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Proof Interpretations with Truth

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This article systematically investigates so-called "truth variants" of several functional interpretations. We start by showing a close relation between two variants of modified realizability, namely modified realizability with truth and q-modified realizability. Both variants are shown to be derived from a single "functional interpretation with truth" of intuitionistic linear logic. This analysis suggests that several functional interpretations have truth and q-variants. These variants, however, require a more involved modification than the ones previously considered. Following this lead we present truth and q-variants of the Diller-Nahm interpretation, the bounded modified realizability and the bounded functional interpretation.

1 Introduction

Modified realizability associates to each formula A of Heyting arithmetic in all finite types a new formula "$\boldsymbol{x}$ mr A" by induction on the structure of A as:

$$
\begin{aligned}
\boldsymbol{x}, \boldsymbol{y} \text{ mr } A \wedge B &:\equiv (\boldsymbol{x} \text{ mr } A) \wedge (\boldsymbol{y} \text{ mr } B) \\
z, \boldsymbol{x}, \boldsymbol{y} \text{ mr } A \vee B &:\equiv (z = 0 \to (\boldsymbol{x} \text{ mr } A)) \wedge (z \neq 0 \to (\boldsymbol{y} \text{ mr } B)) \\
\boldsymbol{f} \text{ mr } A \to B &:\equiv \forall \boldsymbol{x}((\boldsymbol{x} \text{ mr } A) \to (\boldsymbol{f}\boldsymbol{x} \text{ mr } B)) \\
\boldsymbol{f} \text{ mr } \forall z A &:\equiv \forall z(\boldsymbol{f} z \text{ mr } A) \\
z, \boldsymbol{x} \text{ mr } \exists z A &:\equiv \boldsymbol{x} \text{ mr } A.
\end{aligned}
$$

Intuitively, the fresh tuple of free-variables $\boldsymbol{x}$ in the formula "$\boldsymbol{x}$ mr A" captures the existential information of the formula A, in the style of the informal BHK interpretation. The formula A is reduced to $\exists \boldsymbol{x}(\boldsymbol{x} \text{ mr } A)$ in such a way that if A is provable (in Heyting arithmetic, for instance) then a tuple of terms $\boldsymbol{t}$ (in Gödel's T) can be extracted from this proof such that "$\boldsymbol{t}$ mr A".

Note that this tuple of terms $\boldsymbol{t}$ extracted from the proof of A is a witness for the statement $\exists \boldsymbol{x}(\boldsymbol{x} \text{ mr } A)$. One may then wonder what has been gained, since ideally we would like to obtain witnesses for the original theorem A (and not necessarily for its interpretation $\exists \boldsymbol{x}(\boldsymbol{x} \text{ mr } A)$). As it turns out, the modified realizability of a formula A retains a relation with the original formula A for the small class of $\exists$-free formulas (that is, formulas without disjunctions and existential quantifiers), in the sense that the *truth property*

$$(\boldsymbol{x} \text{ mr } A_{\text{ef}}) \to A_{\text{ef}}$$

holds for all $\exists$-free formulas A_{ef}. In this way, from a proof of $\exists z A_{\mathrm{ef}}$, where A_{ef} is existential-free, one obtains a tuple of terms such that $s, \boldsymbol{t}$ mr $\exists z A_{\mathrm{ef}} \equiv \boldsymbol{t}$ mr $A_{\mathrm{ef}}[s/z]$, which given the truth property

$$(\boldsymbol{t} \text{ mr } A_{\mathrm{ef}}[s/z]) \to A_{\mathrm{ef}}[s/z]$$

implies that s is a witness for $\exists z A_{\mathrm{ef}}$.

One desires, however, to extend this property to larger classes. This was achived by Kleene and Grayson by adapting the modified realizability clauses for disjunction, existential quantification and implication. Kleene extended the truth property to *disjunctive and existential formulas* by considering a variant mq of mr, called q-modified realizability, where one adds A and B to the disjunction and existential quantifier clauses:

$$z, \boldsymbol{x}, \boldsymbol{y} \text{ mq } A \vee B :\equiv (z = 0 \to (\boldsymbol{x} \text{ mq } A) \wedge A) \wedge (z \neq 0 \to (\boldsymbol{y} \text{ mq } B) \wedge B)$$
$$z, \boldsymbol{x} \text{ mq } \exists z A :\equiv (\boldsymbol{x} \text{ mq } A) \wedge A.$$

Grayson [7] (based on Aczel's slash translation) extended the truth property to *all formulas* by considering a variant mrt of mr, called modified realizability with truth, where one only adds $A \to B$ to the implication clause:

$$\boldsymbol{f} \text{ mrt } A \to B :\equiv \forall \boldsymbol{x}((\boldsymbol{x} \text{ mrt } A) \to (\boldsymbol{f}\boldsymbol{x} \text{ mrt } B)) \wedge (A \to B).$$

The modified realizability with truth can be quite useful, as demonstrated in [11–13], since the interpretation of A does not lose the relation to the original formula A.

Recently, Jørgensen [10] presented a q-variant of the Diller-Nahm interpretation. In the same paper, Jørgensen shows that simply adding $A \to B$ to the implication clause of the Diller-Nahm interpretation does not lead to a sound "Diller-Nahm interpretation with truth". However, Jørgensen's work does not rule out the possibility that a more comprehensive modification works.

In this paper we make mainly two points. (1) We first show a close relation between the q-modified realizability and the modified realizability with truth. Then we factorise the two in terms of the two standard embeddings of intuitionistic logic into linear logic composed with a single modified realizability with truth of intuitionistic linear logic. (2) The factorisation suggests that the universal quantifier clause of modified realizability with truth should be

$$\boldsymbol{f} \text{ mrt } \forall z A :\equiv \forall z(\boldsymbol{f} z \text{ mrt } A) \wedge \forall z A.$$

For modified realizability with truth the addition of $\forall z A$ is redundant (since $\forall z(\boldsymbol{f} z \text{ mrt } A) \to \forall z A$ by the truth property), but for other proof interpretations this makes a difference. Indeed, following this clue, we show that by adding $\forall z A$ to the universal quantifier clause, we get truth variants of the bounded modified realizability, the Diller-Nahm interpretation, and the bounded functional interpretation.

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1.1 Organisation

Section 2. Modified realizability with truth, relation between q- and truth variants. *Section 3.* Truth interpretation of intuitionistic linear logic, relation to interpretations of intuitionistic logic. *Section 4.* Kleene and Aczel slash translations. *Section 5.* Bounded modified realizability with truth. *Section 6.* Diller-Nahm interpretation with truth. *Section 7.* Bounded functional interpretation with truth. *Section 8.* Conclusion.

2 Modified Realizability with Truth

Let us start with a discussion (and some new results) about the two "truth variants" of modified realizability.

2.1 Definitions

Let us use the abbreviation $A \diamond_b B :\equiv (b = \mathrm{t} \to A) \wedge (b = \mathrm{f} \to B)$, where b is a boolean variable, t and f are respectively the true and false values, and we have equality between booleans. In arithmetic we may take $b = \mathrm{t}$ and $b = \mathrm{f}$ as being synonymous for $b =_0 0$ and $b \neq_0 0$, respectively.

Definition 2.1 (q-variant of modified realizability) Each formula A of IL^ω (intuitionistic logic in the language of finite types, see section 3.1) is associated to a new formula $\boldsymbol{x}$ mq A inductively as follows. If A_{at} is an atomic formula, then $\boldsymbol{x}$ mq $A_{\mathrm{at}} :\equiv A_{\mathrm{at}}$ where $\boldsymbol{x}$ is the empty tuple. Assuming we already have defined $\boldsymbol{x}$ mq A and $\boldsymbol{y}$ mq B, then

$$\begin{aligned}
\boldsymbol{x}, \boldsymbol{y} \text{ mq } A \wedge B &:\equiv (\boldsymbol{x} \text{ mq } A) \wedge (\boldsymbol{y} \text{ mq } B)\\
\boldsymbol{x}, \boldsymbol{y}, b \text{ mq } A \vee B &:\equiv ((\boldsymbol{x} \text{ mq } A) \wedge A) \diamond_b ((\boldsymbol{y} \text{ mq } B) \wedge B)\\
\boldsymbol{f} \text{ mq } A \to B &:\equiv \forall \boldsymbol{x}((\boldsymbol{x} \text{ mq } A) \wedge A \to (\boldsymbol{f}\boldsymbol{x} \text{ mq } B))\\
\boldsymbol{f} \text{ mq } \forall z A &:\equiv \forall z(\boldsymbol{f} z \text{ mq } A)\\
z, \boldsymbol{x} \text{ mq } \exists z A &:\equiv (\boldsymbol{x} \text{ mq } A) \wedge A.
\end{aligned}$$

The *q-variant of modified realizability* of a formula A is then defined as $A^{\mathrm{mq}} :\equiv \exists \boldsymbol{x}(\boldsymbol{x} \text{ mq } A)$.

Remark 2.2 We have $\mathsf{IL}^\omega \vdash (\boldsymbol{x} \text{ mq } A) \to A$ for all disjunctive and existential formulas A of IL^ω.

Definition 2.3 (modified realizability with truth) Each formula A of IL^ω is associated to a new formula $\boldsymbol{x}$ mrt A inductively as follows. If A_{at} is an atomic formula, then $\boldsymbol{x}$ mrt $A_{\mathrm{at}} :\equiv A_{\mathrm{at}}$ where $\boldsymbol{x}$ is the empty tuple. Assuming we already have defined $\boldsymbol{x}$ mrt A and $\boldsymbol{y}$ mrt B, then

$$\begin{aligned}
\boldsymbol{x}, \boldsymbol{y} \text{ mrt } A \wedge B &:\equiv (\boldsymbol{x} \text{ mrt } A) \wedge (\boldsymbol{y} \text{ mrt } B)\\
\boldsymbol{x}, \boldsymbol{y}, b \text{ mrt } A \vee B &:\equiv (\boldsymbol{x} \text{ mrt } A) \diamond_b (\boldsymbol{y} \text{ mrt } B)\\
\boldsymbol{f} \text{ mrt } A \to B &:\equiv \forall \boldsymbol{x}((\boldsymbol{x} \text{ mrt } A) \to (\boldsymbol{f}\boldsymbol{x} \text{ mrt } B)) \wedge (A \to B)\\
\boldsymbol{f} \text{ mrt } \forall z A &:\equiv \forall z(\boldsymbol{f} z \text{ mrt } A)\\
z, \boldsymbol{x} \text{ mrt } \exists z A &:\equiv \boldsymbol{x} \text{ mrt } A.
\end{aligned}$$

The *modified realizability with truth* of a formula A is then defined as $A^{\mathrm{mrt}} :\equiv \exists \boldsymbol{x}(\boldsymbol{x} \text{ mrt } A)$.

Remark 2.4 We have $\mathsf{IL}^\omega \vdash (\boldsymbol{x} \text{ mrt } A) \to A$ for all formulas A of IL^ω.

2.2 Relation between the q- and the truth variants of modified realizability

Intuitively, the truth-variant is stronger than the q-variant, since the truth property of mrt holds for all formulas A, whereas for the q-variant it only holds for disjunctive and existential formulas. The theorem gives a rigorous mathematical meaning to this intuition (cf. also [19], volume I, chapter 4, section 10.3).

Theorem 2.5 *For all formulas A of IL^ω we have*

$$\mathsf{IL}^\omega \vdash (\boldsymbol{x} \text{ mrt } A) \leftrightarrow (\boldsymbol{x} \text{ mq } A) \wedge A.$$

Proof. By an easy induction on the structure of the formulas.

<u>$A \to B$</u>.

$$\begin{aligned}
\boldsymbol{f} \text{ mrt } A \to B &\equiv \forall \boldsymbol{x}((\boldsymbol{x} \text{ mrt } A) \to (\boldsymbol{f}\boldsymbol{x} \text{ mrt } B)) \wedge (A \to B)\\
&\overset{\text{(IH)}}{\leftrightarrow} \forall \boldsymbol{x}((\boldsymbol{x} \text{ mq } A) \wedge A \to (\boldsymbol{f}\boldsymbol{x} \text{ mq } B) \wedge B) \wedge (A \to B)\\
&\leftrightarrow \forall \boldsymbol{x}((\boldsymbol{x} \text{ mq } A) \wedge A \to (\boldsymbol{f}\boldsymbol{x} \text{ mq } B)) \wedge (A \to B)\\
&\equiv (\boldsymbol{f} \text{ mq } A \to B) \wedge (A \to B).
\end{aligned}$$

<u>$\forall z A$</u>.

$$\begin{aligned} \boldsymbol{f} \text{ mrt } \forall z A &\equiv \forall z(\boldsymbol{f} z \text{ mrt } A) \\ &\overset{\text{(IH)}}{\leftrightarrow} \forall z((\boldsymbol{f} z \text{ mq } A) \wedge A) \\ &\leftrightarrow \forall z(\boldsymbol{f} z \text{ mq } A) \wedge \forall z A \\ &\equiv (\boldsymbol{f} \text{ mq } \forall z A) \wedge \forall z A. \end{aligned}$$

The other cases are treated similarly. □

2.3 Characterisation

Although we will show that it is possible to produce truth variants for almost any given functional interpretation, it seems that only realizability interpretations have a "nice" characterisation, in the sense that the truth variant of the interpretation can be characterised by the same principles as the version "without truth". For instance, let $\mathsf{IL}^{\#}$ be the theory IL^{ω} extended with the characterisation principles for the standard modified realizability, namely

$$\begin{aligned} \mathsf{AC}: &\quad \forall \boldsymbol{x} \exists \boldsymbol{y} A(\boldsymbol{x}, \boldsymbol{y}) \to \exists \boldsymbol{f} \forall \boldsymbol{x} A(\boldsymbol{x}, \boldsymbol{f}\boldsymbol{x}) \\ \mathsf{IP}_{\mathrm{ef}}: &\quad (A_{\mathrm{ef}} \to \exists \boldsymbol{x} B) \to \exists \boldsymbol{x}(A_{\mathrm{ef}} \to B), \end{aligned}$$

where A_{ef} is an existential-free formula.

