}$ such that $\vdash_{\mathsf{LL}^\omega + P} |P|^{\boldsymbol{t}}_{\boldsymbol{y}}$. Clearly, every interpretable principle is self-interpretable. Not every principle, however, is self-interpretable, since the hybrid interpretation may lead to a strict strengthening of P. For instance, the following principle

$$\forall F^{(\mathbb{N}\to\mathbb{N})\to\mathbb{N}} \forall f, g \leq 1(!_g \forall n(fn = gn) \to Ff = Fg)$$

is not self-interpretable, since the hybrid interpretation will ask for a close primitive recursive term t satisfying

$$\forall F^{(\mathbb{N}\to\mathbb{N})\to\mathbb{N}}, f, g \leq 1(!_g(f(tF) = g(tF)) \to Ff = Fg),$$

which (as shown by Howard [13]) is impossible. We list bellow some principles which are self-interpretable for the hybrid interpretation presented above:

$$\begin{array}{lll} \mathsf{EXT} & : & !_k(x \overset{\rho}{=} y) \multimap fx \overset{\tau}{=} fy \\ \mathsf{IND} & : & !_* \forall n(A(n) \multimap A(n+1)) \multimap \forall k(A(0) \multimap A(k)) \quad (* \in \{k, d, g\}) \end{array}$$

where $x = y$ is as defined at the end of Section 1.1.

2.3 Simple applications of hybrid interpretation

In this section we list some classes of theorems where it might pay off to analyse proofs using a combination of the Kreisel, Diller-Nahm and Gödel modalities. We focus on theorems where using only one single interpretation would not *directly* yield the desired program. If some arbitrary pre-processing of the given proof is allowed, it might be possible that one can always obtain the same result *indirectly* using a single functional interpretation on the pre-processed proof. For instance, in Example 1, if A is quantifier-free one could take $\forall x A$ is a Π_1 axiom, and analyse the proof of $\forall y B \to \forall z C$ using the Dialectica interpretation, obtaining the same result. Using the hybrid interpretation directly allows us to make full use of the modularity of functional interpretations, which does not seem to be possible when (different parts of) given proofs are allowed to be modified (in different ways) before a single functional interpretation is applied. Moreover, the inter-dependencies between variables which are witness and those which not witnesses can be very subtle. This begs for an automated process which only seems to be possible via the hybrid interpretation (see Section 3, in particular the example in Section 3.1).

Example 1

Consider theorems of the form

$$\forall x A \to \forall y B \to \forall z C \tag{6}$$

possibly with parameters, where the negative information on x is irrelevant, while the one on y is of our interest. In this case, we would rather view this theorem as

$$!_k \forall x A \multimap !_g \forall y B \multimap \forall z C \ . \tag{7}$$

For instance, consider the simple intuitionistic theorem

$$\forall x(f(x) \leq 1) \to \forall y(f(y) \neq f(y+1)) \to \forall z(f(z) = f(z+2)). \tag{8}$$

From a proof of this, using labelling (7), our hybrid interpretation extracts a realizer $\Phi(f, z)$ s.t.

$$\forall z(\forall x(f(x) \leq 1) \to (f(\Phi(f,z)) \neq f(\Phi(f,z)+1)) \to (f(z) = f(z+2))).$$

Indeed, one such witness is $\Phi(f, z) := \mathsf{if}\ (f(z+1) = f(z+2))\ \mathsf{then}\ z\ \mathsf{else}\ z+1$. The modified realizability of (8) would not yield any information, since the theorem is existential-free. On the other hand, the Dialectica interpretation of (8) would witness both x and y, giving rise to two programs $\Phi(f, z)$ and $\Psi(f, z)$ satisfying the stronger statement

$$\forall z((f(\Psi(f,z)) \leq 1) \to (f(\Phi(f,z)) \neq f(\Phi(f,z)+1)) \to (f(z) = f(z+2))).$$

For a further example of a concrete theorem having the form (6) see Section 3.1.

Example 2

Examples of the form (6) above can come up when analysing classical proofs of theorems[8]

$$\forall x A \to \forall y \exists z B \tag{9}$$

[8] Many thanks to Mircea-Dan Hernest for suggesting this class of examples, and in particular the theorem about Fibonacci sequences.

since these can be translated into intuitionistic proofs of

$$\forall y(\forall x A \to \forall z \neg B \to \perp) \tag{10}$$

which again has the form (6). One such example is that of the classical existence proof of the Fibonacci sequence, first used in [1] to illustrate the so-called "refined A-translation" and then in [10] to illustrate the light Dialectica (see also Section 4.3 of [9]). The semi-classical Fibonacci proof is a minimal-logic proof of $\forall y \tilde{\exists} z B(y,z)$, where

$$\tilde{\exists} z B(y,z) :\equiv \forall z(B(y,z) \to \perp) \to \perp$$

from assumptions expressing that B is the graph of the Fibonacci function (B is viewed as a predicate constant without computational content), i.e., $B(0,0), B(1,1)$ and

$$\forall x_1, x_2, x_3(B(x_1,x_2) \to B(x_1+1,x_3) \to B(x_1+2,x_2+x_3)).$$

Note that such a specification fits into the form (6) (with $C :\equiv \perp$).

Example 3

Consider also theorems of the form

$$\forall x \forall y A \to B \tag{11}$$

where x can be witnessed precisely but y can only be approximated by a finite set. So, this would be translated as

$$!_g \forall x !_d \forall y A \multimap B.$$

For instance, consider the following simple theorem:

$$\forall x, y(f(x+y) \geq x f(y)) \to \forall n^{\mathsf{even}}(f(n) \geq 2^n f(0)).$$

It is easy to see that only $x = 2$ is needed from the assumption, whereas $y \in \{0, 2, \ldots, n\}$ must be used. Therefore, we have the following stronger theorem

$$\forall n^{\mathsf{even}}(\forall y \in \{0, 2, \ldots, n-2\}(f(2+y) \geq 2f(y)) \to f(n) \geq 2^n f(0)).$$

Example 4

Real numbers are normally represented in formal systems as Cauchy sequences of rationals with a fixed rate of convergence. A real number being positive carries the extra information of a lower bound on how far from zero the limit of the sequence can be (cf. [16]). In order to avoid going into the representation level, when analysing the proof that a certain real function f is positive at x, i.e. $f(x) >_{\mathbb{R}} 0$, it is often useful to view this as $\exists l(f(x) >_{\mathbb{R}} 2^{-l})$. Although witnessing l gives us some lower bound on the value of $f(x)$ the formula $f(x) >_{\mathbb{R}} 2^{-l}$ still carries information on how far above 2^{-l} the value of $f(x)$ is. This extra information is usually irrelevant in practice and the purely existential matrix can be treated as quantifier-free, given that we can always forget these witnesses later. When automatising program extraction, it thus proves to be useful to make sure that the interpretation will not witness the innermost existential quantifier at all. This can be achieved by viewing the statement $f(x) >_{\mathbb{R}} 0$ as $\exists l ?_k(f(x) >_{\mathbb{R}} 2^{-l})$.

Consider the following example

$$\forall f^{\mathbb{N}\to\mathbb{R}}(\forall m(f(m) <_{\mathbb{R}} f(m+1)) \to \forall n(f(n) <_{\mathbb{R}} f(n+2))).$$

Note that $<_{\mathbb{R}}$ is an undecidable relation, but assume we are not interested in the information hidden within $f(m) <_{\mathbb{R}} f(m+1)$. For the sake of program-extraction, the formula above is thus better labelled as

$$\forall f^{\mathbb{N}\to\mathbb{R}}(!_d \forall m ?_k (f(m) <_{\mathbb{R}} f(m+1)) \multimap \forall n ?_k (f(n) <_{\mathbb{R}} f(n+2)))$$

We can produce a finite collection of witnesses for m as $\Phi(f,n) := \{n, n+1\}$ so that

$$\forall f^{\mathbb{N}\to\mathbb{R}}, n(\forall m \in \Phi(f,n)\,(f(m) <_{\mathbb{R}} f(m+1)) \to (f(n) <_{\mathbb{R}} f(n+2))).$$

Example 5

The Dialectica interpretation and modified realizability also treat the induction rule[9]

$$\frac{A(0) \quad A(n) \to A(n+1)}{A(k)}\ (\mathsf{IND})$$

in slightly different ways. In both cases, the proofs of $A(0)$ and $A(n) \to A(n+1)$ provide a realiser $\boldsymbol{t}[k]$ for the witnessing variables of $A(k)$, i.e., $|A|^{\boldsymbol{t}}_{\boldsymbol{y}}$. However, only during the extraction of $\boldsymbol{t}$ via Dialectica interpretation a functional which refutes $A(n)$ when given a refutation for $A(n+1)$ will also be extracted. Such realizer is nonetheless not used in the construction of the desired term $\boldsymbol{t}$. Therefore we could choose to always treat induction in the way modified realizability does, even when constructing a Dialectica witness. In our multi-modal setting, this can be achieved by formulating induction as

$$\frac{A(0) \quad !_k A(n) \multimap A(n+1)}{A(k)}\ (\mathsf{IND})$$

since the Kreisel modality blocks the witnessing of counter-example flows.

3 Hybrid Interpretation Applied to Intuitionistic Logic

Recall that intuitionistic logic can be embedded into linear logic as follows:

Definition 3.1 ([7]) *For any formula A of intuitionistic logic its linear translation A^* is defined inductively as*

$$\begin{array}{lcl} A^*_{\mathsf{at}} & :\equiv & A_{\mathsf{at}} \\ (A \diamond_z B)^* & :\equiv & A^* \diamond_z B^* \\ (A \to B)^* & :\equiv & !A^* \multimap B^* \\ (\forall x A)^* & :\equiv & \forall x A^* \\ (\exists x A)^* & :\equiv & \exists x ! A^*. \end{array}$$

[9] The induction stated here corresponds to the induction rule with no open assumption in natural deduction systems.

If A is provable in intuitionistic logic then A^* is provable in linear logic.