Theorem 2.6 *For all formulas A of IL^{ω} we have*

(i) $\mathsf{IL}^{\#} \vdash \exists \boldsymbol{x}(\boldsymbol{x} \text{ mq } A) \leftrightarrow A$

(ii) $\mathsf{IL}^{\#} \vdash \exists \boldsymbol{x}(\boldsymbol{x} \text{ mrt } A) \leftrightarrow A$.

P r o o f. By the characterisation of modified realizability we have

$$\mathsf{IL}^{\#} \vdash \exists \boldsymbol{x}(\boldsymbol{x} \text{ mr } A) \leftrightarrow A,$$

for any formula A in the language of IL^{ω}. In order to prove (i), one can easily show by induction on A that

$$\mathsf{IL}^{\#} \vdash (\boldsymbol{x} \text{ mq } A) \leftrightarrow (\boldsymbol{x} \text{ mr } A),$$

since $\mathsf{IL}^{\#} \vdash (\boldsymbol{x} \text{ mr } A) \to A$. Point (ii) follows from (i) by Theorem 2.5. □

3 A Truth Interpretation of Linear Logic

The second author [15] has shown that modified realizability of IL^{ω} can be decomposed into a modified realizability of *classical* linear logic together with Girard's embedding of IL^{ω} into linear logic. Recently [5], this has been turned into a finer decomposition in terms of *intuitionistic* linear logic ILL^{ω}. In this section we present a truth variant for this functional interpretation of ILL^{ω}. We will then show how the q- and truth-variants of the modified realizability correspond to the truth interpretation of linear logic via two of Girard's embeddings of IL^{ω} into ILL^{ω}.

3.1 Intuitionistic logic and intuitionistic linear logic

Our languages for intuitionistic logic IL^{ω} and intuitionistic linear logic ILL^{ω} are typed over the ground base types b (booleans) and i (integers) and have two ground constants t^{b} (true) and f^{b} (false) and a ground constant of type i (to ensure that for every type ρ there is a closed term $\boldsymbol{0}^{\rho}$). Also they have term application $(t^{\rho\to\sigma} q^{\rho})^{\sigma}$, λ-abstraction $(\lambda \boldsymbol{x}^{\rho} \,.\, \boldsymbol{t}^{\sigma})^{\rho\to\sigma}$ and definition of terms by cases $(b)(\boldsymbol{c}_0, \boldsymbol{c}_1)$, with axioms

$$A_{\mathrm{at}}[(\lambda \boldsymbol{x} \,.\, \boldsymbol{t}[\boldsymbol{x}])\boldsymbol{q}] \multimapboth A_{\mathrm{at}}[\boldsymbol{t}[\boldsymbol{q}]] \qquad A_{\mathrm{at}}[(\mathrm{t})(\boldsymbol{c}_0, \boldsymbol{c}_1)] \multimapboth A_{\mathrm{at}}[\boldsymbol{c}_0]$$

$$A_{\mathrm{at}}[\lambda \boldsymbol{x} \,.\, \boldsymbol{t}\boldsymbol{x}] \multimapboth A_{\mathrm{at}}[\boldsymbol{t}] \qquad A_{\mathrm{at}}[(\mathrm{f})(\boldsymbol{c}_0, \boldsymbol{c}_1)] \multimapboth A_{\mathrm{at}}[\boldsymbol{c}_1]$$

$P \vdash P$ (id)	$\dfrac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$ (cut)
$\Gamma, 0 \vdash A$ (0L)	$\Gamma \vdash \top$ ($\top$R)
$\dfrac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$ ($\otimes$L)	$\dfrac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$ ($\otimes$R)
$\dfrac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}$ ($\oplus$L)	$\dfrac{\Gamma \vdash A_i}{\Gamma \vdash A_0 \oplus A_1}$ ($\oplus$R)
$\dfrac{\Gamma, A_i \vdash C}{\Gamma, A_0 \,\&\, A_1 \vdash C}$ (&L)	$\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}$ (&R)
$\dfrac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}$ ($\multimap$L)	$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$ ($\multimap$R)
$\dfrac{\Gamma, A[t/x] \vdash B}{\Gamma, \forall x A \vdash B}$ ($\forall$L)	$\dfrac{\Gamma \vdash A}{\Gamma \vdash \forall x A}$ ($\forall$R)
$\dfrac{\Gamma, A \vdash B}{\Gamma, \exists x A \vdash B}$ ($\exists$L)	$\dfrac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x A}$ ($\exists$R)
$\dfrac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}$ (con)	$\dfrac{\Gamma \vdash B}{\Gamma, !A \vdash B}$ (wkn)
$\dfrac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}$ (!L)	$\dfrac{!\Gamma \vdash A}{!\Gamma \vdash !A}$ (!R)

Table 1 Rules of ILL^ω

(restricted to atomic formulas A_{at}, but can be generalised to arbitrary formulas). We assume only equality between booleans, for which we require the following axioms [5]:

$$\vdash b = b \qquad\qquad !(b_0 = b_1) \vdash b_1 = b_0$$
$$!(b_0 = b_1), !(b_1 = b_2) \vdash b_0 = b_2 \qquad\qquad !(b_0 = b_1), A[b_0] \vdash A[b_1]$$
$$\Gamma, !(\mathrm{t} = \mathrm{f}) \vdash 0 \qquad\qquad \vdash !(b = \mathrm{t}) \oplus !(b = \mathrm{f}).$$

For IL^ω we drop the bangs and replace $\oplus$ by $\vee$, $\multimapboth$ by $\leftrightarrow$ and 0 by $\bot$ in the axioms above, and write them in the form $\Gamma \to A$ instead of $\Gamma \vdash A$ (to fit the one-sided axiomatization in [1], section 2.1).

IL^ω is based on $\wedge, \vee, \to, \forall, \exists, \bot$. Its pure logical axioms are the standard ones (cf. [1], section 2.1). It has an if-then-else logical constructor $A \diamond_b B$ defined as

$$A \diamond_b B :\equiv (b = \mathrm{t} \to A) \wedge (b = \mathrm{f} \to B).$$

ILL^ω is based on $\otimes, \&, \oplus, \multimap, \forall, \exists, !, 0, \top$. We define $1 :\equiv !\top$. Its pure logical axioms are shown in Table 1. It also has an if-then-else defined as

$$A \diamond_b B :\equiv (!(b = \mathrm{t}) \multimap A) \,\&\, (!(b = \mathrm{f}) \multimap B).$$

$$\frac{\Gamma[\gamma_0], A \vdash C[c_0] \quad \Gamma[\gamma_1], B \vdash C[c_1]}{\Gamma[(b)(\gamma_0,\gamma_1)], A \diamond_b B \vdash C[(b)(c_0,c_1)]}\ (\diamond_b \mathrm{L}) \qquad \frac{\Gamma[\gamma_0] \vdash A \quad \Gamma[\gamma_1] \vdash B}{\Gamma[(b)(\gamma_0,\gamma_1)] \vdash A \diamond_b B}\ (\diamond_b \mathrm{R})$$

$$\frac{\Gamma, A \vdash C}{\Gamma, A \diamond_{\mathrm{t}} B \vdash C}\ (\diamond_{\mathrm{t}} \mathrm{L}) \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \diamond_{\mathrm{t}} B}\ (\diamond_{\mathrm{t}} \mathrm{R})$$

$$\frac{\Gamma, B \vdash C}{\Gamma, A \diamond_{\mathrm{f}} B \vdash C}\ (\diamond_{\mathrm{f}} \mathrm{L}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \diamond_{\mathrm{f}} B}\ (\diamond_{\mathrm{f}} \mathrm{R})$$

Table 2 Rules of $\diamond$

One can prove that the if-then-else is well-behaved, in the sense that the rules shown in table 2 hold in ILL^ω.

IL^ω can be embedded into ILL^ω via any of the two Girard translations $(\cdot)^*$ and $(\cdot)^\circ$.

Definition 3.1 ([6]) Let us define two embeddings of IL^ω into ILL^ω. Each associates to any formula A of intuitionistic logic its linear translation A^* and A° inductively as

$$\begin{aligned}
A^*_{\mathrm{at}} &:\equiv A_{\mathrm{at}}, \quad \text{if } A_{\mathrm{at}} \not\equiv \bot & A^\circ_{\mathrm{at}} &:\equiv\ !A_{\mathrm{at}}, \quad \text{if } A_{\mathrm{at}} \not\equiv \bot\\
\bot^* &:\equiv 0 & \bot^\circ &:\equiv 0\\
(A \wedge B)^* &:\equiv A^* \,\&\, B^* & (A \wedge B)^\circ &:\equiv A^\circ \otimes B^\circ\\
(A \vee B)^* &:\equiv\ !A^* \oplus !B^* & (A \vee B)^\circ &:\equiv A^\circ \oplus B^\circ\\
(A \to B)^* &:\equiv\ !A^* \multimap B^* & (A \to B)^\circ &:\equiv\ !(A^\circ \multimap B^\circ)\\
(\forall x A)^* &:\equiv \forall x A^* & (\forall x A)^\circ &:\equiv\ !\forall x A^\circ\\
(\exists x A)^* &:\equiv \exists x !A^* & (\exists x A)^\circ &:\equiv \exists x A^\circ.
\end{aligned}$$

The translations are sound in the following sense: if $\mathsf{IL}^\omega \vdash A$, then $\mathsf{ILL}^\omega \vdash A^*$ and $\mathsf{ILL}^\omega \vdash A^\circ$ (soundness for the translation $(\cdot)^\circ$ is proved by noticing that each A° is equivalent to $!A^*$, as the following lemma shows). The two embeddings are related to each other as follows:

Lemma 3.2 ([5]) *For any formula A of IL^ω we have $\mathsf{ILL}^\omega \vdash A^\circ \multimapboth\ !A^*$.*

Proof. By a simple structural induction on the formula A noting that the equivalences

$$\begin{aligned}
0 &\multimapboth\ !0 & !A \otimes !B &\multimapboth\ !(A \,\&\, B)\\
!A \oplus !B &\multimapboth\ !(!A \oplus !B) & !(!A \multimap !B) &\multimapboth\ !(!A \multimap B)\\
!\forall x !A &\multimapboth\ !\forall x A & \exists x !A &\multimapboth\ !\exists x !A
\end{aligned}$$

hold in ILL^ω. □

3.2 Definition

The following definition presents (in a uniform way) both the modified realizability interpretation of intuitionistic linear logic [5] and its truth variant.

Definition 3.3 (Modified realizability of ILL^ω [5] and its truth variant) Let F be a mapping of formulas of ILL^ω such that

($\star_1$) $F(A)[t/z] \equiv F(A[t/z])$

($\star_2$) if $!\Gamma \vdash A$ then $!F(\Gamma) \vdash F(A)$.

Each formula A of ILL^ω is associated to a new formula $\{A\}^{\boldsymbol{x}}_{\boldsymbol{y}}$ (where $\boldsymbol{x}$ and $\boldsymbol{y}$ are tuples of variables) inductively as follows. If $A_{\rm at}$ is an atomic formula, then $\{A\}^{\boldsymbol{x}}_{\boldsymbol{y}} :\equiv A_{\rm at}$ where $\boldsymbol{x}$ and $\boldsymbol{y}$ are the empty tuple. Assuming we already have defined $\{A\}^{\boldsymbol{x}}_{\boldsymbol{y}}$ and $\{B\}^{\boldsymbol{v}}_{\boldsymbol{w}}$, then

$$
\begin{aligned}
\{A \otimes B\}^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} &:\equiv \{A\}^{\boldsymbol{x}}_{\boldsymbol{y}} \otimes \{B\}^{\boldsymbol{v}}_{\boldsymbol{w}} \\
\{A \mathbin{\&} B\}^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w},b} &:\equiv \{A\}^{\boldsymbol{x}}_{\boldsymbol{y}} \Diamond_b \{B\}^{\boldsymbol{v}}_{\boldsymbol{w}} \\
\{A \oplus B\}^{\boldsymbol{x},\boldsymbol{v},b}_{\boldsymbol{y},\boldsymbol{w}} &:\equiv \{A\}^{\boldsymbol{x}}_{\boldsymbol{y}} \Diamond_b \{B\}^{\boldsymbol{v}}_{\boldsymbol{w}} \\
\{A \multimap B\}^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} &:\equiv \{A\}^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{x}\boldsymbol{w}} \multimap \{B\}^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{w}} \\
\{\forall z A(z)\}^{\boldsymbol{f}}_{\boldsymbol{y},z} &:\equiv \{A(z)\}^{\boldsymbol{f}z}_{\boldsymbol{y}} \\
\{\exists z A(z)\}^{\boldsymbol{x},z}_{\boldsymbol{y}} &:\equiv \{A(z)\}^{\boldsymbol{x}}_{\boldsymbol{y}} \\
\{!A\}^{\boldsymbol{x}} &:\equiv \,!\forall \boldsymbol{y} \{A\}^{\boldsymbol{x}}_{\boldsymbol{y}} \otimes \,!F(A).
\end{aligned}
$$

Intuitively, the interpretation above associates a two-player (say Eloise/Abelard) game to each formula A. The binary relation $\{A\}^{\boldsymbol{x}}_{\boldsymbol{y}}$ should be thought of as the "outcome function" which, given higher-order moves $\boldsymbol{x}$ and $\boldsymbol{y}$ by both players, determines which player wins.