In this section we discuss how the hybrid interpretation of $\mathsf{LL}_{\mathsf{h}}^{\omega}$ can be combined (via the embedding above) to yield a hybrid interpretation of IL^{ω}. Let us assume we are starting with an intuitionistic proof π of a theorem A, together with the desired information (i.e., quantified variables of A to be realized) in the form of a labelling of the linear translation of A, i.e. A^*. We wish to show how the proof π can be automatically translated into a multi-modal linear logic proof, with the modalities decorated in such way that a proof analysis (via the hybrid interpretation) will give us the information requested about A (whenever this is possible). For instance, in a theorem of the form (9) it could be that we are interested only in the negative universal information x, and not in the positive existential information z. Hence we rather present (9) as a specification in multi-modal linear logic decorated like

$$!_g \forall x A \multimap \forall y ?_k \exists z B.$$

In Table 3 we describe an algorithm which can ascertain whether such labelling of the theorem can be propagated through the whole proof or not. If the algorithm succeeds, the hybrid interpretation can then be applied and will return a realizer t and a linear logic proof of

$$\forall y(!A[ty/x] \multimap ?\exists z B)$$

which can finally be translated back to an intuitionistic proof of

$$\forall y(A[ty/x] \rightarrow \exists z B).$$

Theorem 3.2 *Let π be a proof of A in intuitionistic logic. Let A_l^* be a labelling in $\mathsf{LL}_{\mathsf{h}}^{\omega}$ of A^*. If it is possbile for the modalities in π^* (the translation of the proof π into linear logic) to be labelled in such way that it yields a proof of A_l^* in $\mathsf{LL}_{\mathsf{h}}^{\omega}$, then the algorithm of Table 3 will return one such labelling.*

Proof. Assume a possible labelling of π^* exists. Then, it is easy to check that it must satisfy all the equations and inequalities generated by the algorithm of Table 3. Therefore, the equations and inequalities generated are solvable, and the solution we get must also yield a (possibly different) correct labelling. $\square$

3.1 Example illustrating labelling algorithm

In this section we show (through a simple example) how the labelling algorithm described above works.

Theorem. Let $f^{\mathbb{N}\to\mathbb{N}}$ be a function such that

$$A \equiv \forall x^{\mathbb{N}} \underbrace{(f(x) = 3f(x+1))}_{A_0(x)},$$

and assume $F^{(\mathbb{N}\to\mathbb{N})\to\mathbb{N}}$ is an operator satisfying

Input: Intuitionistic proof π of theorem A plus decoration of A^*

Output: Decoration of π^*, if possible, which respects given decoration of A^*

Decorate whole proof π^* bottom up, starting from theorem and working towards the axioms

For the modalities in A^* not yet labelled, associate fresh label variable $X, Y, \ldots$

For each rule encountered do the following:

- Connectives/quantifiers: simply propagate assignment from conclusion to premise(s)
- Cut rule: assign the modalities in the cut formula with fresh variable labels $X, Y, \ldots$
- Promotion rule: if conclusion is $?_{r_0}B_0, \ldots, ?_{r_n}B_n, !_tA$ premise becomes $?_{r_0}B_0, \ldots, ?_{r_n}B_n, A$
  generate inequality $r_0 \leq t$ and equalities $r_i = r_j$
- Contraction rule: if conclusion is $\Gamma, ?_rA$ then premise becomes $\Gamma, ?_{Z_0}A, ?_{Z_1}A$
  generated inequalities ($r \leq Z_i$)
  if $?_r$ is the Gödel modality, check that condition ($\star$) is satisfied
- Dereliction and weakening rule: simply propagate assignment
- Axiom: generate unification equations for the assignments of A and $A^\perp$

Solve generated set of equations and inequalities

If assignment found return it, otherwise return "not possible"

Table 3: Labelling algorithm

$$B \equiv \; \forall y^{\mathbb{N}} \forall g \underbrace{(F(yg) = yF(g))}_{B_0(y,g)},$$

$$C \equiv \; \forall z^{\mathbb{N}} \underbrace{(F(f) \leq F(\lambda x.f(x+z)) + z)}_{B_0(z)}.$$

From this we can conclude

$$D \equiv \; F(2f) \leq 3.$$

Proof. By assumption A and extensionality we get $F(\lambda x.3f(x+1)) = F(f)$. By B we have $3F(\lambda x.f(x+1)) = F(f)$, whereas by C we get $2F(f) \leq 3$. Finally, by B again we get $F(2f) \leq 3$. The theorem we have proved is

$$\forall x A_0(x) \wedge \forall y, g B_0(y,g) \wedge \forall z C_0(z) \to D.$$

Despite the use of extensionality over the assumption A, it is clear that assumptions B and C can be weakened. In fact, the following stronger theorem holds

$$\forall x A_0(x) \wedge \forall y \in \{2,3\} \forall g \in \{f, \lambda x.f(x+1)\} B_0(y) \wedge C_0(1) \to D.$$

We will now show how this stronger theorem can be obtained from the proof above by the hybrid functional interpretation, using the labelling algorithm described in Table 3.

We start by translating the theorem into linear logic, and choosing an appropriate labelling of the modalities. In our case, a successful labelling would be

$$!_k \forall x A_0(x) \otimes !_d \forall y, g B_0(y, g) \otimes !_g \forall z C_0(z) \multimap D.$$

From this initial labelling, following the algorithm of Table 3 we get a successful labelling of the (linear translation of) intuitionistic proof as[10]

$$
\begin{prooftree}
\AxiomC{$!_k A \vdash A$}
\UnaryInfC{$!_k A \vdash !_k A$}
\RightLabel{(E)}
\UnaryInfC{$!_k A \vdash E$}
\AxiomC{$!_d B \vdash B$}
\UnaryInfC{$!_d B \vdash B_0(3, \lambda . f(x+1))$}
\BinaryInfC{$!_k A, !_d B \vdash F$}
\AxiomC{$!_g C \vdash C$}
\UnaryInfC{$!_g C \vdash C_0(1)$}
\BinaryInfC{$!_k A, !_d B, !_k C \vdash G$}
\UnaryInfC{$!_k A, !_d B, !_k C \vdash 2F(f) \leq 3$}
\AxiomC{$!_d B \vdash B$}
\UnaryInfC{$!_d B \vdash B_0(2, f)$}
\BinaryInfC{$!_k A, !_d B, !_d B, !_k C \vdash D$}
\RightLabel{(con_d)}
\UnaryInfC{$!_k A, !_d B, !_k C \vdash D$}
\end{prooftree}
$$

where we have used the following additional abbreviations

$$E \equiv \; F(\lambda x. 3f(x+1)) = F(f),$$

$$F \equiv \; 3F(\lambda x. f(x+1)) = F(f),$$

$$G \equiv \; 3F(f) - 3 \leq F(f).$$

Note that alternative labellings such as

$$!_k \forall x A_0(x) \otimes !_d \forall y, g B_0(y, g) \otimes !_d \forall z C_0(z) \multimap D,$$

and

$$!_k \forall x A_0(x) \otimes !_g \forall y, g B_0(y, g) \otimes !_g \forall z C_0(z) \multimap D,$$

would also work; whereas, a labelling such as

$$!_d \forall x A_0(x) \otimes !_g \forall y, g B_0(y, g) \otimes !_g \forall z C_0(z) \multimap D,$$

would fail to be propagated upwards through the proof, since the extensionality axiom requires a Kreisel modality.

[10] For the sake of intuition we write $A_0^\perp, \ldots, A_n^\perp, B$ as $A_0, \ldots, A_n \vdash B$.

3.2 On a direct hybrid interpretation of IL^ω

One could think of developing a hybrid interpretation of IL^ω directly, without going through the use of linear logic, by noticing that in the translated IL^ω-proof the modality ! only appears in the premise of an implication[11]. Therefore, we could work with a "multi-implication" intuitionistic logic ($\to_k$, $\to_d$ and $\to_g$), and define the interpretation of each implication as

$$\begin{aligned} |A \to_k B|^{\boldsymbol{f}}_{\boldsymbol{x},\boldsymbol{w}} &\equiv \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to |B|^{\boldsymbol{fx}}_{\boldsymbol{w}} \\ |A \to_d B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} &\equiv \forall \boldsymbol{y} \in \boldsymbol{gxw}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to |B|^{\boldsymbol{fx}}_{\boldsymbol{w}} \\ |A \to_g B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} &\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{gxw}} \to |B|^{\boldsymbol{fx}}_{\boldsymbol{w}}. \end{aligned}$$

The first thing to notice is that, unlike the modalities in linear logic, the intuitionistic implication is canonical. More precisely, simply adding two or more distinct arrows, e.g. $A \to_k B$ and $A \to_g B$, with the same usual rules would not work, since we would be able to derive their equivalence: $A \to_k B \vdash A \to_g B$ and $A \to_g B \vdash A \to_k B$. Another problem is the interpretation of the provability sign ($\vdash$), which would have to correspond to one of the particular implications. A way to remedy this would be to work with labelled contexts, where in a sequent such as $\Gamma \vdash A$, each assumption B_i in the context Γ will be either a "Gödel's", "Diller-Nahm" or "Kreisel" assumption. The sequents would then be of the form $[\Gamma]_k; [\Delta]_d; [\Theta]_k \vdash A$. That would give rise to a form of multi-implication intuitionistic logic, together with a hybrid interpretation. The full flexibility of working with linear logic, however, does not seem to be achievable, since an implication of the form $!_d \forall x !_k \forall y A \multimap B$ would not have a correspondence in the multi-implication intuitionistic logic, for instance.

4 Variants of Hybrid Interpretation

Let us now look at two variants of functional interpretations which make use of Howard/Bezem's notion of (strongly) majorizable functionals [2, 13]. These are the "monotone" [14, 15] and "bounded" interpretations [4, 5]. For simplicity we will use here Bezem's strong notion of majorizability, although the monotone interpretations also work with Howard's notion.

For the following two sub-sections, let us assume that our language contains the usual $\leq$ relation on natural numbers. We then assume the following abbreviations[12]

- $n \leq^*_{\mathbb{N}} m :\equiv n \leq m$ and
- $f \leq^* g :\equiv !_k \forall y \forall x \leq^* y\, ((fx \leq^* gy) \otimes (fx \leq^* fy))$.

Lemma 4.1 *For each closed term t of $\mathsf{LL}^\omega_{\mathsf{h}}$ there exists a closed term t^* such that $\vdash_{\mathsf{LL}^\omega_{\mathsf{h}}} t \leq^* t^*$.*

---

[11] Actually, the embedding also makes use of a further ! when treating the existential quantifier. This, however, is inessential for the functional interpretations, given principle TA (see Section 2.1).

[12] Note that we use Kreisel's modality in the definition of majorizability, so that $x \leq^* y$ is both refutation and computation irrelevant (a fixed formula). In this way, bounded quantifiers of the form $\forall x \leq^* t\, A$ can denote precisely $\forall x((x \leq^* t) \multimap A)$ and the interpretation of the equivalence is still interpretable (for a suitable interpretation of (bounded) quantifiers, see Section 4.2). The same would not happen had we used a Gödel modality in the definition of the majorizability relation (cf. [5, 6]).

4.1 Monotone hybrid interpretation

As shown in [18], the monotone variant of both the Dialectica and modified realizability interpretations come from a "monotone" soundness theorem, rather than a new interpretation of formulas. In this section we show that these monotone variants can also be combined in a single hybrid monotone soundness theorem. For simplicity, assume that the Diller-Nahm modality has been dropped, so that we avoid having to extend the majorizability relation to the type of finite sets.

Lemma 4.2 *Let* $\exists\!\!\forall_{\boldsymbol{y}}^{\boldsymbol{x}\leq^*\boldsymbol{a}} A$ *be an abbreviation for* $\exists\!\!\forall_{\boldsymbol{y}}^{\boldsymbol{x}}((\boldsymbol{x}\leq^*\boldsymbol{a})\wedge A)$. *The following are derivable in* LL^ω

(i) $\exists\!\!\forall_{\boldsymbol{y}_0,\boldsymbol{y}_1}^{\boldsymbol{x}\leq^*\boldsymbol{a}} A(\boldsymbol{y}_0,\boldsymbol{y}_1) \multimap \exists\!\!\forall_{\boldsymbol{y}}^{\boldsymbol{x}\leq^*\boldsymbol{a}} A(\boldsymbol{y},\boldsymbol{y})$

(ii) if $\boldsymbol{a}\leq^*\boldsymbol{a}^*$ *is derivable then* $\exists\!\!\forall_{\boldsymbol{y}}^{\boldsymbol{x}\leq^*\boldsymbol{a}} A \multimap \exists\!\!\forall_{\boldsymbol{y}}^{\boldsymbol{x}\leq^*\boldsymbol{a}^*} A$.

Theorem 4.3 (Monotone soundness of hybrid interpretation) *Let* $A_0,\ldots,A_n$ *be a sequence of formulas of* $\mathsf{LL}^\omega_{\mathsf{h}}$, *with* $\boldsymbol{u}$ *as the only free-variables. If*

$$\vdash_{\mathsf{LL}^\omega_{\mathsf{h}}} A_0,\ldots,A_n$$

then monotone closed terms $\boldsymbol{a}^*\equiv\boldsymbol{a}_0^*,\ldots,\boldsymbol{a}_n^*$ *can be automatically synthesised from its formal proof, such that*

$$\vdash_{\mathsf{LL}^\omega} \exists\!\!\forall_{\boldsymbol{y},\boldsymbol{u}}^{\boldsymbol{a}\leq^*\boldsymbol{a}^*}(|A_0 \parr \ldots \parr A_n|_{\boldsymbol{y}}^{\boldsymbol{a}\boldsymbol{u}}).$$

Proof. By induction on the given derivation. For this proof only we deviate from our convention and use $\boldsymbol{a},\boldsymbol{b},\ldots$ to stand for variables and $\boldsymbol{a}^*,\boldsymbol{b}^*,\ldots$ to stand for terms. We will consider the free variables $\boldsymbol{u}$ only when treating the quantifiers, where they matter the most. Multiple steps of derivation are denotes by $(*)$. Let us consider a few cases:

Cut rule

$$\dfrac{\dfrac{\dfrac{\exists\!\!\forall_{\boldsymbol{w},\boldsymbol{y}}^{\boldsymbol{\gamma}\leq^*\boldsymbol{\gamma}^*,\boldsymbol{a}\leq^*\boldsymbol{a}^*}|\Gamma \parr A|_{\boldsymbol{w},\boldsymbol{y}}^{\boldsymbol{\gamma},\boldsymbol{a}}}{\exists\!\!\forall_{\boldsymbol{w},\boldsymbol{y}}^{\boldsymbol{\gamma}\leq^*\boldsymbol{\gamma}^*,\boldsymbol{a}\leq^*\boldsymbol{a}^*}(|\Gamma|_{\boldsymbol{w}}^{\boldsymbol{\gamma}\boldsymbol{y}} \parr |A|_{\boldsymbol{y}}^{\boldsymbol{a}\boldsymbol{w}})}\,\text{D2.1}}{\exists\!\!\forall_{\boldsymbol{w},\boldsymbol{b},\boldsymbol{z}}^{\boldsymbol{\gamma}\leq^*\boldsymbol{\gamma}^*,\boldsymbol{a}\leq^*\boldsymbol{a}^*}(|\Gamma|_{\boldsymbol{w}}^{\boldsymbol{\gamma}(\boldsymbol{b}\boldsymbol{z})} \parr |A|_{\boldsymbol{b}\boldsymbol{z}}^{\boldsymbol{a}\boldsymbol{w}})} \qquad \dfrac{\dfrac{\exists\!\!\forall_{\boldsymbol{z},\boldsymbol{x}}^{\boldsymbol{\delta}\leq^*\boldsymbol{\delta}^*,\boldsymbol{b}\leq^*\boldsymbol{b}^*}|\Delta \parr A^\perp|_{\boldsymbol{z},\boldsymbol{x}}^{\boldsymbol{\delta},\boldsymbol{b}}}{\exists\!\!\forall_{\boldsymbol{z},\boldsymbol{x}}^{\boldsymbol{\delta}\leq^*\boldsymbol{\delta}^*,\boldsymbol{b}\leq^*\boldsymbol{b}^*}(|\Delta|_{\boldsymbol{z}}^{\boldsymbol{\delta}\boldsymbol{x}} \parr |A^\perp|_{\boldsymbol{x}}^{\boldsymbol{b}\boldsymbol{z}})}\,\text{D2.1}}{\exists\!\!\forall_{\boldsymbol{z},\boldsymbol{a},\boldsymbol{w}}^{\boldsymbol{\delta}\leq^*\boldsymbol{\delta}^*,\boldsymbol{b}\leq^*\boldsymbol{b}^*}(|\Delta|_{\boldsymbol{z}}^{\boldsymbol{\delta}(\boldsymbol{a}\boldsymbol{w})} \parr |A^\perp|_{\boldsymbol{a}\boldsymbol{w}}^{\boldsymbol{b}\boldsymbol{z}})}}{\exists\!\!\forall_{\boldsymbol{w},\boldsymbol{z}}^{\boldsymbol{\gamma}\leq^*\boldsymbol{\gamma}^*,\boldsymbol{\delta}\leq^*\boldsymbol{\delta}^*}|\Gamma \parr \Delta|_{\boldsymbol{w},\boldsymbol{z}}^{\boldsymbol{\gamma},\boldsymbol{\delta}}}\,(*)$$

Tensor

$$\dfrac{\dfrac{\dfrac{\dfrac{\exists\!\!\forall_{\boldsymbol{w},\boldsymbol{y}}^{\boldsymbol{\gamma}\leq^*\boldsymbol{\gamma}^*,\boldsymbol{a}\leq^*\boldsymbol{a}^*}|\Gamma \parr A|_{\boldsymbol{w},\boldsymbol{y}}^{\boldsymbol{\gamma},\boldsymbol{a}}}{\exists\!\!\forall_{\boldsymbol{w},\boldsymbol{y}}^{\boldsymbol{\gamma}\leq^*\boldsymbol{\gamma}^*,\boldsymbol{a}\leq^*\boldsymbol{a}^*}(|\Gamma|_{\boldsymbol{w}}^{\boldsymbol{\gamma}\boldsymbol{y}} \parr |A|_{\boldsymbol{y}}^{\boldsymbol{a}\boldsymbol{w}})}\,\text{D2.1}}{\exists\!\!\forall_{\boldsymbol{w},\boldsymbol{f},\boldsymbol{b},\boldsymbol{z}}^{\boldsymbol{\gamma}\leq^*\boldsymbol{\gamma}^*,\boldsymbol{a}\leq^*\boldsymbol{a}^*}(|\Gamma|_{\boldsymbol{w}}^{\boldsymbol{\gamma}\boldsymbol{f}(\boldsymbol{b}\boldsymbol{z})} \parr |A|_{\boldsymbol{f}(\boldsymbol{b}\boldsymbol{z})}^{\boldsymbol{a}\boldsymbol{w}})} \qquad \dfrac{\dfrac{\exists\!\!\forall_{\boldsymbol{z},\boldsymbol{x}}^{\boldsymbol{\delta}\leq^*\boldsymbol{\delta}^*,\boldsymbol{b}\leq^*\boldsymbol{b}^*}|\Delta \parr B|_{\boldsymbol{z},\boldsymbol{x}}^{\boldsymbol{\delta},\boldsymbol{b}}}{\exists\!\!\forall_{\boldsymbol{z},\boldsymbol{x}}^{\boldsymbol{\delta}\leq^*\boldsymbol{\delta}^*,\boldsymbol{b}\leq^*\boldsymbol{b}^*}(|\Delta|_{\boldsymbol{z}}^{\boldsymbol{\delta}\boldsymbol{x}} \parr |B|_{\boldsymbol{x}}^{\boldsymbol{b}\boldsymbol{z}})}\,\text{D2.1}}{\exists\!\!\forall_{\boldsymbol{z},\boldsymbol{g},\boldsymbol{a},\boldsymbol{w}}^{\boldsymbol{\delta}\leq^*\boldsymbol{\delta}^*,\boldsymbol{b}\leq^*\boldsymbol{b}^*}(|\Delta|_{\boldsymbol{z}}^{\boldsymbol{\delta}\boldsymbol{g}(\boldsymbol{a}\boldsymbol{w})} \parr |B|_{\boldsymbol{g}(\boldsymbol{a}\boldsymbol{w})}^{\boldsymbol{b}\boldsymbol{z}})}}{\exists\!\!\forall_{\boldsymbol{w},\boldsymbol{z},\boldsymbol{f},\boldsymbol{g}}^{\boldsymbol{\gamma}\leq^*\boldsymbol{\gamma}^*,\boldsymbol{\delta}\leq^*\boldsymbol{\delta}^*,\boldsymbol{a}\leq^*\boldsymbol{a}^*,\boldsymbol{b}\leq^*\boldsymbol{b}^*}(|\Gamma|_{\boldsymbol{w}}^{\boldsymbol{\gamma}\boldsymbol{f}(\boldsymbol{b}\boldsymbol{z})} \parr |\Delta|_{\boldsymbol{z}}^{\boldsymbol{\delta}\boldsymbol{g}(\boldsymbol{a}\boldsymbol{w})} \parr (|A|_{\boldsymbol{f}(\boldsymbol{b}\boldsymbol{z})}^{\boldsymbol{a}\boldsymbol{w}} \otimes |B|_{\boldsymbol{g}(\boldsymbol{a}\boldsymbol{w})}^{\boldsymbol{b}\boldsymbol{z}}))}\,(*)}{\exists\!\!\forall_{\boldsymbol{w},\boldsymbol{z},\boldsymbol{f},\boldsymbol{g}}^{\boldsymbol{\gamma}\leq^*\boldsymbol{\gamma}^*,\boldsymbol{\delta}\leq^*\boldsymbol{\delta}^*}(|\Gamma|_{\boldsymbol{w}}^{\boldsymbol{\gamma}\boldsymbol{z}} \parr |\Delta|_{\boldsymbol{z}}^{\boldsymbol{\delta}\boldsymbol{w}} \parr |A\otimes B|_{\boldsymbol{f},\boldsymbol{g}}^{\boldsymbol{a}\boldsymbol{w},\boldsymbol{b}\boldsymbol{z}})}$$