If $F(A) :\equiv 1$, then $\{\cdot\}$ is called *modified realizability* and the "$\otimes\, !F(A)$" in the clause of bang becomes superfluous (as $!A$ is equivalent to $!A \otimes !1$) and so may be omitted. If $F(A) :\equiv A$, then $\{\cdot\}$ is called *modified realizability with truth*.

Remark 3.4 We have $\mathsf{ILL}^\omega \vdash \{!A\}^{\boldsymbol{x}} \multimap \,!F(A)$ for all formulas A of ILL^ω.

In the case $F(A) \equiv A$, the idea is that in a functional interpretation with truth we want Eloise to win the game $!A$ only when A is "true". We will show that these variants of the modified realizability of ILL^ω are also sound, and that moreover they refine both the truth and q-variants of modified realizability (cf. Proposition 3.7).

3.3 Soundness theorem

Theorem 3.5 (Soundness for modified realizability of ILL^ω) *Let $A_0, \ldots, A_n$ and B be arbitrary formulas of ILL^ω and let $\boldsymbol{z}$ be all their free variables. From any proof of $A_0(\boldsymbol{z}), \ldots, A_n(\boldsymbol{z}) \vdash B(\boldsymbol{z})$ in ILL^ω we can effectively extract terms $\boldsymbol{a}_0, \ldots, \boldsymbol{a}_n$ and $\boldsymbol{b}$ of ILL^ω such that $\{A_0(\boldsymbol{z})\}^{\boldsymbol{x}_0}_{\boldsymbol{a}_0}, \ldots, \{A_n(\boldsymbol{z})\}^{\boldsymbol{x}_n}_{\boldsymbol{a}_n} \vdash \{B(\boldsymbol{z})\}^{\boldsymbol{b}}_{\boldsymbol{w}}$ is provable in ILL^ω, where $FV(\boldsymbol{a}_i) \subseteq \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_n, \boldsymbol{z}, \boldsymbol{w}\}$ and $FV(\boldsymbol{b}) \subseteq \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_n, \boldsymbol{z}\}$.*

P r o o f. By induction on the derivation of $A_0(\boldsymbol{z}), \ldots, A_n(\boldsymbol{z}) \vdash B(\boldsymbol{z})$. Since the truth interpretation coincides with the standard interpretation for pure ILL^ω, we only need to check the rules for the modality $!A$ (see [5] for the soundness proof for the pure fragment of ILL^ω).

(con).

$$
\dfrac{\dfrac{\dfrac{\dfrac{\{\Gamma\}^{\boldsymbol{u}}_{\boldsymbol{\gamma}[\boldsymbol{x}_0,\boldsymbol{x}_1]}, \{!A\}^{\boldsymbol{x}_0}, \{!A\}^{\boldsymbol{x}_1} \vdash \{B\}^{\boldsymbol{b}[\boldsymbol{x}_0,\boldsymbol{x}_1]}_{\boldsymbol{w}}}{\{\Gamma\}^{\boldsymbol{u}}_{\boldsymbol{\gamma}[\boldsymbol{x}_0,\boldsymbol{x}_1]}, !\forall\boldsymbol{y}\{A\}^{\boldsymbol{x}_0}_{\boldsymbol{y}}, !F(A), !\forall\boldsymbol{y}\{A\}^{\boldsymbol{x}_1}_{\boldsymbol{y}}, !F(A) \vdash \{B\}^{\boldsymbol{b}[\boldsymbol{x}_0,\boldsymbol{x}_1]}_{\boldsymbol{w}}}\,(\mathrm{D3.3})}{\{\Gamma\}^{\boldsymbol{u}}_{\boldsymbol{\gamma}[\boldsymbol{x},\boldsymbol{x}]}, !\forall\boldsymbol{y}\{A\}^{\boldsymbol{x}}_{\boldsymbol{y}}, !F(A), !\forall\boldsymbol{y}\{A\}^{\boldsymbol{x}}_{\boldsymbol{y}}, !F(A) \vdash \{B\}^{\boldsymbol{b}[\boldsymbol{x},\boldsymbol{x}]}_{\boldsymbol{w}}}\,[\tfrac{\boldsymbol{x}}{\boldsymbol{x}_0}, \tfrac{\boldsymbol{x}}{\boldsymbol{x}_1}]}{\{\Gamma\}^{\boldsymbol{u}}_{\boldsymbol{\gamma}[\boldsymbol{x},\boldsymbol{x}]}, !\forall\boldsymbol{y}\{A\}^{\boldsymbol{x}}_{\boldsymbol{y}}, !F(A) \vdash \{B\}^{\boldsymbol{b}[\boldsymbol{x},\boldsymbol{x}]}_{\boldsymbol{w}}}\,(\mathrm{con})}{\{\Gamma\}^{\boldsymbol{u}}_{\boldsymbol{\gamma}[\boldsymbol{x},\boldsymbol{x}]}, \{!A\}^{\boldsymbol{x}} \vdash \{B\}^{\boldsymbol{b}[\boldsymbol{x},\boldsymbol{x}]}_{\boldsymbol{w}}}\,(\otimes\mathrm{L}, \mathrm{D3.3})
$$

(wkn).

$$
\dfrac{\dfrac{\{\Gamma\}^{\boldsymbol{u}}_{\boldsymbol{\gamma}} \vdash \{B\}^{\boldsymbol{b}}_{\boldsymbol{w}}}{\{\Gamma\}^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, !\forall\boldsymbol{y}\{A\}^{\boldsymbol{x}}_{\boldsymbol{y}}, !F(A) \vdash \{B\}^{\boldsymbol{b}}_{\boldsymbol{w}}}\,(\mathrm{wkn})}{\{\Gamma\}^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, \{!A\}^{\boldsymbol{x}} \vdash \{B\}^{\boldsymbol{b}}_{\boldsymbol{w}}}\,(\otimes\mathrm{L}, \mathrm{D3.3})
$$

<u>(!R)</u>.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\{!\Gamma\}^{u} \vdash \{A\}^{a}_{y}}{!\forall \boldsymbol{v}\{\Gamma\}^{u}_{v}, !F(\Gamma) \vdash \{A\}^{a}_{y}}\,(\mathrm{D3.3})}{!\forall \boldsymbol{v}\{\Gamma\}^{u}_{v}, !F(\Gamma) \vdash \forall \boldsymbol{y}\{A\}^{a}_{y}}\,(\forall\mathrm{R})}{!\forall \boldsymbol{v}\{\Gamma\}^{u}_{v}, !F(\Gamma) \vdash !\forall \boldsymbol{y}\{A\}^{a}_{y}}\,(!\mathrm{R}) \qquad \dfrac{\dfrac{!\Gamma \vdash A}{!F(\Gamma) \vdash F(A)}\,(\star_2)}{!F(\Gamma) \vdash !F(A)}\,(!\mathrm{R})}{!\forall \boldsymbol{v}\{\Gamma\}^{u}_{v}, !F(\Gamma) \vdash !\forall \boldsymbol{y}\{A\}^{a}_{y} \otimes !F(A)}\,(\mathrm{con}, \otimes\mathrm{R})}{\{!\Gamma\}^{u} \vdash \{!A\}^{a}}\,(\otimes\mathrm{L}, \mathrm{D3.3})$$

<u>(!L)</u>.

$$\dfrac{\dfrac{\dfrac{\dfrac{\{\Gamma\}^{v}_{\gamma}, \{A\}^{x}_{a} \vdash \{B\}^{b}_{w}}{\{\Gamma\}^{v}_{\gamma}, \forall \boldsymbol{y}\{A\}^{x}_{y} \vdash \{B\}^{b}_{w}}\,(\forall\mathrm{L})}{\{\Gamma\}^{v}_{\gamma}, !\forall \boldsymbol{y}\{A\}^{x}_{y} \vdash \{B\}^{b}_{w}}\,(!\mathrm{L})}{\{\Gamma\}^{v}_{\gamma}, !\forall \boldsymbol{y}\{A\}^{x}_{y}, !F(A) \vdash \{B\}^{b}_{w}}\,(\mathrm{wkn})}{\{\Gamma\}^{v}_{\gamma}, \{!A\}^{x} \vdash \{B\}^{b}_{w}}\,(\otimes\mathrm{R}, \mathrm{D3.3})$$

<u>$\vdash !(b = \mathrm{t}) \oplus !(b = \mathrm{f})$</u>.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash \mathrm{t} = \mathrm{t}}{\vdash !(\mathrm{t} = \mathrm{t})}\,(!\mathrm{R}) \qquad \dfrac{\dfrac{\vdash \mathrm{t} = \mathrm{t}}{\vdash F(\mathrm{t} = \mathrm{t})}\,(\star_2)}{\vdash !F(\mathrm{t} = \mathrm{t})}\,(!\mathrm{R})}{\vdash !(\mathrm{t} = \mathrm{t}) \otimes !F(\mathrm{t} = \mathrm{t})}\,(\otimes\mathrm{R})}{\vdash (!(b = \mathrm{t}) \otimes !F(b = \mathrm{t}))[\mathrm{t}/b]}\,(\star_1)}{\vdash !(b = \mathrm{t}) \multimap (!(b = \mathrm{t}) \otimes !F(b = \mathrm{t}))} \qquad \dfrac{\vdots}{\vdash !(b = \mathrm{f}) \multimap (!(b = \mathrm{f}) \otimes !F(b = \mathrm{f}))}}{(!(b = \mathrm{t}) \otimes !F(b = \mathrm{t})) \diamond_b (!(b = \mathrm{f}) \otimes !F(b = \mathrm{f}))}\,(\&\,\mathrm{R})}{\{!(b = \mathrm{t}) \oplus !(b = \mathrm{f})\}^{b}}\,(\mathrm{D3.3})$$

This concludes the proof. □

Corollary 3.6 ILL^{ω} *has the following disjunction and existence properties:*

(i) *If* $\mathsf{ILL}^{\omega} \vdash \exists x!A$, *then there is a term t such that* $\mathsf{ILL}^{\omega} \vdash !A[t/x]$

(ii) *If* $\mathsf{ILL}^{\omega} \vdash !A \oplus !B$, *then there is a term t such that* $\mathsf{ILL}^{\omega} \vdash !A \diamond_t !B$.

We have discussed two possible choices of F which satisfy conditions $(\star_1)$ and $(\star_2)$, namely

(1) $F(A) :\equiv 1$

(2) $F(A) :\equiv A$.

As it turns out, the following other choices also satisfy the necessary condition, and hence lead to different variants of modified realizability:

(3) $F(A) :\equiv A^{*}$, where $(\cdot)^{*}$ is any proof translation satisfying $(\star_1)$ and $(\star_2)$

(4) $F(A) :\equiv !B \multimap A$, for a fixed closed formula B.

3.4 Relation to interpretations of intuitionistic logic

We now show that the two correspondences of Lemma 3.2 and Theorem 2.5 are deeply related. More precisely, the q-variant of modified realizability corresponds to Girard's $(\cdot)^*$ embedding, whereas the truth-variant corresponds to the $(\cdot)^\circ$ embedding. Similar work was done for interpretations without truth in [5].

Proposition 3.7 *Let $F(A) :\equiv A$. For all formulas A of IL^ω we have*

(i) $\mathsf{ILL}^\omega \vdash \{A^\circ\}^{\boldsymbol{x}} \multimapboth (\boldsymbol{x} \text{ mrt } A)^\circ$

(ii) $\mathsf{ILL}^\omega \vdash !\forall \boldsymbol{y}\{A^*\}^{\boldsymbol{x}}_{\boldsymbol{y}} \multimapboth !(\boldsymbol{x} \text{ mq } A)^*$.

Proof. The proof is by induction on the structure of formulas. We consider a few cases. For point (i):

$\underline{A \vee B}$. We can prove (a) $\mathsf{ILL}^\omega \vdash\ !A \diamond_b\ !B \multimapboth\ !(!A \diamond_b\ !B)$. Also we can prove by induction on the structure of formulas (b) $\mathsf{ILL}^\omega \vdash \{A^\circ\}^{\boldsymbol{x}} \multimapboth\ !\{A^\circ\}^{\boldsymbol{x}}$ (or, equivalently but easier, that for all formula A of ILL^ω there exists a formula B of ILL^ω such that $\mathsf{ILL}^\omega \vdash \{A^\circ\}^{\boldsymbol{x}} \multimapboth\ !B$).

$$\begin{aligned}
\{(A \vee B)^\circ\}^{\boldsymbol{x},\boldsymbol{y},b} &\equiv \{A^\circ\}^{\boldsymbol{x}} \diamond_b \{B^\circ\}^{\boldsymbol{y}} \\
&\overset{\text{(b)}}{\multimapboth}\ !\{A^\circ\}^{\boldsymbol{x}} \diamond_b\ !\{B^\circ\}^{\boldsymbol{y}} \\
&\overset{\text{(a)}}{\multimapboth}\ !(!\{A^\circ\}^{\boldsymbol{x}} \diamond_b\ !\{B^\circ\}^{\boldsymbol{y}}) \\
&\overset{\text{(b)}}{\multimapboth}\ !(\{A^\circ\}^{\boldsymbol{x}} \diamond_b \{B^\circ\}^{\boldsymbol{y}}) \\
&\overset{\text{(IH)}}{\multimapboth}\ !((\boldsymbol{x} \text{ mrt } A)^\circ \diamond_b (\boldsymbol{y} \text{ mrt } B)^\circ) \\
&\equiv\ !((!(b = \text{t}) \multimap (\boldsymbol{x} \text{ mrt } A)^\circ) \,\&\, (!(b = \text{f}) \multimap (\boldsymbol{y} \text{ mrt } B)^\circ)) \\
&\multimapboth\ !(!(b = \text{t}) \multimap (\boldsymbol{x} \text{ mrt } A)^\circ) \otimes !(!(b = \text{f}) \multimap (\boldsymbol{y} \text{ mrt } B)^\circ) \\
&\equiv (\boldsymbol{x}, \boldsymbol{y}, b \text{ mrt } (A \vee B))^\circ.
\end{aligned}$$