Existential quantifier

$$
\begin{array}{cl}
\exists\!\!\!\forall^{\gamma \leq^* \gamma^*, \boldsymbol{a} \leq^* \boldsymbol{a}^*}_{\boldsymbol{w},\boldsymbol{y},\boldsymbol{u}} |\Gamma \parr A[t[\boldsymbol{u}]/z]|^{\gamma \boldsymbol{u}, \boldsymbol{a} \boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{y}} & \\ \hline
\exists\!\!\!\forall^{\gamma \leq^* \gamma^*, \boldsymbol{a} \leq^* \boldsymbol{a}^*}_{\boldsymbol{w},\boldsymbol{y},\boldsymbol{u}} (|\Gamma|^{\gamma \boldsymbol{u} \boldsymbol{y}}_{\boldsymbol{w}} \parr |A[t[\boldsymbol{u}]/z]|^{\boldsymbol{a} \boldsymbol{u} \boldsymbol{w}}_{\boldsymbol{y}}) & \text{D2.1} \\ \hline
\exists\!\!\!\forall^{\gamma \leq^* \gamma^*, \boldsymbol{a} \leq^* \boldsymbol{a}^*}_{\boldsymbol{w},\boldsymbol{y},\boldsymbol{u}} (|\Gamma|^{\gamma \boldsymbol{u} \boldsymbol{y}}_{\boldsymbol{w}} \parr |\exists z A|^{\boldsymbol{a} \boldsymbol{u} \boldsymbol{w}, t[\boldsymbol{u}]}_{\boldsymbol{y}}) & \text{D2.1} \\ \hline
\exists\!\!\!\forall^{\gamma \leq^* \gamma^*, \boldsymbol{a} \leq^* \boldsymbol{a}^*, h \leq^* t^*}_{\boldsymbol{w},\boldsymbol{y},\boldsymbol{u}} (|\Gamma|^{\gamma \boldsymbol{u} \boldsymbol{y}}_{\boldsymbol{w}} \parr |\exists z A|^{\boldsymbol{a} \boldsymbol{u} \boldsymbol{w}, h \boldsymbol{u}}_{\boldsymbol{y}}) & (*)
\end{array}
$$

where $\lambda \boldsymbol{u}.t[\boldsymbol{u}] \leq^* t^*$.

Universal quantifier

$$
\begin{array}{cl}
\exists\!\!\!\forall^{\gamma \leq^* \gamma^*, \boldsymbol{a} \leq^* \boldsymbol{a}^*}_{\boldsymbol{w},\boldsymbol{y},\boldsymbol{u}} |\Gamma \parr A|^{\gamma \boldsymbol{u}, \boldsymbol{a} \boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{y}} & \\ \hline
\exists\!\!\!\forall^{\gamma \leq^* \gamma^*, \boldsymbol{a} \leq^* \boldsymbol{a}^*}_{\boldsymbol{w},\boldsymbol{y},\boldsymbol{u}} (|\Gamma|^{\gamma \boldsymbol{u} \boldsymbol{y}}_{\boldsymbol{w}} \parr |A|^{\boldsymbol{a} \boldsymbol{u} \boldsymbol{w}}_{\boldsymbol{y}}) & \text{D2.1} \\ \hline
\exists\!\!\!\forall^{\gamma \leq^* \gamma^*, \boldsymbol{a} \leq^* \boldsymbol{a}^*}_{\boldsymbol{w},\boldsymbol{y},\boldsymbol{u}} (|\Gamma|^{\gamma \boldsymbol{u} \boldsymbol{y}}_{\boldsymbol{w}} \parr |\forall u_0 A|^{\boldsymbol{a} \hat{\boldsymbol{u}} \boldsymbol{w}}_{\boldsymbol{y}, u_0}) & \text{D2.1}
\end{array}
$$

where $\hat{\boldsymbol{u}}$ denotes the vector $\boldsymbol{u}$ minus the variable u_0.

Contraction (Gödel modality)

$$
\begin{array}{cl}
\exists\!\!\!\forall^{\gamma \leq^* \gamma^*, \boldsymbol{a}_0 \leq^* \boldsymbol{a}^*_0, \boldsymbol{a}_1 \leq^* \boldsymbol{a}^*_1}_{\boldsymbol{v},\boldsymbol{y}_0,\boldsymbol{y}_1} |\Gamma \parr ?_g A \parr ?_g A|^{\gamma, \boldsymbol{a}_0, \boldsymbol{a}_1}_{\boldsymbol{v},\boldsymbol{y}_0,\boldsymbol{y}_1} & \\ \hline
\exists\!\!\!\forall^{\gamma \leq^* \gamma^*, \boldsymbol{a}_0 \leq^* \boldsymbol{a}^*_0, \boldsymbol{a}_1 \leq^* \boldsymbol{a}^*_1}_{\boldsymbol{v},\boldsymbol{y}_0,\boldsymbol{y}_1} (|\Gamma|^{\gamma \boldsymbol{y}_0 \boldsymbol{y}_1}_{\boldsymbol{v}} \parr ?|A|^{\boldsymbol{a}_0 \boldsymbol{v} \boldsymbol{y}_1 \boldsymbol{y}_0}_{\boldsymbol{y}_0} \parr ?|A|^{\boldsymbol{a}_1 \boldsymbol{v} \boldsymbol{y}_0 \boldsymbol{y}_1}_{\boldsymbol{y}_1}) & \text{D2.1} \\ \hline
\exists\!\!\!\forall^{\gamma \leq^* \gamma^*, \boldsymbol{a}_0 \leq^* \boldsymbol{a}^*_0, \boldsymbol{a}_1 \leq^* \boldsymbol{a}^*_1}_{\boldsymbol{v},\boldsymbol{y}} (|\Gamma|^{\gamma \boldsymbol{y} \boldsymbol{y}}_{\boldsymbol{v}} \parr ?|A|^{\boldsymbol{a}_0 \boldsymbol{v} \boldsymbol{y} \boldsymbol{y}}_{\boldsymbol{y}} \parr ?|A|^{\boldsymbol{a}_1 \boldsymbol{v} \boldsymbol{y} \boldsymbol{y}}_{\boldsymbol{y}}) & \text{L4.2}(i) \\ \hline
\exists\!\!\!\forall^{\gamma \leq^* \gamma^*, \boldsymbol{a}_0 \leq^* \max\{\boldsymbol{a}^*_0, \boldsymbol{a}^*_1\}, \boldsymbol{a}_1 \leq^* \max\{\boldsymbol{a}^*_0, \boldsymbol{a}^*_1\}}_{\boldsymbol{v},\boldsymbol{y}} (|\Gamma|^{\gamma \boldsymbol{y} \boldsymbol{y}}_{\boldsymbol{v}} \parr ?|A|^{\boldsymbol{a}_0 \boldsymbol{v} \boldsymbol{y} \boldsymbol{y}}_{\boldsymbol{y}} \parr ?|A|^{\boldsymbol{a}_1 \boldsymbol{v} \boldsymbol{y} \boldsymbol{y}}_{\boldsymbol{y}}) & \text{L4.2}(ii) \\ \hline
\exists\!\!\!\forall^{\gamma \leq^* \gamma^*, \boldsymbol{a} \leq^* \max\{\boldsymbol{a}^*_0, \boldsymbol{a}^*_1\}}_{\boldsymbol{v},\boldsymbol{y}} (|\Gamma|^{\gamma \boldsymbol{y} \boldsymbol{y}}_{\boldsymbol{v}} \parr ?|A|^{\boldsymbol{a} \boldsymbol{v} \boldsymbol{y} \boldsymbol{y}}_{\boldsymbol{y}}) & (*)
\end{array}
$$