$\underline{A \to B}$.

$$\begin{aligned}
\{(A \to B)^\circ\}^{\boldsymbol{f}} &\equiv\ !\forall \boldsymbol{x}(\{A^\circ\}^{\boldsymbol{x}} \multimap \{B^\circ\}^{\boldsymbol{f}\boldsymbol{x}}) \otimes (A \to B)^\circ \\
&\overset{\text{(IH)}}{\multimapboth}\ !\forall \boldsymbol{x}((\boldsymbol{x} \text{ mrt } A)^\circ \multimap (\boldsymbol{f}\boldsymbol{x} \text{ mrt } B)^\circ) \otimes (A \to B)^\circ \\
&\multimapboth\ !\forall \boldsymbol{x}!((\boldsymbol{x} \text{ mrt } A)^\circ \multimap (\boldsymbol{f}\boldsymbol{x} \text{ mrt } B)^\circ) \otimes (A \to B)^\circ \\
&\equiv (\boldsymbol{f} \text{ mrt } (A \to B))^\circ.
\end{aligned}$$

$\underline{\forall z A}$. We can prove that (a) if $\mathsf{IL}^\omega \vdash A \leftrightarrow B$ then $\mathsf{ILL}^\omega \vdash A^\circ \multimapboth B^\circ$, by applying the soundness theorem of $(\cdot)^\circ$ to $\mathsf{IL}^\omega \vdash A \to B$ and $\mathsf{IL}^\omega \vdash B \to A$. We can also prove that (b) $\mathsf{IL}^\omega \vdash \forall z(\boldsymbol{f}z \text{ mrt } A) \leftrightarrow \forall z(\boldsymbol{f}z \text{ mrt } A) \wedge \forall z A$, using the instance $\mathsf{IL}^\omega \vdash \boldsymbol{f}z \text{ mrt } A \to A$ of the truth property of mrt.

$$\begin{aligned}
\{(\forall z A)^\circ\}^{\boldsymbol{f}} &\equiv\ !\forall z\{A^\circ\}^{\boldsymbol{f}z} \otimes !\forall z A^\circ \\
&\overset{\text{(IH)}}{\multimapboth}\ !\forall z(\boldsymbol{f}z \text{ mrt } A)^\circ \otimes !\forall z A^\circ \\
&\equiv (\forall z(\boldsymbol{f}z \text{ mrt } A) \wedge \forall z A)^\circ \\
&\overset{\text{(a,b)}}{\multimapboth} (\forall z(\boldsymbol{f}z \text{ mrt } A))^\circ \\
&\equiv (\boldsymbol{f} \text{ mrt } \forall z A)^\circ.
\end{aligned}$$

For point (ii):

$\underline{A \wedge B}$.

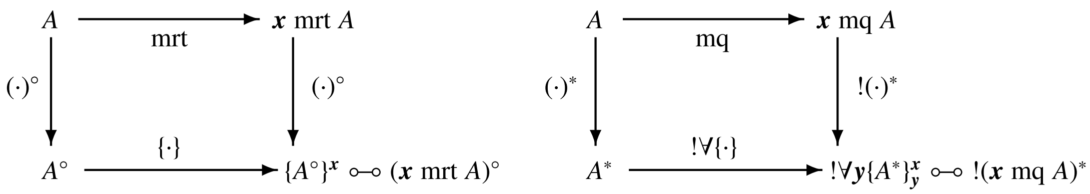

Fig. 1 Diagrams of Proposition 3.7

$$
\begin{aligned}
!\forall \boldsymbol{y}, \boldsymbol{w}, b\{(A \wedge B)^*\}^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w},b} &\equiv\ !\forall \boldsymbol{y}, \boldsymbol{w}, b(\{A^*\}^{\boldsymbol{x}}_{\boldsymbol{y}} \diamond_b \{B^*\}^{\boldsymbol{v}}_{\boldsymbol{w}}) \\
&\multimapboth\ !\forall \boldsymbol{y}, \boldsymbol{w}(\{A^*\}^{\boldsymbol{x}}_{\boldsymbol{y}} \,\&\, \{B^*\}^{\boldsymbol{v}}_{\boldsymbol{w}}) \\
&\multimapboth\ !(\forall \boldsymbol{y}\{A^*\}^{\boldsymbol{x}}_{\boldsymbol{y}} \,\&\, \forall \boldsymbol{w}\{B^*\}^{\boldsymbol{v}}_{\boldsymbol{w}}) \\
&\multimapboth\ !(!\forall \boldsymbol{y}\{A^*\}^{\boldsymbol{x}}_{\boldsymbol{y}} \,\&\, !\forall \boldsymbol{w}\{B^*\}^{\boldsymbol{v}}_{\boldsymbol{w}}) \\
&\overset{\text{(IH)}}{\multimapboth}\ !(!(\boldsymbol{x} \text{ mq } A)^* \,\&\, !(\boldsymbol{v} \text{ mq } B)^*) \\
&\multimapboth\ !((\boldsymbol{x} \text{ mq } A)^* \,\&\, (\boldsymbol{v} \text{ mq } B)^*) \\
&\equiv\ !(\boldsymbol{x}, \boldsymbol{v} \text{ mq } (A \wedge B))^*.
\end{aligned}
$$

$\underline{A \to B}$.

$$
\begin{aligned}
!\forall \boldsymbol{x}, \boldsymbol{w}\{(A \to B)^*\}^{\boldsymbol{f}}_{\boldsymbol{x},\boldsymbol{w}} &\equiv\ !\forall \boldsymbol{x}, \boldsymbol{w}(!\forall y\{A^*\}^{\boldsymbol{x}}_{\boldsymbol{y}} \otimes !A^* \multimap \{B^*\}^{\boldsymbol{f}\boldsymbol{x}}_{\boldsymbol{w}}) \\
&\multimapboth\ !\forall \boldsymbol{x}(!\forall y\{A^*\}^{\boldsymbol{x}}_{\boldsymbol{y}} \otimes !A^* \multimap \forall \boldsymbol{w}\{B^*\}^{\boldsymbol{f}\boldsymbol{x}}_{\boldsymbol{w}}) \\
&\overset{\text{(IH)}}{\multimapboth}\ !\forall \boldsymbol{x}(!(\boldsymbol{x} \text{ mq } A)^* \otimes !A^* \multimap \forall \boldsymbol{w}\{B^*\}^{\boldsymbol{f}\boldsymbol{x}}_{\boldsymbol{w}}) \\
&\multimapboth\ !\forall \boldsymbol{x}(!((\boldsymbol{x} \text{ mq } A)^* \,\&\, A^*) \multimap \forall \boldsymbol{w}\{B^*\}^{\boldsymbol{f}\boldsymbol{x}}_{\boldsymbol{w}}) \\
&\multimapboth\ !\forall \boldsymbol{x}!(!((\boldsymbol{x} \text{ mq } A)^* \,\&\, A^*) \multimap \forall \boldsymbol{w}\{B^*\}^{\boldsymbol{f}\boldsymbol{x}}_{\boldsymbol{w}}) \\
&\multimapboth\ !\forall \boldsymbol{x}!(!((\boldsymbol{x} \text{ mq } A)^* \,\&\, A^*) \multimap !\forall \boldsymbol{w}\{B^*\}^{\boldsymbol{f}\boldsymbol{x}}_{\boldsymbol{w}}) \\
&\overset{\text{(IH)}}{\multimapboth}\ !\forall \boldsymbol{x}!(!((\boldsymbol{x} \text{ mq } A)^* \,\&\, A^*) \multimap !(\boldsymbol{f}\boldsymbol{x} \text{ mq } B)^*) \\
&\multimapboth\ !\forall \boldsymbol{x}!(!((\boldsymbol{x} \text{ mq } A)^* \,\&\, A^*) \multimap (\boldsymbol{f}\boldsymbol{x} \text{ mq } B)^*) \\
&\multimapboth\ !\forall \boldsymbol{x}(!((\boldsymbol{x} \text{ mq } A)^* \,\&\, A^*) \multimap (\boldsymbol{f}\boldsymbol{x} \text{ mq } B)^*) \\
&\equiv\ !(\boldsymbol{f} \text{ mq } (A \to B))^*.
\end{aligned}
$$

The remaining cases can be checked in a similar way. □

Remark 3.8 Although factorisation (ii) seems less clean than factorisation (i), due to the presence of bangs and universal quantifiers, they are actually perfectly analogous. This is because bangs and universal quantifiers are also present in factorisation (i) hidden behind the notation: A° already includes a bang as it is equivalent to a banged formula, and $\{A^\circ\}^{\boldsymbol{x}}$ has in it quantifications $\forall \boldsymbol{y}$ that are responsible for the variables $\boldsymbol{y}$ not being shown.

3.5 Extrapolating on linear-intuitionistic relation

Note that we have a single soundness theorem (Theorem 3.5) for both the usual functional interpretation and its truth variant, with a single procedure for constructing witnesses. This shows that the witnesses extracted via a standard functional interpretation are precisely the same witnesses obtained via its truth variant. In fact, the truth variant of the interpretation is in a precise sense only enriching the verification proof $\{A_0(z)\}^{\boldsymbol{x}_0}_{\boldsymbol{a}_0}, \ldots, \{A_n(z)\}^{\boldsymbol{x}_n}_{\boldsymbol{a}_n} \vdash \{B(z)\}^{\boldsymbol{b}}_{\boldsymbol{w}}$, maintaining some of the information which was available from the given proof $A_0, \ldots, A_n \vdash B$.

We sketch an alternative way of looking at functional interpretations with truth which makes this even more explicit. Let T^ω be a given theory for which we have a functional interpretation $|\cdot| : \mathsf{T}^\omega \to \mathsf{T}^\omega$. In order to

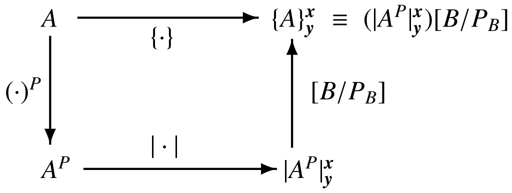

Fig. 2 Alternative view on truth variants

obtain a truth variant of the interpretation $|\cdot|$ we do the following. First, define an extension of T^ω (call it T^ω_P) by: Adding new atomic formulas P_A for each formula A of the original theory T^ω, except for atomic formulas A for which we take $P_A :\equiv A$; assuming $P_A[t/x] \equiv P_{A[t/x]}$; assuming that the free variables of P_A are the same as those free in A; for each theorem $A_0, \ldots, A_n \vdash B$ of T^ω we add axioms $P_{A_0}, \ldots, P_{A_n} \vdash P_B$ in T^ω_P; and we add axioms $P_{!A} \multimap\!\!\circ\ !P_A$ to T^ω_P. Assuming $|\cdot|$ is trivial for the new axioms just added, the original interpretation extends trivially[1] to an interpretation $|\cdot| : \mathsf{T}^\omega_P \to \mathsf{T}^\omega_P$. Now, define a formula translation $(\cdot)^P : \mathsf{T}^\omega \mapsto \mathsf{T}^\omega_P$ by simply replacing $!A$ by $!A \otimes !P_A$ inductively. This translation is sound due to the new axioms of T^ω_P. It is easy to show that the truth variant $\{\cdot\}$ is related to the standard functional interpretation $|\cdot|$ as

$$\{A\}^x_y \equiv (|A^P|^x_y)[B/P_B].$$

This correspondence is illustrated in Figure 2. The same comments also hold for truth variants in the intuitionistic context. We have shown in Proposition 3.7 that both the q- and truth variants of modified realizability arise from a single truth interpretation of linear logic combined with the two different embeddings $(\cdot)^*$ and $(\cdot)^\circ$ of IL^ω into ILL^ω. As the truth interpretation of linear logic adds a "copy" P_A of the original formula $!A$ in parallel to interpreting $!A$ itself, this is also what is happening behind the scenes in the q-variant of modified realizability. A copy of A is created precisely at the places where the embedding $(\cdot)^*$ introduces a $!A$, i.e. in the clauses for $A \to B, A \vee B$ and $\exists z A$. Modified realizability with truth, on the other hand, introduces a copy of A in the places where the $(\cdot)^\circ$ introduces a $!A$, i.e. $A \to B$ and $\forall z A$. In the case of modified realizability, creating a copy of $\forall z A$ in the interpretation of the universal quantifier is redundant, since $\forall z (\boldsymbol{f} z \text{ mrt } A)$ already implies $\forall z A$, because $(\boldsymbol{f} z \text{ mrt } A) \to A$.

This leads us to a heuristic for obtaining truth and q-variants of an interpretation: For the q-variant we add a copy of the formula being interpreted in the clauses for $A \to B$ (in the premise) and $A \vee B$ and $\exists z A$; whereas for the truth variant we add a copy of A in the clauses for $A \to B$ and $\forall z A$. A second heuristic is suggested by Theorem 2.5: Once we got a q-variant $(\cdot)^{\mathrm{Iq}}$ of an interpretation $(\cdot)^{\mathrm{I}}$ (what can usually be done just by copying what is done for modified realizability) we may get a variant with truth $(\cdot)^{\mathrm{It}}$ by defining it as $A^{\mathrm{It}} :\equiv A^{\mathrm{Iq}} \wedge A$.

In the remaining sections we show how these methodologies led us to new truth variants of bounded modified realizability, the Diller-Nahm interpretation, and the bounded functional interpretation. We start, however, by recasting Aczel's and Kleene's slash translations in terms of these two heuristics.