Contraction (Kreisel modality)

$$
\begin{array}{cl}
\exists\!\!\!\forall^{\boldsymbol{\delta} \leq^* \boldsymbol{\delta}^*}_{\boldsymbol{u},\boldsymbol{y}_0,\boldsymbol{y}_1} |\Delta \parr ?_k A \parr ?_k A|^{\boldsymbol{\delta}}_{\boldsymbol{u},\boldsymbol{y}_0,\boldsymbol{y}_1} & \\ \hline
\exists\!\!\!\forall^{\boldsymbol{\delta} \leq^* \boldsymbol{\delta}^*}_{\boldsymbol{u},\boldsymbol{y}_0,\boldsymbol{y}_1} (|\Delta|^{\boldsymbol{\delta}}_{\boldsymbol{u}} \parr ?\exists \boldsymbol{x} |A|^{\boldsymbol{x}}_{\boldsymbol{y}_0} \parr ?\exists \boldsymbol{x} |A|^{\boldsymbol{x}}_{\boldsymbol{y}_1}) & \text{D2.1} \\ \hline
\exists\!\!\!\forall^{\boldsymbol{\delta} \leq^* \boldsymbol{\delta}^*}_{\boldsymbol{u},\boldsymbol{y}} (|\Delta|^{\boldsymbol{\delta}}_{\boldsymbol{u}} \parr ?\exists \boldsymbol{x} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \parr ?\exists \boldsymbol{x} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}) & \text{L4.2}(i) \\ \hline
\exists\!\!\!\forall^{\boldsymbol{\delta} \leq^* \boldsymbol{\delta}^*}_{\boldsymbol{u},\boldsymbol{y}} (|\Delta|^{\boldsymbol{\delta}}_{\boldsymbol{u}} \parr ?\exists \boldsymbol{x} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}) & (*)
\end{array}
$$

The other cases are treated similarly. $\square$

Note that in the soundness of the Gödel contraction rule only the existence (provable in the verifying system LL^ω) of a decision function is required. The majorant for the decision function, however, needs to be part of the term language. Such majorant is normally taken to be the computable term $\lambda x, y. \max\{x, y\}$. This means that the Gödel contraction can be allowed for a larger class of formulas A, as long as the verifying system is able to prove that a definition-by-cases function exists for conditions of the form $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$.

The main importance of the monotone soundness, however, comes from the ability to produce bounds for the conclusion given bounds for the premise. When premises do not have computable witnesses, but have computable bounds on these, the full power of the monotone soundness comes into force. To see this, let Δ consist the class of formulas of the form[13]

$$A \equiv\, !_k \forall \boldsymbol{x} \exists \boldsymbol{y} \leq \boldsymbol{t}\boldsymbol{x} !_k \forall \boldsymbol{z} A_{\mathsf{qf}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$$

and let $\tilde{\Delta}$ denote the class of Skolem normal forms of Δ, i.e.

$$\tilde{A} \equiv \exists \boldsymbol{f} \leq \boldsymbol{t} !_k \forall \boldsymbol{x}, \boldsymbol{z} A_{\mathsf{qf}}(\boldsymbol{x}, \boldsymbol{f}\boldsymbol{x}, \boldsymbol{z}).$$

Kohlenbach has shown that weak König's lemma falls into the class of formulas Δ, and hence, the monotone interpretation provides a way to extract bounds from ineffective proofs in mathematics.

Corollary 4.4 *Let $A_0, \ldots, A_n$ be a sequence of formulas of $\mathsf{LL}^\omega_{\mathsf{h}}$, with $\boldsymbol{u}$ as the only free-variables. If*

$$\vdash_{\mathsf{LL}^\omega_{\mathsf{h}} + \Delta} A_0, \ldots, A_n$$

then monotone closed terms $\boldsymbol{a}^* \equiv \boldsymbol{a}^*_0, \ldots, \boldsymbol{a}^*_n$ can be automatically synthesised from its formal proof, such that

$$\vdash_{\mathsf{LL}^\omega + \tilde{\Delta}} \exists\!\!\!/_{\boldsymbol{y}, \boldsymbol{u}}^{\boldsymbol{a} \leq^* \boldsymbol{a}^*} (|A_0 \parr \ldots \parr A_n|^{\boldsymbol{a}\boldsymbol{u}}_{\boldsymbol{y}}).$$

Proof. For simplicity, assume only one formula from the class Δ is used to derive a particular formula A, i.e.

$$!_k \forall \boldsymbol{x} \exists \boldsymbol{y} \leq \boldsymbol{t}\boldsymbol{x} !_k \forall \boldsymbol{z} B_{\mathsf{qf}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \multimap A.$$

Theorem 4.3 implies that closed monotone terms $\boldsymbol{a}^*$ can be extracted such that

$$\exists\!\!\!/_{\boldsymbol{f}, \boldsymbol{w}, \boldsymbol{u}}^{\boldsymbol{F} \leq^* \boldsymbol{a}^*} (!_k \forall \boldsymbol{x} (!_k (\boldsymbol{f}\boldsymbol{x} \leq \boldsymbol{t}\boldsymbol{x}) \wedge !_k \forall \boldsymbol{z} B_{\mathsf{qf}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})) \multimap |A|^{\boldsymbol{F}\boldsymbol{f}\boldsymbol{u}}_{\boldsymbol{w}}).$$

In particular, this implies

$$\exists\!\!\!/_{\boldsymbol{f}, \boldsymbol{w}, \boldsymbol{u}}^{\boldsymbol{F} \leq^* \boldsymbol{a}^*} (!_k (\boldsymbol{f} \leq \boldsymbol{t}) \wedge !_k \forall \boldsymbol{x}, \boldsymbol{z} B_{\mathsf{qf}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})) \multimap |A|^{\boldsymbol{F}\boldsymbol{f}\boldsymbol{u}}_{\boldsymbol{w}}).$$

and hence

$$\exists \boldsymbol{f} \leq \boldsymbol{t} \, !_k \forall \boldsymbol{x}, \boldsymbol{z} B_{\mathsf{qf}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \multimap \exists\!\!\!/_{\boldsymbol{w}, \boldsymbol{u}}^{\boldsymbol{a} \leq^* \boldsymbol{a}^* \boldsymbol{t}^*} |A|^{\boldsymbol{a}\boldsymbol{u}}_{\boldsymbol{w}}$$

where $\boldsymbol{t} \leq^* \boldsymbol{t}^*$ (using that $\boldsymbol{f} \leq \boldsymbol{t} \wedge \boldsymbol{t} \leq^* \boldsymbol{t}^*$ implies $\boldsymbol{f} \leq^* \boldsymbol{t}^*$). That concludes the proof. □

---

[13]Here $\leq$ denotes pointwise inequality, i.e. $f \overset{\rho \to \tau}{\leq} g :\equiv\, !_k \forall x^\rho (fx \overset{\tau}{\leq} gx)$.

4.2 Hybrid bounded interpretation

We have conjectured in [11] that "Howard modalities" ($!_h, ?_h$) could also be added to the hybrid interpretation setting, in order to incorporate the bounded functional interpretation [5, 6]. Combined with the Kreisel modality, that would lead to a hybrid bounded interpretation which incorporated both the bounded functional interpretation and the bounded modified realizability [4]. In this section we look at this problem in more details, and explain what we have achieved so far, and what problems we have encountered.

Assume we add a fourth set of modalities ($!_h, ?_h$) to the multi-modal linear logic and interpret these using the majorisability relation $\leq^*$ as

$$\begin{array}{lcl} |!_h A|^{\boldsymbol{x}}_{\boldsymbol{f}} & :\equiv & !\tilde{\forall} \boldsymbol{y} \leq^* \boldsymbol{f}\boldsymbol{x} \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \\ |?_h A|^{\boldsymbol{f}}_{\boldsymbol{y}} & :\equiv & ?\tilde{\exists} \boldsymbol{x} \leq^* \boldsymbol{f}\boldsymbol{y} \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}, \end{array}$$

where $\tilde{\forall} \boldsymbol{y} A$ abbreviates $\forall \boldsymbol{y}((\boldsymbol{y} \leq^* \boldsymbol{y}) \multimap A)$ (similarly for $\tilde{\exists} \boldsymbol{x} A$). As argued before (cf. [18, 19]), this interpretation only works if all objects involved are monotone (i.e. self-majorisable). That can be obtained by interpreting quantifiers as

$$\begin{array}{lcl} |\exists z A(z)|^{\boldsymbol{x}, a}_{\boldsymbol{f}} & :\equiv & \exists z \leq^* a \, |A(z)|^{\boldsymbol{x}}_{\boldsymbol{f} a} \\ |\forall z A(z)|^{\boldsymbol{f}}_{\boldsymbol{y}, b} & :\equiv & \forall z \leq^* b \, |A(z)|^{\boldsymbol{f} b}_{\boldsymbol{y}}, \end{array}$$

since an existential (respectively, universal) quantification over a (respectively, b) can be restricted to monotone objects. Once we can confine ourselves to monotone objects, bounded quantifiers can be viewed as a particular case of unbounded quantifiers where $\boldsymbol{f}$ can be chosen uniformly, i.e.

$$\begin{array}{lcl} |\exists z \leq^* c \, A(z)|^{\boldsymbol{x}}_{\boldsymbol{y}} & :\equiv & \exists z \leq^* c \, |A(z)|^{\boldsymbol{x}}_{\boldsymbol{y}} \\ |\forall z \leq^* c \, A(z)|^{\boldsymbol{x}}_{\boldsymbol{y}} & :\equiv & \forall z \leq^* c \, |A(z)|^{\boldsymbol{x}}_{\boldsymbol{y}}. \end{array}$$

Unfortunately, there is a problem with this interpretation when dealing with the additive connectives. In order the obtain the full bounded functional interpretation, we would like to interpret these as

$$\begin{array}{lcl} |A \wedge B|^{\boldsymbol{x}, \boldsymbol{y}}_{\boldsymbol{v}, \boldsymbol{w}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \wedge |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\ |A \vee B|^{\boldsymbol{x}, \boldsymbol{y}}_{\boldsymbol{v}, \boldsymbol{w}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \vee |B|^{\boldsymbol{v}}_{\boldsymbol{w}}. \end{array}$$

Consider, however, the derivable rule (in sequent style)

$$\frac{A \vdash C \quad B \vdash C}{A \vee B \vdash C}$$

Each of the two proofs in the premise of the rule might give rise to two different witnesses ($\boldsymbol{c}_0$ and $\boldsymbol{c}_1$) for C, depending on the assumption used (either A or B). In the conclusion, however, we must choose a single witness, given witnesses for $A \vee B$. When interpreting intuitionistic logic, this problem is solved by making use of the following monotonicity property

$$\text{if } |C|^{\boldsymbol{x}}_{\boldsymbol{y}} \text{ and } \boldsymbol{x} \leq^* \boldsymbol{x}^* \text{ and } \boldsymbol{y} \leq^* \boldsymbol{y} \text{ then } |C|^{\boldsymbol{x}^*}_{\boldsymbol{y}}. \tag{12}$$

If (12) holds for all formulas C, then each $|C|^{\boldsymbol{c}_i}_{\boldsymbol{u}}$ implies $|C|^{\max\{\boldsymbol{c}_0,\boldsymbol{c}_1\}}_{\boldsymbol{u}}$, so we can pick $\max\{\boldsymbol{c}_0,\boldsymbol{c}_1\}$ for the witness of C in the conclusion $A \vee B \vdash C$.