[1] Assuming the interpretation $|\cdot|$ does not rely on particular properties of the system T^ω which fail in the extended systems T^ω_P. This is in fact why the heuristic above does not apply to Gödel's dialectica interpretation.

4 The Slash Translation and its Variants

We recall some facts about three different slash translations and show how they also fit our heuristics. Let a finite sequence of sentences Γ be given. To each formula A we assign a relation $\Gamma \mid A$ by induction on A as follows:

$$\begin{aligned} \Gamma \mid A_{\mathrm{at}} &:\equiv \Gamma \vdash A_{\mathrm{at}} \\ \Gamma \mid A \wedge B &:\equiv \Gamma \mid A \text{ and } \Gamma \mid B \\ \Gamma \mid A \vee B &:\equiv \Gamma \mid A \text{ or } \Gamma \mid B \\ \Gamma \mid A \to B &:\equiv \text{if } \Gamma \mid A \text{ then } \Gamma \mid B \\ \Gamma \mid \exists z A &:\equiv \Gamma \mid A[n/z] \text{ for some numeral } n \\ \Gamma \mid \forall z A &:\equiv \Gamma \mid A[n/z] \text{ for all numerals } n, \end{aligned}$$

where $\Gamma \vdash A$ denotes provability in a given fixed formal system. Intuitively, this is closely related Tarski's semantics, viewing Γ as a "structure" and $\Gamma \mid A$ as "A is valid in Γ". Formally, this translation is sound in the sense that if $\Gamma \mid \Gamma$ and $\Gamma \vdash A$ then $\Gamma \mid \forall \boldsymbol{n} A$, where $\mathrm{FV}(A) = \{\boldsymbol{n}\}$ and $\Gamma \mid \Gamma$ means $\Gamma \mid A$ for each $A \in \Gamma$. We demonstrate now how the slash translation above is related to Kleene's and Aczel's slash translations in the same way that modified realizability is related to its q- and truth variants (cf. Section 2).

Definition 4.1 (q-variant of slash translation) Kleene's slash $\Gamma \mid_{\mathrm{k}} A$ is defined (cf. [18], definition 3.1.2) by induction on A as

$$\begin{aligned} \Gamma \mid_{\mathrm{k}} A_{\mathrm{at}} &:\equiv \Gamma \vdash A_{\mathrm{at}} \\ \Gamma \mid_{\mathrm{k}} A \wedge B &:\equiv \Gamma \mid_{\mathrm{k}} A \text{ and } \Gamma \mid_{\mathrm{k}} B \\ \Gamma \mid_{\mathrm{k}} A \vee B &:\equiv (\Gamma \mid_{\mathrm{k}} A \text{ and } \Gamma \vdash A) \text{ or } (\Gamma \mid_{\mathrm{k}} B \text{ and } \Gamma \vdash B) \\ \Gamma \mid_{\mathrm{k}} A \to B &:\equiv \text{if } (\Gamma \mid_{\mathrm{k}} A \text{ and } \Gamma \vdash A) \text{ then } \Gamma \mid_{\mathrm{k}} B \\ \Gamma \mid_{\mathrm{k}} \forall z A &:\equiv \Gamma \mid_{\mathrm{k}} A[n/z] \text{ for all numerals } n \\ \Gamma \mid_{\mathrm{k}} \exists z A &:\equiv (\Gamma \mid_{\mathrm{k}} A[n/z] \text{ and } \Gamma \vdash A[n/z]) \text{ for some numeral } n. \end{aligned}$$

Remark 4.2 We have that $\Gamma \mid_{\mathrm{k}} A$ implies $\Gamma \vdash A$ for all disjunctive and existential formulas A.

Definition 4.3 (Truth variant of slash translation) Aczel's slash $\Gamma \mid_{\mathrm{a}} A$ is defined (cf. [18], theorem 5.1.18) by induction on A as

$$\begin{aligned} \Gamma \mid_{\mathrm{a}} A_{\mathrm{at}} &:\equiv \Gamma \vdash A_{\mathrm{at}} \\ \Gamma \mid_{\mathrm{a}} A \wedge B &:\equiv \Gamma \mid_{\mathrm{a}} A \text{ and } \Gamma \mid_{\mathrm{a}} B \\ \Gamma \mid_{\mathrm{a}} A \vee B &:\equiv \Gamma \mid_{\mathrm{a}} A \text{ or } \Gamma \mid_{\mathrm{a}} B \\ \Gamma \mid_{\mathrm{a}} A \to B &:\equiv (\text{if } \Gamma \mid_{\mathrm{a}} A \text{ then } \Gamma \mid_{\mathrm{a}} B) \text{ and } \Gamma \vdash A \to B \\ \Gamma \mid_{\mathrm{a}} \forall z A &:\equiv \Gamma \vdash \forall z A \text{ and } \Gamma \mid_{\mathrm{a}} A[n/z] \text{ for all numerals } n \\ \Gamma \mid_{\mathrm{a}} \exists z A &:\equiv \Gamma \mid_{\mathrm{a}} A[n/z] \text{ for some numeral } n. \end{aligned}$$

Remark 4.4 We have that $\Gamma \mid_{\mathrm{a}} A$ implies $\Gamma \vdash A$ for all formulas A.

The next proposition shows that Kleene's slash is related to Aczel's slash in the same way that the q-variant of modified realizability is related to its truth variant.

Proposition 4.5 (See [19], exercise 3.5.3) *In both definitions above, assume* $\Gamma \vdash A$ *denotes provability in some extension of intuitionistic logic. For all formulas A we have* $\Gamma \mid_{\mathrm{a}} A$ *if and only if* ($\Gamma \mid_{\mathrm{k}} A$ *and* $\Gamma \vdash A$)*.*

P r o o f. By an easy induction on the structure of A. □

Corollary 4.6 (Strengthening of [18], 5.1.17) *If* $\Gamma \vdash A$ *then* $\Gamma \mid_{\mathrm{k}} A$ *iff* $\Gamma \mid_{\mathrm{a}} A$.

5 Bounded Modified Realizability with Truth

Let us consider now the q- and truth variants of the bounded modified realizability [3]. In the following, $\mathsf{IL}^{\omega}_{\leq^*}$ denotes the formal system described in [3] containing an extensional majorizability relation $\leq^*$ and primitive bounded quantifications $\forall z \leq^* tA$ and $\exists z \leq^* tA$.

5.1 Definitions

Definition 5.1 (q-variant of bounded modified realizability) Each formula A of $\mathsf{IL}^{\omega}_{\leq^*}$ is associated to a new formula $\boldsymbol{x}$ brq A inductively as follows. If A_{at} is an atomic formula, then $\boldsymbol{x}$ brq $A_{\text{at}} :\equiv A_{\text{at}}$ where $\boldsymbol{x}$ is the empty tuple. Assuming we already have defined $\boldsymbol{x}$ brq A and $\boldsymbol{y}$ brq B, then

$$\begin{aligned}
\boldsymbol{x}, \boldsymbol{y} \text{ brq } (A \wedge B) &:\equiv (\boldsymbol{x} \text{ brq } A) \wedge (\boldsymbol{y} \text{ brq } B)\\
\boldsymbol{x}, \boldsymbol{y} \text{ brq } (A \vee B) &:\equiv ((\boldsymbol{x} \text{ brq } A) \wedge A) \vee ((\boldsymbol{y} \text{ brq } B) \wedge B)\\
\boldsymbol{f} \text{ brq } (A \to B) &:\equiv \tilde{\forall} \boldsymbol{x}((\boldsymbol{x} \text{ brq } A) \wedge A \to (\boldsymbol{f}\boldsymbol{x} \text{ brq } B))\\
\boldsymbol{x} \text{ brq } \forall z \leq^* tA &:\equiv \forall z \leq^* t(\boldsymbol{x} \text{ brq } A)\\
\boldsymbol{x} \text{ brq } \exists z \leq^* tA &:\equiv \exists z \leq^* t((\boldsymbol{x} \text{ brq } A) \wedge A)\\
\boldsymbol{f} \text{ brq } \forall z A &:\equiv \tilde{\forall} u \forall z \leq^* u(\boldsymbol{f}u \text{ brq } A)\\
u, \boldsymbol{x} \text{ brq } \exists z A &:\equiv \exists z \leq^* u((\boldsymbol{x} \text{ brq } A) \wedge A).
\end{aligned}$$

The *q-variant of bounded modified realizability* of a formula A is then defined as $A^{\text{brq}} :\equiv \tilde{\exists}\boldsymbol{x}(\boldsymbol{x} \text{ brq } A)$.

Remark 5.2 We have $\mathsf{IL}^{\omega}_{\leq^*} \vdash (\boldsymbol{x} \text{ brq } A) \to A$ for all disjunctive and (bounded and unbounded) existential formulas A of $\mathsf{IL}^{\omega}_{\leq^*}$.

Definition 5.3 (Bounded modified realizability with truth) To each formula A of $\mathsf{IL}^{\omega}_{\leq^*}$ let us associate a new formula $\boldsymbol{x}$ brt A inductively as follows. If A_{at} is an atomic formula, then $\boldsymbol{x}$ brt $A_{\text{at}} :\equiv A_{\text{at}}$ where $\boldsymbol{x}$ is the empty tuple. Assuming that we already defined $\boldsymbol{x}$ brt A and $\boldsymbol{y}$ brt B, then

$$\begin{aligned}
\boldsymbol{x}, \boldsymbol{y} \text{ brt } (A \wedge B) &:\equiv (\boldsymbol{x} \text{ brt } A) \wedge (\boldsymbol{y} \text{ brt } B)\\
\boldsymbol{x}, \boldsymbol{y} \text{ brt } (A \vee B) &:\equiv (\boldsymbol{x} \text{ brt } A) \vee (\boldsymbol{y} \text{ brt } B)\\
\boldsymbol{f} \text{ brt } (A \to B) &:\equiv \tilde{\forall} \boldsymbol{x}((\boldsymbol{x} \text{ brt } A) \to (\boldsymbol{f}\boldsymbol{x} \text{ brt } B)) \wedge (A \to B)\\
\boldsymbol{x} \text{ brt } \forall z \leq^* tA &:\equiv \forall z \leq^* t(\boldsymbol{x} \text{ brt } A)\\
\boldsymbol{x} \text{ brt } \exists z \leq^* tA &:\equiv \exists z \leq^* t(\boldsymbol{x} \text{ brt } A)\\
\boldsymbol{f} \text{ brt } \forall z A &:\equiv \tilde{\forall} u \forall z \leq^* u(\boldsymbol{f}u \text{ brt } A) \wedge \forall z A\\
u, \boldsymbol{x} \text{ brt } \exists z A &:\equiv \exists z \leq^* u(\boldsymbol{x} \text{ brt } A).
\end{aligned}$$

The *bounded modified realizability with truth* of a formula A is then defined as $A^{\text{brt}} :\equiv \tilde{\exists}\boldsymbol{x}(\boldsymbol{x} \text{ brt } A)$.

Remark 5.4 We have $\mathsf{IL}^{\omega}_{\leq^*} \vdash (\boldsymbol{x} \text{ brt } A) \to A$ for all formulas A of $\mathsf{IL}^{\omega}_{\leq^*}$.

5.2 Soundness theorems

Theorem 5.5 (Soundness for q-variant of bounded modified realizability) *Let A be an arbitrary formula of $\mathsf{IL}^{\omega}_{\leq^*}$ and let $\boldsymbol{z}$ be all of its free variables. If $\mathsf{IL}^{\omega}_{\leq^*} \vdash A$, then we can extract from a proof of A closed monotone (that is, $\mathsf{IL}^{\omega}_{\leq^*} \vdash \boldsymbol{t} \leq^* \boldsymbol{t}$) terms $\boldsymbol{t}$ of $\mathsf{IL}^{\omega}_{\leq^*}$ such that $\mathsf{IL}^{\omega}_{\leq^*} \vdash \tilde{\forall}\boldsymbol{a}\forall \boldsymbol{z} \leq^* \boldsymbol{a}(\boldsymbol{t}\boldsymbol{a} \text{ brq } A)$.*

P r o o f. The proof is by induction on a proof of A. We assume the logical rules and axioms of intuitionistic logic as described in [1], section 2.1.

<u>$A \vee A \to A$</u>. Here we use that $\mathsf{IL}^{\omega}_{\leq^*} \vdash \forall \boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x} \leq^* \boldsymbol{y} \wedge (\boldsymbol{x} \text{ brq } A) \to (\boldsymbol{y} \text{ brq } A))$. The interpretation of this axiom is

$$\tilde{\exists}\boldsymbol{f}\tilde{\forall}\boldsymbol{x}_1, \boldsymbol{x}_2((((\boldsymbol{x}_1 \text{ brq } A) \wedge A) \vee ((\boldsymbol{x}_2 \text{ brq } A) \wedge A)) \wedge (A \vee A) \to \boldsymbol{f}\boldsymbol{x}_1\boldsymbol{x}_2 \text{ brq } A).$$

The witness for $\boldsymbol{f}$ can be taken as $\lambda \boldsymbol{a}, \boldsymbol{x}_1, \boldsymbol{x}_2 \,.\, \max(\boldsymbol{x}_1, \boldsymbol{x}_2)$.

$\underline{\forall z A \to A[q/z]}$. Here we use $\boldsymbol{x}$ brq $(A[q/z]) \equiv (\boldsymbol{x}$ brq $A)[q/z]$. Also we use that for all terms $q[\boldsymbol{z}']$, where $\boldsymbol{z}'$ are all the variables in q, there exists a term $q^*[\boldsymbol{z}']$ with exactly the same variables $\boldsymbol{z}'$ such that $\mathsf{IL}^{\omega}_{\leq^*} \vdash \tilde{\forall} \boldsymbol{a}' \forall \boldsymbol{z}' \leq^* \boldsymbol{a}'(q[\boldsymbol{z}'] \leq^* q^*[\boldsymbol{a}'])$. The realizability of this axioms is

$$\tilde{\exists} \boldsymbol{F} \tilde{\forall} \boldsymbol{f} (\tilde{\forall} u \forall z \leq^* u(\boldsymbol{f} u \text{ brq } A) \wedge \forall z A \to \boldsymbol{F} \boldsymbol{f} \text{ brq } A[q/z]).$$

$\boldsymbol{F}$ can be taken as $\lambda \boldsymbol{a}, \boldsymbol{f} \,.\, \boldsymbol{f}(q^*[\boldsymbol{a}'])$ (if it is not closed, we close it by replacing its variables by $\mathbf{0}$).

$\underline{A, A \to B \Rightarrow B}$. Here we use the assumption that $\mathsf{IL}^{\omega}_{\leq^*} \vdash A$. Let $\boldsymbol{z}_A$, $\boldsymbol{z}_{AB}$ and $\boldsymbol{z}_B$ be all the free variables of A, $A \to B$ and B, respectively. By induction hypothesis we have witnesses $\boldsymbol{q}$ and $\boldsymbol{r}$ for the realizability of the premises of the rule, that is,

$$\tilde{\forall} \boldsymbol{a}_A \forall \boldsymbol{z}_A \leq^* \boldsymbol{a}_A(\boldsymbol{q}\boldsymbol{a}_A \text{ brq } A)$$
$$\tilde{\forall} \boldsymbol{a}_{AB} \forall \boldsymbol{z}_{AB} \leq^* \boldsymbol{a}_{AB} \tilde{\forall} \boldsymbol{x}((\boldsymbol{x} \text{ brq } A) \wedge A \to \boldsymbol{r}\boldsymbol{a}_{AB}\boldsymbol{x} \text{ brq } B).$$

A witness for the realizability of the conclusion of the rule can be taken as $\lambda \boldsymbol{a}_B \,.\, \boldsymbol{r}\boldsymbol{a}_{AB}(\boldsymbol{q}\boldsymbol{a}_A)$ (if it is not closed, we close it by replacing its variables by $\mathbf{0}$).

$\underline{\exists z \leq^* qA \to \exists z(z \leq^* q \wedge A)}$. Here we use the fact that the formula $(z \leq^* q)$ is $\tilde{\exists}$-free (that is, all its unbounded quantifications are universal monotone) and that for all $\tilde{\exists}$-free formulas A we have $\mathsf{IL}^{\omega}_{\leq^*} \vdash A \leftrightarrow (\boldsymbol{x}$ brq $A)$, where $\boldsymbol{x}$ is the empty tuple that may be omitted. The interpretation of this axiom is

$$\tilde{\exists} \boldsymbol{f}, g \tilde{\forall} \boldsymbol{x}(\exists z \leq^* q(\boldsymbol{x} \text{ brq } A \wedge A) \wedge \exists z \leq^* qA \to \exists z \leq^* g\boldsymbol{x}(z \leq^* q \wedge \boldsymbol{f}\boldsymbol{x} \text{ brq } A \wedge A)).$$

The witnesses for $\boldsymbol{f}$ and g can be taken as $\lambda \boldsymbol{a}, \boldsymbol{x} \,.\, \boldsymbol{x}$ and $\lambda \boldsymbol{a}, \boldsymbol{x} \,.\, q^*[\boldsymbol{a}']$, respectively. The other cases are treated similarly. □

Proposition 5.6 *For all formulas A of $\mathsf{IL}^{\omega}_{\trianglelefteq}$ we have*

$$\mathsf{IL}^{\omega}_{\leq^*} \vdash (\boldsymbol{x} \text{ brt } A) \leftrightarrow (\boldsymbol{x} \text{ brq } A) \wedge A.$$

P r o o f. Easy induction on the structure of formulas.

$\underline{\forall z \leq^* tA}$.

$$\begin{aligned} \boldsymbol{x} \text{ brt } \forall z \leq^* tA &\equiv \forall z \leq^* t(\boldsymbol{x} \text{ brt } A) \\ &\leftrightarrow \forall z \leq^* t((\boldsymbol{x} \text{ brq } A) \wedge A) \\ &\leftrightarrow \forall z \leq^* t(\boldsymbol{x} \text{ brq } A) \wedge \forall z \leq^* tA \\ &\equiv (\boldsymbol{x} \text{ brq } \forall z \leq^* tA) \wedge \forall z \leq^* tA. \end{aligned}$$

$\underline{\forall z A}$.

$$\begin{aligned} \boldsymbol{f} \text{ brt } \forall z A &\equiv \tilde{\forall} u \forall z \leq^* u(\boldsymbol{f}u \text{ brt } A) \wedge \forall z A \\ &\leftrightarrow \tilde{\forall} u \forall z \leq^* u((\boldsymbol{f}u \text{ brq } A) \wedge A) \wedge \forall z A \\ &\leftrightarrow \tilde{\forall} u \forall z \leq^* u(\boldsymbol{f}u \text{ brq } A) \wedge \forall z A \\ &\equiv (\boldsymbol{f} \text{ brq } \forall z A) \wedge \forall z A. \end{aligned}$$

The other cases are treated similarly. □

Theorem 5.7 (Soundness for bounded modified realizability with truth) *Let A be an arbitrary formula of $\mathsf{IL}^{\omega}_{\leq^*}$ and let $\boldsymbol{z}$ be all its free variables. If $\mathsf{IL}^{\omega}_{\leq^*} \vdash A$ then we can extract from a proof of A closed monotone terms $\boldsymbol{t}$ of $\mathsf{IL}^{\omega}_{\leq^*}$ such that $\mathsf{IL}^{\omega}_{\leq^*} \vdash \tilde{\forall} \boldsymbol{a} \forall \boldsymbol{z} \leq^* \boldsymbol{a}(\boldsymbol{t}\boldsymbol{a}$ brt $A)$.*

P r o o f. Follows from Theorem 5.5 and Proposition 5.6. □

5.3 Characterisation theorems

As shown in Section 2.3, the characterisation of the modified realizability with truth uses the same principles for the characterisation of the standard modified realizability. We now show that the same happens with the truth variant of the bounded modified realizability. Let $\mathsf{IL}^{\#}_{\leq^*}$ denote the theory $\mathsf{IL}^{\omega}_{\leq^*}$ extended with the characterisation principles of the bounded modified realizability, i.e. bounded choice principle, bounded independence of premise and the majorizability axioms (cf. [3]).

Theorem 5.8 *For all formulas A of $\mathsf{IL}^{\omega}_{\leq^*}$ we have*

(i) $\mathsf{IL}^{\#}_{\leq^*} \vdash \tilde{\exists} \boldsymbol{x}(\boldsymbol{x} \text{ brq } A) \leftrightarrow A$

(ii) $\mathsf{IL}^{\#}_{\leq^*} \vdash \tilde{\exists} \boldsymbol{x}(\boldsymbol{x} \text{ brt } A) \leftrightarrow A.$

P r o o f. The proof is similar to that of Theorem 2.6. The characterisation theorem for the bounded modified realizability gives us

$$\mathsf{IL}^{\#}_{\leq^*} \vdash \tilde{\exists} \boldsymbol{x}(\boldsymbol{x} \text{ br } A) \leftrightarrow A.$$

In order to prove (i) one shows by induction on the structure of A that $\mathsf{IL}^{\#}_{\leq^*} \vdash \tilde{\forall} \boldsymbol{x}((\boldsymbol{x} \text{ brt } A) \leftrightarrow (\boldsymbol{x} \text{ br } A))$. Point (ii) then follows from (i) and Proposition 5.6. □

6 Diller-Nahm Interpretation with Truth

We have seen above how the (bounded) modified realizability with truth and q-variant of (bounded) modified realizability are closely related (Theorem 2.5 and Proposition 5.6), and how the unbounded ones can be seen to come from a common truth interpretation of linear logic (Proposition 3.7). Following these ideas, in this section we show how a Diller-Nahm interpretation with truth interpretation can be obtained from Jørgensen's recent q-variant of the Diller-Nahm interpretation.

6.1 Definitions

Let us use the abbreviation $x \in a$ to mean that x is a member of the (suitably coded in the language of arithmetic in all finite types) finite non-empty set a.

Definition 6.1 (q-variant of Diller-Nahm interpretation, [9]) Each formula A of intuitionistic logic is associated to a new formula $A_{\mathrm{Q}}(\boldsymbol{x};\boldsymbol{y})$ inductively as follows. If A_{at} is an atomic formula, then $(A_{\mathrm{at}})_{\mathrm{Q}}(;) :\equiv A_{\mathrm{at}}$. Assuming the interpretations of A and B are already given we define:

$$\begin{aligned}
(A \wedge B)_{\mathrm{Q}}(\boldsymbol{x},\boldsymbol{v};\boldsymbol{y},\boldsymbol{w}) &:\equiv A_{\mathrm{Q}}(\boldsymbol{x};\boldsymbol{y}) \wedge B_{\mathrm{Q}}(\boldsymbol{v};\boldsymbol{w}),\\
(A \vee B)_{\mathrm{Q}}(\boldsymbol{x},\boldsymbol{v},b;\boldsymbol{y},\boldsymbol{w}) &:\equiv (A_{\mathrm{Q}}(\boldsymbol{x};\boldsymbol{y}) \wedge A) \diamond_b (B_{\mathrm{Q}}(\boldsymbol{v};\boldsymbol{w}) \wedge B),\\
(A \rightarrow B)_{\mathrm{Q}}(\boldsymbol{f},\boldsymbol{g};\boldsymbol{x},\boldsymbol{w}) &:\equiv \forall \boldsymbol{y} \in \boldsymbol{f}\boldsymbol{x}\boldsymbol{w} A_{\mathrm{Q}}(\boldsymbol{x};\boldsymbol{y}) \wedge A \rightarrow B_{\mathrm{Q}}(\boldsymbol{g}\boldsymbol{x};\boldsymbol{w}),\\
(\forall z A)_{\mathrm{Q}}(\boldsymbol{f};\boldsymbol{y},z) &:\equiv A_{\mathrm{Q}}(\boldsymbol{f}z;\boldsymbol{y}),\\
(\exists z A)_{\mathrm{Q}}(\boldsymbol{x},z;\boldsymbol{y}) &:\equiv A_{\mathrm{Q}}(\boldsymbol{x};\boldsymbol{y}) \wedge A.
\end{aligned}$$

The *q-variant of the Diller-Nahm interpretation* is then defined as $A^{\mathrm{Q}} :\equiv \exists \boldsymbol{x} \forall \boldsymbol{y} A_{\mathrm{Q}}(\boldsymbol{x};\boldsymbol{y})$.

Remark 6.2 We have $\mathsf{IL}^{\omega} \vdash A_{\mathrm{Q}}(\boldsymbol{x};\boldsymbol{y}) \rightarrow A$ for all disjunctive and existential formulas A of IL^{ω}.

In the same paper, Jørgensen argues that only changing the clause on implication doesn't result in a sound Diller-Nahm interpretation with truth. We show that this can be fixed by also changing the clause on universal quantifier.

Definition 6.3 (Diller-Nahm interpretation with truth, for IL^{ω}) Each formula A of intuitionistic logic is associated to a new formula $A_{\mathrm{Dt}}(\boldsymbol{x};\boldsymbol{y})$ inductively as follows. If A_{at} is an atomic formula, then $(A_{\mathrm{at}})_{\mathrm{Dt}}(;) :\equiv A_{\mathrm{at}}$.

Assuming the interpretations of A and B are already given we define:

$$\begin{aligned}
(A \wedge B)_{\mathrm{Dt}}(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{y}, \boldsymbol{w}) &:\equiv A_{\mathrm{Dt}}(\boldsymbol{x}; \boldsymbol{y}) \wedge B_{\mathrm{Dt}}(\boldsymbol{v}; \boldsymbol{w}), \\
(A \vee B)_{\mathrm{Dt}}(\boldsymbol{x}, \boldsymbol{v}, b; \boldsymbol{y}, \boldsymbol{w}) &:\equiv A_{\mathrm{Dt}}(\boldsymbol{x}; \boldsymbol{y}) \diamond_b B_{\mathrm{Dt}}(\boldsymbol{v}; \boldsymbol{w}), \\
(A \to B)_{\mathrm{Dt}}(\boldsymbol{f}, \boldsymbol{g}; \boldsymbol{x}, \boldsymbol{w}) &:\equiv (\forall \boldsymbol{y} \in \boldsymbol{f}\boldsymbol{x}\boldsymbol{w} A_{\mathrm{Dt}}(\boldsymbol{x}; \boldsymbol{y}) \to B_{\mathrm{Dt}}(\boldsymbol{g}\boldsymbol{x}; \boldsymbol{w})) \wedge (A \to B), \\
(\forall z A)_{\mathrm{Dt}}(\boldsymbol{f}; \boldsymbol{y}, z) &:\equiv A_{\mathrm{Dt}}(\boldsymbol{f}z; \boldsymbol{y}) \wedge \forall z A, \\
(\exists z A)_{\mathrm{Dt}}(\boldsymbol{x}, z; \boldsymbol{y}) &:\equiv A_{\mathrm{Dt}}(\boldsymbol{x}; \boldsymbol{y}).
\end{aligned}$$

The *Diller-Nahm interpretation with truth* is then defined as $A^{\mathrm{Dt}} :\equiv \exists \boldsymbol{x} \forall \boldsymbol{y} A_{\mathrm{Dt}}(\boldsymbol{x}; \boldsymbol{y})$.