The (asymmetric) monotonicity property (12), however, does not hold in the very symmetric setting of linear logic. See, for instance, the interpretation of $!_h A$, where a "bigger" $\boldsymbol{x}$ no longer guarantees the truth of $\tilde{\forall}\boldsymbol{y} \leq^* \boldsymbol{f}\boldsymbol{x}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$, since the range of the bounded quantification over $\boldsymbol{y}$ might also increase. A similar problem occurs in the interpretation of $\exists z A$.

This indicates that a sound bounded functional interpretation of the additives in the linear logic context is unlike. On the other hand, property (12) is "only" needed for the treatment for the additive connectives. Therefore, we present below a sound interpretation for the multiplicative exponential fragment of linear logic (MELL). We consider the "Howard" and "Kreisel" sets of modalities, which enables us to incorporate in one interpretation (the additive-free fragment of) both the bounded functional interpretation [5] and the bounded modified realizability [4].

Definition 4.5 (Hybrid bounded interpretation for MELL) *The interpretation of atomic formulas are the atomic formulas themselves, with empty sets of witnessing and challenge variables, i.e.* $|A_{\mathsf{at}}| :\equiv A_{\mathsf{at}}$ *and* $|A_{\mathsf{at}}^{\perp}| :\equiv A_{\mathsf{at}}^{\perp}$. *Assuming* $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ *and* $|B|^{\boldsymbol{v}}_{\boldsymbol{w}}$ *already defined, we define*[14]

$$
\begin{array}{lcl}
|A \parr B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{y},\boldsymbol{w}} & :\equiv & |A|^{\boldsymbol{f}\boldsymbol{w}}_{\boldsymbol{y}} \vee |B|^{\boldsymbol{g}\boldsymbol{y}}_{\boldsymbol{w}} \\
|A \otimes B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{f},\boldsymbol{g}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{v}} \wedge |B|^{\boldsymbol{v}}_{\boldsymbol{g}\boldsymbol{x}} \\
|\exists z A|^{\boldsymbol{x},a}_{\boldsymbol{f}} & :\equiv & \exists z \leq^* a\, |A|^{\boldsymbol{x}}_{\boldsymbol{f}a} \\
|\forall z A|^{\boldsymbol{f}}_{\boldsymbol{y},b} & :\equiv & \forall z \leq^* b\, |A|^{\boldsymbol{f}b}_{\boldsymbol{y}} \\
|\exists z \leq^* s\, A|^{\boldsymbol{x}}_{\boldsymbol{y}} & :\equiv & \exists z \leq^* s\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \\
|\forall z \leq^* s\, A|^{\boldsymbol{x}}_{\boldsymbol{y}} & :\equiv & \forall z \leq^* s\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.
\end{array}
$$

The Kreisel and Howard modalities are interpreted as

$$
\begin{array}{lclclcl}
|!_k A|^{\boldsymbol{x}} & :\equiv & \tilde{\forall}\boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} & \qquad & |?_k A|_{\boldsymbol{y}} & :\equiv & \tilde{\exists}\boldsymbol{x} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \\
|!_h A|^{\boldsymbol{x}}_{\boldsymbol{f}} & :\equiv & \tilde{\forall}\boldsymbol{y} \leq^* \boldsymbol{f}\boldsymbol{x}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} & \qquad & |?_h A|^{\boldsymbol{f}}_{\boldsymbol{y}} & :\equiv & \tilde{\exists}\boldsymbol{x} \leq^* \boldsymbol{f}\boldsymbol{y}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.
\end{array}
$$

Note that the hybrid bounded interpretation above is not an extension of hybrid interpretation (Def 2.1), but rather a new distinct interpretation, due to the distinguished treatment of quantifiers. Before we proceed let us state a second possibility for embedding intuitionistic logic into linear logic, which we will make use of next.

Definition 4.6 ([7]) *For any formula* A *of intuitionistic logic its linear translation* A° *is defined inductively as*

$$
\begin{array}{lcl}
A^\circ_{\mathsf{at}} & :\equiv & !A_{\mathsf{at}} \\
(A \diamond_z B)^\circ & :\equiv & A^\circ \diamond_z B^\circ \\
(A \to B)^\circ & :\equiv & !(A^\circ \multimap B^\circ) \\
(\forall x A)^\circ & :\equiv & !\forall x A^\circ \\
(\exists x A)^\circ & :\equiv & \exists x A^\circ.
\end{array}
$$

[14]For simplicity we will perform the verification of soundness in classical logic, rather than inside classical linear logic.

The following theorem shows how we can combine the hybrid bounded interpretation above (Def 4.5) with the two possible embeddings (Defs 3.1 and 4.6) of IL^ω into LL^ω in order to obtain the additive-free fragment of both the bounded modified realizability and (a variant of the) bounded functional interpretation.

Proposition 4.7 *Let us define*

- $A_{\tilde{\mathrm{B}}}(\boldsymbol{x};\boldsymbol{y}) :\equiv (|A^*|^{\boldsymbol{x}}_{\boldsymbol{y}})^i$ *(embedding $(\cdot)^*$ using Howard modalities)*
- $A_{\tilde{\mathrm{br}}}(\boldsymbol{x}) :\equiv (|A^\circ|^{\boldsymbol{x}})^i$ *(embedding $(\cdot)^\circ$ using Kreisel modalities)*

where $(\cdot)^i$ denotes the translation of LL^ω back into IL^ω. The following holds

$$\begin{array}{lcl}
(A \to B)_{\tilde{\mathrm{B}}}(\boldsymbol{f},\boldsymbol{g};\boldsymbol{x},\boldsymbol{w}) & \equiv & \tilde{\forall}\boldsymbol{y} \leq^* \boldsymbol{f}\boldsymbol{x}\boldsymbol{w}\, A_{\tilde{\mathrm{B}}}(\boldsymbol{x};\boldsymbol{y}) \to B_{\tilde{\mathrm{B}}}(\boldsymbol{g}\boldsymbol{x};\boldsymbol{w}) \\
(\exists z A)_{\tilde{\mathrm{B}}}(\boldsymbol{x},a;\boldsymbol{f}) & \equiv & \exists z \leq^* a \tilde{\forall}\boldsymbol{y} \leq^* \boldsymbol{f}\boldsymbol{x}a\, A_{\tilde{\mathrm{B}}}(\boldsymbol{x};\boldsymbol{y}) \\
(\forall z A)_{\tilde{\mathrm{B}}}(\boldsymbol{f};\boldsymbol{y},b) & \equiv & \forall z \leq^* b\, A_{\tilde{\mathrm{B}}}(\boldsymbol{f}b;\boldsymbol{y}) \\
(\exists z \leq^* c\, A)_{\tilde{\mathrm{B}}}(\boldsymbol{x};\boldsymbol{f}) & \equiv & \exists z \leq^* c \tilde{\forall}\boldsymbol{y} \leq^* \boldsymbol{f}\boldsymbol{x}\, A_{\tilde{\mathrm{B}}}(\boldsymbol{x};\boldsymbol{y}) \\
(\forall z \leq^* c\, A)_{\tilde{\mathrm{B}}}(\boldsymbol{x};\boldsymbol{y}) & \equiv & \forall z \leq^* c\, A_{\tilde{\mathrm{B}}}(\boldsymbol{x};\boldsymbol{y}),
\end{array}$$

and

$$\begin{array}{lcl}
(A \to B)_{\tilde{\mathrm{br}}}(\boldsymbol{f}) & \equiv & \tilde{\forall}\boldsymbol{x}(A_{\tilde{\mathrm{br}}}(\boldsymbol{x}) \to B_{\tilde{\mathrm{br}}}(\boldsymbol{f}\boldsymbol{x})) \\
(\exists z A)_{\tilde{\mathrm{br}}}(\boldsymbol{x},a) & \equiv & \exists z \leq^* a\, A_{\tilde{\mathrm{br}}}(\boldsymbol{x}) \\
(\forall z A)_{\tilde{\mathrm{br}}}(\boldsymbol{f}) & \equiv & \tilde{\forall} b \forall z \leq^* b\, A_{\tilde{\mathrm{br}}}(\boldsymbol{f}b) \\
(\exists z \leq^* c\, A)_{\tilde{\mathrm{br}}}(\boldsymbol{x}) & \equiv & \exists z \leq^* c\, A_{\tilde{\mathrm{br}}}(\boldsymbol{x}) \\
(\forall z \leq^* c\, A)_{\tilde{\mathrm{br}}}(\boldsymbol{x}) & \equiv & \forall z \leq^* c\, A_{\tilde{\mathrm{br}}}(\boldsymbol{x}),
\end{array}$$

where $A \equiv B$ denotes that A is syntactically equal to B.

Although we obtain precisely the bounded modified realizability, the treatment of the existential (and bounded existential) quantifier in $(\cdot)_{\tilde{\mathrm{B}}}(\cdot\,;\cdot)$ differs from the original bounded functional interpretation. Note, however, that the following equivalence (similarly for the bounded existential quantifier) holds

$$\begin{array}{rcl}
\exists \boldsymbol{x}, a \forall \boldsymbol{f} (\exists z A)_{\tilde{\mathrm{B}}}(\boldsymbol{x},a;\boldsymbol{f}) & \equiv & \exists \boldsymbol{x}, a \forall \boldsymbol{f} \exists z \leq^* a \forall \boldsymbol{y} \leq^* \boldsymbol{f}\boldsymbol{x}a\, A_{\tilde{\mathrm{B}}}(\boldsymbol{x};\boldsymbol{y}) \\
& \Leftrightarrow & \exists \boldsymbol{x}, a \forall \boldsymbol{y} \exists z \leq^* a \forall \boldsymbol{y}' \leq^* \boldsymbol{y}\, A_{\tilde{\mathrm{B}}}(\boldsymbol{x};\boldsymbol{y}').
\end{array}$$

Finally, let us present the soundness theorem for our hybrid bounded interpretation of MELL.