Remark 6.4 We have $\mathsf{IL}^\omega \vdash A_{\mathrm{Dt}}(\boldsymbol{x}; \boldsymbol{y}) \to A$ for all formulas A of IL^ω.

6.2 Soundness

Proposition 6.5 *For all formulas A of IL^ω we have*

$$\mathsf{IL}^\omega \vdash A_{\mathrm{Dt}}(\boldsymbol{x}; \boldsymbol{y}) \leftrightarrow A_{\mathrm{Q}}(\boldsymbol{x}; \boldsymbol{y}) \wedge A.$$

Proof. By induction on the logical structure of A.

$\underline{A \to B}$.

$$\begin{aligned}
(A \to B)_{\mathrm{Dt}}(\boldsymbol{f}, \boldsymbol{g}; \boldsymbol{x}, \boldsymbol{w}) &\equiv (\forall \boldsymbol{y} \in \boldsymbol{f}\boldsymbol{x}\boldsymbol{w} A_{\mathrm{Dt}}(\boldsymbol{x}; \boldsymbol{y}) \to B_{\mathrm{Dt}}(\boldsymbol{g}\boldsymbol{x}; \boldsymbol{w})) \wedge (A \to B) \\
&\overset{\text{(IH)}}{\leftrightarrow} (\forall \boldsymbol{y} \in \boldsymbol{f}\boldsymbol{x}\boldsymbol{w} (A_{\mathrm{Q}}(\boldsymbol{x}; \boldsymbol{y}) \wedge A) \to B_{\mathrm{Q}}(\boldsymbol{g}\boldsymbol{x}; \boldsymbol{w}) \wedge B) \wedge (A \to B) \\
&\leftrightarrow (\forall \boldsymbol{y} \in \boldsymbol{f}\boldsymbol{x}\boldsymbol{w} A_{\mathrm{Q}}(\boldsymbol{x}; \boldsymbol{y}) \wedge A \to B_{\mathrm{Q}}(\boldsymbol{g}\boldsymbol{x}; \boldsymbol{w}) \wedge B) \wedge (A \to B) \\
&\leftrightarrow (\forall \boldsymbol{y} \in \boldsymbol{f}\boldsymbol{x}\boldsymbol{w} A_{\mathrm{Q}}(\boldsymbol{x}; \boldsymbol{y}) \wedge A \to B_{\mathrm{Q}}(\boldsymbol{g}\boldsymbol{x}; \boldsymbol{w})) \wedge (A \to B) \\
&\equiv (A \to B)_{\mathrm{Q}}(\boldsymbol{f}, \boldsymbol{g}; \boldsymbol{x}, \boldsymbol{w}) \wedge (A \to B).
\end{aligned}$$

It should be noted that we must work with non-empty sets, so that $\forall x \in a\, (A(x) \wedge B) \leftrightarrow \forall x \in a\, A(x) \wedge B$. The remaining cases can be checked in a similar way. □

Theorem 6.6 (Soundness for Diller-Nahm interpretation with truth) *Let A be an arbitrary formula of IL^ω and let $\boldsymbol{z}$ be all its free variables. If $\mathsf{IL}^\omega \vdash A$ then we can extract from a proof of A closed terms $\boldsymbol{t}$ of IL^ω such that $\mathsf{IL}^\omega \vdash A_{\mathrm{Dt}}(\boldsymbol{t}\boldsymbol{z}; \boldsymbol{y})$.*

Proof. Follows from the soundness of the q-variant of Diller-Nahm interpretation (cf. [9]) and Proposition 6.5. □

6.3 Around Jørgensen's counter-example

Note that in order to obtain a Diller-Nahm interpretation with truth we have changed both the clauses on implication and universal quantifier from Diller-Nahm's original definition [2]. Jørgensen [9] showed that by only changing the implication clause one obtains an unsound Diller-Nahm interpretation with truth. The counter-example: take

$$A \equiv \forall z (\forall x \neg B \to \neg \exists x B).$$

A is clearly intuitionistically valid. We have, however, that for $B \equiv Tzzx$, where T is Kleene's famous predicate, there is no term t such that $\forall x, z A_{\mathrm{Dt}}(t; x, z)$, that is

$$\forall x, z (\forall x_0 \in txz \neg Tzzx_0 \to (\neg Tzzx \wedge \neg \exists x_1 Tzzx_1))$$

as such t would provide a solution to the halting problem. In our Diller-Nahm interpretation with truth, however, this formula A is interpreted as

$$\exists f \forall x, z ((\forall x_0 \in fxz \neg Tzzx_0 \wedge \forall x_2 \neg Tzzx_2) \to (\neg Tzzx \wedge \neg \exists x_1 Tzzx_1)),$$

which is equivalent to the provable statement

$$\forall z (\forall x_2 \neg Tzzx_2 \to \neg \exists x_1 Tzzx_1).$$

6.4 Diller-Nahm interpretation with truth versus realizability

One should observe that for logically simple formulas, the Diller-Nahm interpretation with truth in fact coincides with modified realizability with truth. For instance, consider a formula of the kind $\Pi_1 \to \Pi_2$, that is $\forall x P(x) \to \forall v \exists w Q(v,w)$ with P and Q quantifier-free (and, without loss of generality, are $\exists$-free and $v \notin \mathrm{FV}(P)$). Its Diller-Nahm interpretation with truth and modified realizability with truth are both equivalent to

$$\exists g \forall v (\forall x P(x) \to Q(v, gv)) \wedge (\forall x P(x) \to \forall v \exists w Q(v,w)).$$

On the other hand, for formulas of higher logical complexity the Diller-Nahm interpretation with truth is different from realizability. For instance, the Diller-Nahm interpretation with truth of a proof of $\forall x \exists y P(x,y) \to \forall v \exists w Q(v,w)$ (that is $\Pi_2 \to \Pi_2$) gives terms t, s such that in particular

$$\forall f, v (\forall x \in tfv P(x, fx) \wedge \forall x \exists y P(x,y) \to Q(v, sfv))$$

which is seemly stronger than what is obtained via modified realizability with truth, namely

$$\forall f, v (\forall x P(x, fx) \to Q(v, sfv))$$

since $\forall x P(x, fx)$ is stronger than $\forall x \in tfv P(x, fx) \wedge \forall x \exists y P(x,y)$.

6.5 Characterisation

It is usual for functional interpretations that a formula A is interpreted as the existence of a term $\boldsymbol{t}$ such that $\forall \boldsymbol{y} |A|^{\boldsymbol{t}}_{\boldsymbol{y}}$. Hence, it might seem strange that for the Diller-Nahm interpretation with truth we already have that $A_{\mathrm{Dt}}(\boldsymbol{x};\boldsymbol{y})$ (rather than $\forall \boldsymbol{y} A_{\mathrm{Dt}}(\boldsymbol{x};\boldsymbol{y})$) implies A. Although we have shown that the resulting modification of Diller-Nahm interpretation is still sound, one might wonder whether it is "complete", meaning that only true formulas are witnessed. In this section we show that this is the case, albeit at a heavy price: we seem to need full (classical) choice in order to show that A is equivalent to $\exists \boldsymbol{x} \forall \boldsymbol{y} A_{\mathrm{Dt}}(\boldsymbol{x};\boldsymbol{y})$ (while for the original Diller-Nahm interpretation $\exists \boldsymbol{x} \forall \boldsymbol{y} A_{\mathrm{D}}(\boldsymbol{x};\boldsymbol{y})$ it is not needed because $A_{\mathrm{D}}(\boldsymbol{x};\boldsymbol{y})$ it essentially quantifier-free).

Proposition 6.7

(i) $\mathsf{PA}^\omega + \mathsf{AC} \vdash A \leftrightarrow A^{\mathrm{Qt}}$

(ii) $\mathsf{PA}^\omega + \mathsf{AC} \vdash A \leftrightarrow A^{\mathrm{Dt}}$.

P r o o f. The proof of (i) is by induction on structure of formulas.

<u>$A \to B$</u>.

$$\begin{aligned}
(A \to B) &\leftrightarrow (A \wedge A \to B) \\
&\overset{\text{(IH)}}{\leftrightarrow} \exists \boldsymbol{x} \forall \boldsymbol{y} A_{\mathrm{Q}}(\boldsymbol{x};\boldsymbol{y}) \wedge A \to \exists \boldsymbol{v} \forall \boldsymbol{w} B_{\mathrm{Q}}(\boldsymbol{v};\boldsymbol{w}) \\
&\leftrightarrow \exists \boldsymbol{x} \forall \boldsymbol{a} \forall \boldsymbol{y} \in \boldsymbol{a} A_{\mathrm{Q}}(\boldsymbol{x};\boldsymbol{y}) \wedge A \to \exists \boldsymbol{v} \forall \boldsymbol{w} B_{\mathrm{Q}}(\boldsymbol{v};\boldsymbol{w}) \\
&\overset{(\mathsf{PA}^\omega)}{\leftrightarrow} \forall \boldsymbol{x} \exists \boldsymbol{v} \forall \boldsymbol{w} \exists \boldsymbol{a} (\forall \boldsymbol{y} \in \boldsymbol{a} A_{\mathrm{Q}}(\boldsymbol{x};\boldsymbol{y}) \wedge A \to B_{\mathrm{Q}}(\boldsymbol{v};\boldsymbol{w})) \\
&\overset{(\mathsf{AC})}{\leftrightarrow} \exists \boldsymbol{f}, \boldsymbol{g} \forall \boldsymbol{x}, \boldsymbol{w} (\forall \boldsymbol{y} \in \boldsymbol{f}\boldsymbol{x}\boldsymbol{w} A_{\mathrm{Q}}(\boldsymbol{x};\boldsymbol{y}) \wedge A \to B_{\mathrm{Q}}(\boldsymbol{g}\boldsymbol{x};\boldsymbol{w})) \\
&\equiv (A \to B)^{\mathrm{Qt}}.
\end{aligned}$$

The other cases are treated similarly. Point (ii) then follows from (i) and Proposition 6.5. □

Remark 6.8 One can easily argue that AC is necessary for the characterisation of the Diller-Nahm interpretation with truth as follows. Assume $A^{\mathrm{Dt}} \leftrightarrow A$ for all formulas A. Then, in particular we have

$$\forall x \exists y A(x,y) \to (\forall x \exists y A(x,y))^{\mathrm{Dt}}.$$

AC then follows by observing that $(\forall x \exists y A(x,y))^{\mathrm{Dt}} \to \exists f \forall x A(x,y)$. It is an open question, however, whether full classical logic is indeed necessary or not for the characterisation of the Diller-Nahm interpretation with truth (or its q-variant). A similar phenomena happens with the bounded functional presented in the next section.

7 Bounded Functional Interpretation with Truth

In this section we present q- and truth variants of the bounded functional interpretation [4]. In the following, $\mathsf{IL}^\omega_\trianglelefteq$ denotes the formal system described in [4] containing an intensional majorizability relation $\trianglelefteq$ and primitive bounded quantifications $\forall z \trianglelefteq tA$ and $\exists z \trianglelefteq tA$.

7.1 Definitions

Definition 7.1 (q-variant of bounded functional interpretation) To each formula A of $\mathsf{IL}^\omega_\trianglelefteq$ is associated a new formula $A_{\mathrm{Bq}}(\boldsymbol{x};\boldsymbol{y})$ inductively as follows. If A_{at} is an atomic formula, then $(A_{\mathrm{at}})_{\mathrm{Bq}}(;) :\equiv A_{\mathrm{at}}$. Assuming that we already defined $A_{\mathrm{Bq}}(\boldsymbol{x};\boldsymbol{y})$ and $B_{\mathrm{Bq}}(\boldsymbol{v};\boldsymbol{w})$, then

$$
\begin{aligned}
(A \wedge B)_{\mathrm{Bq}}(\boldsymbol{x},\boldsymbol{v};\boldsymbol{y},\boldsymbol{w}) &:\equiv A_{\mathrm{Bq}}(\boldsymbol{x};\boldsymbol{y}) \wedge B_{\mathrm{Bq}}(\boldsymbol{v};\boldsymbol{w})\\
(A \vee B)_{\mathrm{Bq}}(\boldsymbol{x},\boldsymbol{v};\boldsymbol{y},\boldsymbol{w}) &:\equiv (\tilde{\forall}\tilde{\boldsymbol{y}} \trianglelefteq \boldsymbol{y} A_{\mathrm{Bq}}(\boldsymbol{x};\tilde{\boldsymbol{y}}) \wedge A) \vee (\tilde{\forall}\tilde{\boldsymbol{w}} \trianglelefteq \boldsymbol{w} B_{\mathrm{Bq}}(\boldsymbol{v};\tilde{\boldsymbol{w}}) \wedge B)\\
(A \to B)_{\mathrm{Bq}}(\boldsymbol{f},\boldsymbol{g};\boldsymbol{x},\boldsymbol{w}) &:\equiv \tilde{\forall}\boldsymbol{y} \trianglelefteq \boldsymbol{f}\boldsymbol{x}\boldsymbol{w} A_{\mathrm{Bq}}(\boldsymbol{x};\boldsymbol{y}) \wedge A \to B_{\mathrm{Bq}}(\boldsymbol{g}\boldsymbol{x};\boldsymbol{w})\\
(\forall z \trianglelefteq tA)_{\mathrm{Bq}}(\boldsymbol{x};\boldsymbol{y}) &:\equiv \forall z \trianglelefteq tA_{\mathrm{Bq}}(\boldsymbol{x};\boldsymbol{y})\\
(\exists z \trianglelefteq tA)_{\mathrm{Bq}}(\boldsymbol{x};\boldsymbol{y}) &:\equiv \exists z \trianglelefteq t(\tilde{\forall}\tilde{\boldsymbol{y}} \trianglelefteq \boldsymbol{y} A_{\mathrm{Bq}}(\boldsymbol{x};\tilde{\boldsymbol{y}}) \wedge A)\\
(\forall zA)_{\mathrm{Bq}}(\boldsymbol{f};u,\boldsymbol{y}) &:\equiv \forall z \trianglelefteq uA_{\mathrm{Bq}}(\boldsymbol{f}u;\boldsymbol{y})\\
(\exists zA)_{\mathrm{Bq}}(u,\boldsymbol{x};\boldsymbol{y}) &:\equiv \exists z \trianglelefteq u(\tilde{\forall}\tilde{\boldsymbol{y}} \trianglelefteq \boldsymbol{y} A_{\mathrm{Bq}}(\boldsymbol{x};\tilde{\boldsymbol{y}}) \wedge A).
\end{aligned}
$$

The *q-variant of bounded functional interpretation* of a formula A is then defined as $A^{\mathrm{Bq}} :\equiv \tilde{\exists}\boldsymbol{x}\tilde{\forall}\boldsymbol{y} A_{\mathrm{Bq}}(\boldsymbol{x};\boldsymbol{y})$.