Theorem 4.8 (Soundness of hybrid bounded interpretation) *Let $A_0,\ldots,A_n$ be a sequence of formulas of MELL, with $\boldsymbol{z}$ as the only free-variables. If*

$$\vdash A_0,\ldots,A_n$$

is provable in MELL then monotone terms $\boldsymbol{a}_0,\ldots,\boldsymbol{a}_n$ can be automatically synthesised from its formal proof, such that

$$(\boldsymbol{z} \leq^* \boldsymbol{z}^*), (\boldsymbol{y}_0 \leq^* \boldsymbol{y}_0), \ldots, (\boldsymbol{y}_n \leq^* \boldsymbol{y}_n) \vdash |A_0|^{\boldsymbol{a}_0}_{\boldsymbol{y}_0}, \ldots, |A_n|^{\boldsymbol{a}_n}_{\boldsymbol{y}_n}$$

where $\mathsf{FV}(\boldsymbol{a}_i) \in \{\boldsymbol{z}^*, \boldsymbol{y}_0, \ldots, \boldsymbol{y}_n\} \backslash \{\boldsymbol{y}_i\}$.

Proof. By induction on the given derivation. We assume that bounded quantifiers are axiomatised as

$$\forall x \leq^* t\, A \leftrightarrow \forall x((x \leq^* t) \multimap A)$$

$$\exists x \leq^* t\, A \leftrightarrow \exists x((x \leq^* t) \otimes A)$$

where $\leftrightarrow$ denotes linear equivalence. Note that since the majorazability relation has been defined using the Kreisel modality (see beginning of Section 4), $x \leq^* t$ is a fixed formula and hence these axioms have simple realisers. We work under assumptions

(i) $\boldsymbol{z} \leq^* \boldsymbol{z}^*$

(ii) $\boldsymbol{y}_0 \leq^* \boldsymbol{y}_0, \ldots, \boldsymbol{y}_n \leq^* \boldsymbol{y}_n$.

Whenever some of the components of these assumptions are actively used we will write them explicitly in the proof.

Tensor. Assume monotone terms $\boldsymbol{\gamma}[\boldsymbol{y}], \boldsymbol{\delta}[\boldsymbol{w}], \boldsymbol{a}, \boldsymbol{b}$ have been obtained for the premise of the rule. We construct monotone terms for the conclusion as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{(\boldsymbol{y} \leq^* \boldsymbol{y}) \vdash |\Gamma|^{\boldsymbol{\gamma}[\boldsymbol{y}]}_{\boldsymbol{v}}, |A|^{\boldsymbol{a}}_{\boldsymbol{y}}}{(\boldsymbol{fb} \leq^* \boldsymbol{fb}) \vdash |\Gamma|^{\boldsymbol{\gamma}[\boldsymbol{fb}]}_{\boldsymbol{v}}, |A|^{\boldsymbol{a}}_{\boldsymbol{fb}}}\,[\tfrac{\boldsymbol{fb}}{\boldsymbol{y}}]}{(\boldsymbol{f} \leq^* \boldsymbol{f}) \vdash |\Gamma|^{\boldsymbol{\gamma}[\boldsymbol{fb}]}_{\boldsymbol{v}}, |A|^{\boldsymbol{a}}_{\boldsymbol{fb}}} \qquad \dfrac{\dfrac{(\boldsymbol{w} \leq^* \boldsymbol{w}) \vdash |\Delta|^{\boldsymbol{\delta}[\boldsymbol{w}]}_{\boldsymbol{u}}, |B|^{\boldsymbol{b}}_{\boldsymbol{w}}}{(\boldsymbol{ga} \leq^* \boldsymbol{ga}) \vdash |\Delta|^{\boldsymbol{\delta}[\boldsymbol{ga}]}_{\boldsymbol{u}}, |B|^{\boldsymbol{b}}_{\boldsymbol{ga}}}\,[\tfrac{\boldsymbol{ga}}{\boldsymbol{w}}]}{(\boldsymbol{g} \leq^* \boldsymbol{g}) \vdash |\Delta|^{\boldsymbol{\delta}[\boldsymbol{ga}]}_{\boldsymbol{u}}, |B|^{\boldsymbol{b}}_{\boldsymbol{ga}}}}{(\boldsymbol{f} \leq^* \boldsymbol{f}), (\boldsymbol{g} \leq^* \boldsymbol{g}) \vdash |\Gamma|^{\boldsymbol{\gamma}[\boldsymbol{fb}]}_{\boldsymbol{v}}, |\Delta|^{\boldsymbol{\delta}[\boldsymbol{ga}]}_{\boldsymbol{u}}, |A|^{\boldsymbol{a}}_{\boldsymbol{fb}} \wedge |B|^{\boldsymbol{b}}_{\boldsymbol{ga}}}}{(\boldsymbol{f} \leq^* \boldsymbol{f}), (\boldsymbol{g} \leq^* \boldsymbol{g}) \vdash |\Gamma|^{\boldsymbol{\gamma}[\boldsymbol{fb}]}_{\boldsymbol{v}}, |\Delta|^{\boldsymbol{\delta}[\boldsymbol{ga}]}_{\boldsymbol{u}}, |A \otimes B|^{\boldsymbol{a},\boldsymbol{b}}_{\boldsymbol{f},\boldsymbol{g}}}$$

Given the assumptions, it is clear that $\boldsymbol{\gamma}[\boldsymbol{fb}]$ and $\boldsymbol{\delta}[\boldsymbol{ga}]$ are also monotone. For the rest of proof we omit the straightforward verification that the constructed terms are monotone.

Cut.

$$\dfrac{\dfrac{\dfrac{(\boldsymbol{y} \leq^* \boldsymbol{y}) \vdash |\Gamma|^{\boldsymbol{\gamma}[\boldsymbol{y}]}_{\boldsymbol{w}}, |A|^{\boldsymbol{a}_0[\boldsymbol{w}]}_{\boldsymbol{y}}}{(\boldsymbol{a}_1[\boldsymbol{u}] \leq^* \boldsymbol{a}_1[\boldsymbol{u}]) \vdash |\Gamma|^{\boldsymbol{\gamma}[\boldsymbol{a}_1[\boldsymbol{u}]]}_{\boldsymbol{w}}, |A|^{\boldsymbol{a}_0[\boldsymbol{w}]}_{\boldsymbol{a}_1[\boldsymbol{u}]}} \quad \dfrac{\dfrac{(\boldsymbol{x} \leq^* \boldsymbol{x}) \vdash |\Delta|^{\boldsymbol{\delta}[\boldsymbol{x}]}_{\boldsymbol{u}}, |A^{\perp}|^{\boldsymbol{a}_1[\boldsymbol{u}]}_{\boldsymbol{x}}}{(\boldsymbol{a}_0[\boldsymbol{w}] \leq^* \boldsymbol{a}_0[\boldsymbol{w}]) \vdash |\Delta|^{\boldsymbol{\delta}[\boldsymbol{a}_0[\boldsymbol{w}]]}_{\boldsymbol{u}}, |A^{\perp}|^{\boldsymbol{a}_1[\boldsymbol{u}]}_{\boldsymbol{a}_0[\boldsymbol{w}]}}}{(\boldsymbol{a}_0[\boldsymbol{w}] \leq^* \boldsymbol{a}_0[\boldsymbol{w}]) \vdash |\Delta|^{\boldsymbol{\delta}[\boldsymbol{a}_0[\boldsymbol{w}]]}_{\boldsymbol{u}}, (|A|^{\boldsymbol{a}_0[\boldsymbol{w}]}_{\boldsymbol{a}_1[\boldsymbol{u}]})^{\perp}}}{(\boldsymbol{a}_0[\boldsymbol{w}] \leq^* \boldsymbol{a}_0[\boldsymbol{w}]), (\boldsymbol{a}_1[\boldsymbol{u}] \leq^* \boldsymbol{a}_1[\boldsymbol{u}]) \vdash |\Gamma|^{\boldsymbol{\gamma}[\boldsymbol{a}_1[\boldsymbol{u}]]}_{\boldsymbol{w}}, |\Delta|^{\boldsymbol{\delta}[\boldsymbol{a}_0[\boldsymbol{w}]]}_{\boldsymbol{u}}}}{|\Gamma|^{\boldsymbol{\gamma}[\boldsymbol{a}_1[\boldsymbol{u}]]}_{\boldsymbol{w}}, |\Delta|^{\boldsymbol{\delta}[\boldsymbol{a}_0[\boldsymbol{w}]]}_{\boldsymbol{u}}}$$

since $\boldsymbol{a}_i \leq^* \boldsymbol{a}_i$ follows from assumptions (i) and (ii).

Universal quantifier.

$$\dfrac{\dfrac{\dfrac{z_0 \leq^* z_0^* \vdash |\Gamma|_{\boldsymbol{w}}^{\gamma[\boldsymbol{y},z_0^*]}, |A(z_0)|_{\boldsymbol{y}}^{\boldsymbol{a}[z_0^*]}}{z_0^* \leq^* z_0^*, z_0 \leq^* z_0^* \vdash |\Gamma|_{\boldsymbol{w}}^{\gamma[\boldsymbol{y},z_0^*]}, |A(z_0)|_{\boldsymbol{y}}^{\boldsymbol{a}[z_0^*]}}}{z_0^* \leq^* z_0^* \vdash |\Gamma|_{\boldsymbol{w}}^{\gamma[\boldsymbol{y},z_0^*]}, \forall z_0 \leq^* z_0^* \, |A(z_0)|_{\boldsymbol{y}}^{\boldsymbol{a}[z_0^*]}}}{z_0^* \leq^* z_0^* \vdash |\Gamma|_{\boldsymbol{w}}^{\gamma[\boldsymbol{y},z_0^*]}, |\forall z_0 A(z_0)|_{\boldsymbol{y},z_0^*}^{\lambda z_0 . \boldsymbol{a}[z_0]}}$$

Existential quantifier.

$$\dfrac{\dfrac{\dfrac{\dfrac{\boldsymbol{y} \leq^* \boldsymbol{y}, \boldsymbol{z} \leq^* \boldsymbol{z}^* \vdash |\Gamma|_{\boldsymbol{w}}^{\gamma[\boldsymbol{y}]}, |A(t[\boldsymbol{z}])|_{\boldsymbol{y}}^{\boldsymbol{a}} \qquad \boldsymbol{z} \leq^* \boldsymbol{z}^* \vdash t[\boldsymbol{z}] \leq^* t^*[\boldsymbol{z}^*]}{\boldsymbol{y} \leq^* \boldsymbol{y}, \boldsymbol{z} \leq^* \boldsymbol{z}^* \vdash |\Gamma|_{\boldsymbol{w}}^{\gamma[\boldsymbol{y}]}, \exists x \leq^* t^*[\boldsymbol{z}^*] |A(x)|_{\boldsymbol{y}}^{\boldsymbol{a}}}}{\boldsymbol{f}(t^*[\boldsymbol{z}^*]) \leq^* \boldsymbol{f}(t^*[\boldsymbol{z}^*]), \boldsymbol{z} \leq^* \boldsymbol{z}^* \vdash |\Gamma|_{\boldsymbol{w}}^{\gamma[\boldsymbol{f}(t^*[\boldsymbol{z}^*])]}, \exists x \leq^* t^*[\boldsymbol{z}^*] |A(x)|_{\boldsymbol{f}(t^*[\boldsymbol{z}^*])}^{\boldsymbol{a}}} \, [\tfrac{\boldsymbol{f}(t^*[\boldsymbol{z}^*])}{\boldsymbol{y}}]}{\boldsymbol{z} \leq^* \boldsymbol{z}^*, \boldsymbol{f} \leq^* \boldsymbol{f} \vdash |\Gamma|_{\boldsymbol{w}}^{\gamma[\boldsymbol{f}(t^*[\boldsymbol{z}^*])]}, \exists x \leq^* t^*[\boldsymbol{z}^*] |A(x)|_{\boldsymbol{f}(t^*[\boldsymbol{z}^*])}^{\boldsymbol{a}}}}{\boldsymbol{z} \leq^* \boldsymbol{z}^*, \boldsymbol{f} \leq^* \boldsymbol{f} \vdash |\Gamma|_{\boldsymbol{w}}^{\gamma[\boldsymbol{f}(t^*[\boldsymbol{z}^*])]}, |\exists x A(x)|_{\boldsymbol{f}}^{\boldsymbol{a}, t^*[\boldsymbol{z}^*]}}$$

Howard dereliction and promotion.

$$\dfrac{\dfrac{|\Gamma|_{\boldsymbol{w}}^{\gamma}, |A|_{\boldsymbol{y}}^{\boldsymbol{a}} \quad \vdash \boldsymbol{a} \leq^* \boldsymbol{a}}{|\Gamma|_{\boldsymbol{w}}^{\gamma}, \tilde{\exists} \boldsymbol{x} \leq^* \boldsymbol{a} \, |A|_{\boldsymbol{y}}^{\boldsymbol{x}}}}{|\Gamma|_{\boldsymbol{w}}^{\gamma}, |?_h A|_{\boldsymbol{y}}^{\boldsymbol{a}}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{|?_h \Gamma|_{\boldsymbol{w}}^{\gamma[\boldsymbol{y}]}, |A|_{\boldsymbol{y}}^{\boldsymbol{a}}}{\tilde{\exists} \boldsymbol{v} \leq^* \gamma[\boldsymbol{y}] \, |\Gamma|_{\boldsymbol{w}}^{\boldsymbol{v}}, |A|_{\boldsymbol{y}}^{\boldsymbol{a}}} \qquad \boldsymbol{y} \leq^* \boldsymbol{f}\boldsymbol{a} \vdash \boldsymbol{y} \leq^* \boldsymbol{f}\boldsymbol{a}}{\tilde{\exists} \boldsymbol{y} \leq^* \boldsymbol{f}\boldsymbol{a} \, \tilde{\exists} \boldsymbol{v} \leq^* \gamma[\boldsymbol{y}] \, |\Gamma|_{\boldsymbol{w}}^{\boldsymbol{v}}, \tilde{\forall} \boldsymbol{y} \leq^* \boldsymbol{f}\boldsymbol{a} \, |A|_{\boldsymbol{y}}^{\boldsymbol{a}}}}{\boldsymbol{f} \leq^* \boldsymbol{f} \vdash \tilde{\exists} \boldsymbol{v} \leq^* \gamma[\boldsymbol{f}\boldsymbol{a}] \, |\Gamma|_{\boldsymbol{w}}^{\boldsymbol{v}}, \tilde{\forall} \boldsymbol{y} \leq^* \boldsymbol{f}\boldsymbol{a} \, |A|_{\boldsymbol{y}}^{\boldsymbol{a}}}}{\boldsymbol{f} \leq^* \boldsymbol{f} \vdash |?_h \Gamma|_{\boldsymbol{w}}^{\gamma[\boldsymbol{f}\boldsymbol{a}]}, |!_h A|_{\boldsymbol{f}}^{\boldsymbol{a}}}$$

Howard weakening and contraction.

$$\dfrac{\dfrac{|\Gamma|_{\boldsymbol{w}}^{\gamma}}{|\Gamma|_{\boldsymbol{w}}^{\gamma}, \tilde{\exists} \boldsymbol{x} \leq^* \boldsymbol{c} \, |A|_{\boldsymbol{y}}^{\boldsymbol{x}}}}{|\Gamma|_{\boldsymbol{w}}^{\gamma}, |?_h A|_{\boldsymbol{y}}^{\boldsymbol{c}}} \qquad \dfrac{\dfrac{\dfrac{|\Gamma|_{\boldsymbol{w}}^{\gamma[\boldsymbol{y}_0, \boldsymbol{y}_1]}, |?_h A|_{\boldsymbol{y}_0}^{\boldsymbol{a}_0}, |?_h A|_{\boldsymbol{y}_1}^{\boldsymbol{a}_1}}{|\Gamma|_{\boldsymbol{w}}^{\gamma[\boldsymbol{y}]}, |?_h A|_{\boldsymbol{y}}^{\boldsymbol{a}_0}, |?_h A|_{\boldsymbol{y}}^{\boldsymbol{a}_1}}}{|\Gamma|_{\boldsymbol{w}}^{\gamma[\boldsymbol{y}]}, |?_h A|_{\boldsymbol{y}}^{\max\{\boldsymbol{a}_0, \boldsymbol{a}_1\}}, |?_h A|_{\boldsymbol{y}}^{\max\{\boldsymbol{a}_0, \boldsymbol{a}_1\}}}}{|\Gamma|_{\boldsymbol{w}}^{\gamma[\boldsymbol{y}]}, |?_h A|_{\boldsymbol{y}}^{\max\{\boldsymbol{a}_0, \boldsymbol{a}_1\}}}$$

Kreisel dereliction and promotion.

$$\dfrac{\dfrac{|\Gamma|_{\boldsymbol{w}}^{\gamma[\boldsymbol{y}]}, |A|_{\boldsymbol{y}}^{\boldsymbol{a}} \quad \vdash \boldsymbol{a} \leq^* \boldsymbol{a}}{|\Gamma|_{\boldsymbol{w}}^{\gamma[\boldsymbol{y}]}, \tilde{\exists} \boldsymbol{x} |A|_{\boldsymbol{y}}^{\boldsymbol{x}}}}{|\Gamma|_{\boldsymbol{w}}^{\gamma[\boldsymbol{y}]}, |?_k A|_{\boldsymbol{y}}^{\boldsymbol{a}}} \qquad \dfrac{\dfrac{\boldsymbol{y} \leq^* \boldsymbol{y} \vdash |?_k \Gamma|_{\boldsymbol{w}}, |A|_{\boldsymbol{y}}^{\boldsymbol{a}}}{\vdash |?_k \Gamma|_{\boldsymbol{w}}, \tilde{\forall} \boldsymbol{y} |A|_{\boldsymbol{y}}^{\boldsymbol{a}}}}{\vdash |?_k \Gamma|_{\boldsymbol{w}}, |!_k A|^{\boldsymbol{a}}}$$

Kreisel weakening and contraction.

$$\dfrac{\dfrac{|\Gamma|_{\boldsymbol{w}}^{\gamma}}{|\Gamma|_{\boldsymbol{w}}^{\gamma}, \tilde{\exists} \boldsymbol{x} |A|_{\boldsymbol{y}}^{\boldsymbol{x}}}}{|\Gamma|_{\boldsymbol{w}}^{\gamma}, |?_k A|_{\boldsymbol{y}}} \qquad \dfrac{\dfrac{|\Gamma|_{\boldsymbol{w}}^{\gamma[\boldsymbol{y}_0, \boldsymbol{y}_1]}, |?_k A|_{\boldsymbol{y}_0}, |?_k A|_{\boldsymbol{y}_1}}{|\Gamma|_{\boldsymbol{w}}^{\gamma[\boldsymbol{y}]}, |?_k A|_{\boldsymbol{y}}, |?_k A|_{\boldsymbol{y}}}}{|\Gamma|_{\boldsymbol{w}}^{\gamma[\boldsymbol{y}]}, |?_k A|_{\boldsymbol{y}}}$$

That concludes the proof. $\Box$

Remark 4.9 *A final remark about the additives. Note that we could have defined an interpretation for the if-then-else connective as*

$$|A \diamond_z B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} :\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \diamond_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}}$$

and this would give rise to a formula interpretation of additives (viewing boolean quantifications as bounded quantifications) which indeed corresponds precisely to the formula interpretation *of bounded modified realizability, and to the* formula interpretation *of a variant of the bounded functional interpretation (of intuitionistic logic). The correspondence on the level of formulas, however, does not lift to a sound proof interpretation in linear logic, due to the lack of the monotonicity property (12).*

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