Remark 7.2 We have $\mathsf{IL}^\omega_\trianglelefteq \vdash A_{\mathrm{Bq}}(\boldsymbol{x};\boldsymbol{y}) \to A$ for all disjunctive and (bounded and unbounded) existential formulas A of $\mathsf{IL}^\omega_\trianglelefteq$.

Definition 7.3 (Bounded functional interpretation with truth) To each formula A of $\mathsf{IL}^\omega_\trianglelefteq$ is associated a new formula $A_{\mathrm{Bt}}(\boldsymbol{x};\boldsymbol{y})$ inductively as follows. If A_{at} is an atomic formula, then $(A_{\mathrm{at}})_{\mathrm{Bt}}(;) :\equiv A_{\mathrm{at}}$. Assuming that we already defined $A_{\mathrm{Bt}}(\boldsymbol{x};\boldsymbol{y})$ and $B_{\mathrm{Bt}}(\boldsymbol{v};\boldsymbol{w})$, then

$$
\begin{aligned}
(A \wedge B)_{\mathrm{Bt}}(\boldsymbol{x},\boldsymbol{v};\boldsymbol{y},\boldsymbol{w}) &:\equiv A_{\mathrm{Bt}}(\boldsymbol{x};\boldsymbol{y}) \wedge B_{\mathrm{Bt}}(\boldsymbol{v};\boldsymbol{w})\\
(A \vee B)_{\mathrm{Bt}}(\boldsymbol{x},\boldsymbol{v};\boldsymbol{y},\boldsymbol{w}) &:\equiv \tilde{\forall}\tilde{\boldsymbol{y}} \trianglelefteq \boldsymbol{y} A_{\mathrm{Bt}}(\boldsymbol{x};\tilde{\boldsymbol{y}}) \vee \tilde{\forall}\tilde{\boldsymbol{w}} \trianglelefteq \boldsymbol{w} B_{\mathrm{Bt}}(\boldsymbol{v};\tilde{\boldsymbol{w}})\\
(A \to B)_{\mathrm{Bt}}(\boldsymbol{f},\boldsymbol{g};\boldsymbol{x},\boldsymbol{w}) &:\equiv (\tilde{\forall}\boldsymbol{y} \trianglelefteq \boldsymbol{f}\boldsymbol{x}\boldsymbol{w} A_{\mathrm{Bt}}(\boldsymbol{x};\boldsymbol{y}) \to B_{\mathrm{Bt}}(\boldsymbol{g}\boldsymbol{x};\boldsymbol{w})) \wedge (A \to B)\\
(\forall z \trianglelefteq tA)_{\mathrm{Bt}}(\boldsymbol{x};\boldsymbol{y}) &:\equiv \forall z \trianglelefteq tA_{\mathrm{Bt}}(\boldsymbol{x};\boldsymbol{y})\\
(\exists z \trianglelefteq tA)_{\mathrm{Bt}}(\boldsymbol{x};\boldsymbol{y}) &:\equiv \exists z \trianglelefteq t\tilde{\forall}\tilde{\boldsymbol{y}} \trianglelefteq \boldsymbol{y} A_{\mathrm{Bt}}(\boldsymbol{x};\tilde{\boldsymbol{y}})\\
(\forall zA)_{\mathrm{Bt}}(\boldsymbol{f};u,\boldsymbol{y}) &:\equiv \forall z \trianglelefteq uA_{\mathrm{Bt}}(\boldsymbol{f}u;\boldsymbol{y}) \wedge \forall zA\\
(\exists zA)_{\mathrm{Bt}}(u,\boldsymbol{x};\boldsymbol{y}) &:\equiv \exists z \trianglelefteq u\tilde{\forall}\tilde{\boldsymbol{y}} \trianglelefteq \boldsymbol{y} A_{\mathrm{Bt}}(\boldsymbol{x};\tilde{\boldsymbol{y}}).
\end{aligned}
$$

The *bounded functional interpretation with truth* of a formula A is then defined as $A^{\mathrm{Bt}} :\equiv \tilde{\exists}\boldsymbol{x}\tilde{\forall}\boldsymbol{y} A_{\mathrm{Bt}}(\boldsymbol{x};\boldsymbol{y})$.

Remark 7.4 We have $\mathsf{IL}^\omega_\trianglelefteq \vdash A_{\mathrm{Bt}}(\boldsymbol{x};\boldsymbol{y}) \to A$ for all formulas A of $\mathsf{IL}^\omega_\trianglelefteq$.

7.2 Soundness theorems

Theorem 7.5 (Soundness of q-variant of bounded functional interpretation) *Let A be an arbitrary formula of $\mathsf{IL}^\omega_\trianglelefteq$ and let $\boldsymbol{z}$ be all its free variables. If $\mathsf{IL}^\omega_\trianglelefteq \vdash A$, then we can extract from a proof of A closed monotone (that is, $\mathsf{IL}^\omega_\trianglelefteq \vdash \boldsymbol{t} \trianglelefteq \boldsymbol{t}$) terms $\boldsymbol{t}$ of $\mathsf{IL}^\omega_\trianglelefteq$ such that $\mathsf{IL}^\omega_\trianglelefteq \vdash \tilde{\forall}\boldsymbol{a}\forall\boldsymbol{z} \trianglelefteq \boldsymbol{a}\tilde{\forall}\boldsymbol{y} A_{\mathrm{Bq}}(\boldsymbol{t}\boldsymbol{a};\boldsymbol{y})$.*

P r o o f. The proof is by induction on a proof of A.

<u>$A \to A \wedge A$</u>. The interpretation of this axiom is

$$\tilde{\exists}\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}\tilde{\forall}\boldsymbol{x},\boldsymbol{y}_2,\boldsymbol{y}_3(\tilde{\forall}\boldsymbol{y}_1 \trianglelefteq \boldsymbol{f}\boldsymbol{x}\boldsymbol{y}_2\boldsymbol{y}_3 A_{\mathrm{Bq}}(\boldsymbol{x};\boldsymbol{y}_1) \wedge A \to A_{\mathrm{Bq}}(\boldsymbol{g}\boldsymbol{x};\boldsymbol{y}_2) \wedge A_{\mathrm{Bq}}(\boldsymbol{h}\boldsymbol{x};\boldsymbol{y}_3)).$$

The witnesses for $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$ can be taken as $\lambda\boldsymbol{a},\boldsymbol{x},\boldsymbol{y}_2,\boldsymbol{y}_3 .\ \max(\boldsymbol{y}_2,\boldsymbol{y}_3)$, $\lambda\boldsymbol{a},\boldsymbol{x}.\boldsymbol{x}$ and $\lambda\boldsymbol{a},\boldsymbol{x}.\boldsymbol{x}$, respectively. Here we use that $\mathsf{IL}^\omega_\trianglelefteq \vdash \tilde{\forall}\boldsymbol{y}_2,\boldsymbol{y}_3(\boldsymbol{y}_2 \trianglelefteq \max(\boldsymbol{y}_2,\boldsymbol{y}_3) \wedge \boldsymbol{y}_3 \trianglelefteq \max(\boldsymbol{y}_2,\boldsymbol{y}_3))$.

$\underline{A \vee A \to A}$. The interpretation of this axiom is

$$\tilde{\exists} \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} \tilde{\forall} \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_3 (\tilde{\forall} \boldsymbol{y}_1, \boldsymbol{y}_2 \trianglelefteq \boldsymbol{f} \boldsymbol{x}_1 \boldsymbol{x}_2 \boldsymbol{y}_3, \boldsymbol{g} \boldsymbol{x}_1 \boldsymbol{x}_2 \boldsymbol{y}_3 (\bigvee_{i=1}^{2} (\tilde{\forall} \tilde{\boldsymbol{y}}_i \trianglelefteq \boldsymbol{y}_i A_{\mathrm{Bq}}(\boldsymbol{x}_i; \tilde{\boldsymbol{y}}_i) \wedge A) \wedge (A \vee A) \to A_{\mathrm{Bq}}(\boldsymbol{h} \boldsymbol{x}_1 \boldsymbol{x}_2; \boldsymbol{y}_3))).$$

The witnesses for $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$ can be taken as $\lambda \boldsymbol{a}, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_3 \,.\, \boldsymbol{y}_3$, again $\lambda \boldsymbol{a}, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_3 \,.\, \boldsymbol{y}_3$ and $\lambda \boldsymbol{a}, \boldsymbol{x}_1, \boldsymbol{x}_2 \,.\, \max(\boldsymbol{x}_1, \boldsymbol{x}_2)$, respectively. Here we use that $\mathsf{IL}^{\omega}_{\trianglelefteq} \vdash \tilde{\forall} \boldsymbol{x} \forall \boldsymbol{y}, \boldsymbol{z} (\boldsymbol{y} \trianglelefteq \boldsymbol{z} \wedge A_{\mathrm{Bq}}(\boldsymbol{y}; \boldsymbol{x}) \to A_{\mathrm{Bq}}(\boldsymbol{z}; \boldsymbol{x}))$.

$\underline{A[q/z] \to \exists z A}$. Here we use $A[q/z]_{\mathrm{Bq}}(\boldsymbol{x}; \boldsymbol{y}) \equiv A_{\mathrm{Bq}}(\boldsymbol{x}; \boldsymbol{y})[q/z]$. Also we use that for all terms $q[\boldsymbol{z}']$, where $\boldsymbol{z}'$ are all the variables in q, there exists a term $q^*[\boldsymbol{z}']$ with exactly the same variables $\boldsymbol{z}'$ such that $\mathsf{IL}^{\omega}_{\trianglelefteq} \vdash \tilde{\forall} \boldsymbol{a}' \forall \boldsymbol{z}' \trianglelefteq \boldsymbol{a}' (q[\boldsymbol{z}'] \trianglelefteq q^*[\boldsymbol{a}'])$. The interpretation of the axiom is

$$\tilde{\exists} \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} \tilde{\forall} \boldsymbol{x}, \boldsymbol{w} (\tilde{\forall} \boldsymbol{y} \trianglelefteq \boldsymbol{f} \boldsymbol{x} \boldsymbol{w} A[q/z]_{\mathrm{Bq}}(\boldsymbol{x}; \boldsymbol{y}) \wedge A[q/z] \to \exists z \trianglelefteq \boldsymbol{g} \boldsymbol{x} (\tilde{\forall} \tilde{\boldsymbol{w}} \trianglelefteq \boldsymbol{w} A_{\mathrm{Bq}}(\boldsymbol{h} \boldsymbol{x}; \tilde{\boldsymbol{w}}) \wedge A).$$

The witnesses for $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$ can be taken as $\lambda \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{w} \,.\, \boldsymbol{w}$, $\lambda \boldsymbol{a}, \boldsymbol{x} \,.\, q^*[\boldsymbol{a}']$ (if $z \in \mathrm{FV}(A)$, then $\boldsymbol{z}'$ are among $\boldsymbol{z}$, so $\boldsymbol{a}'$ are among $\boldsymbol{a}$, thus the term is closed; otherwise take $\mathbf{0}$) and $\lambda \boldsymbol{a}, \boldsymbol{x} \,.\, \boldsymbol{x}$, respectively.

$\underline{A, A \to B \Rightarrow B}$. Here we use the assumption that $\mathsf{IL}^{\omega}_{\trianglelefteq} \vdash A$. Let $\boldsymbol{z}_A$, $\boldsymbol{z}_{AB}$ and $\boldsymbol{z}_B$ be all the free variables of A, $A \to B$ and B, respectively. By induction hypothesis we have witnesses $\boldsymbol{q}$, $\boldsymbol{r}$ and $\boldsymbol{s}$ for the interpretation of the premises of the rule, that is,

$$\mathsf{IL}^{\omega}_{\trianglelefteq} \vdash \tilde{\forall} \boldsymbol{a}_A \forall \boldsymbol{z}_A \trianglelefteq \boldsymbol{a}_A \tilde{\forall} \boldsymbol{y}_1 A_{\mathrm{Bq}}(\boldsymbol{q} \boldsymbol{a}_A; \boldsymbol{y}_1)$$
$$\mathsf{IL}^{\omega}_{\trianglelefteq} \vdash \tilde{\forall} \boldsymbol{a}_{AB} \forall \boldsymbol{z}_{AB} \trianglelefteq \boldsymbol{a}_{AB} \tilde{\forall} \boldsymbol{x}_1, \boldsymbol{y}_2 (\tilde{\forall} \boldsymbol{y}_1 \trianglelefteq \boldsymbol{r} \boldsymbol{a}_{AB} \boldsymbol{x}_1 \boldsymbol{y}_2 A_{\mathrm{Bq}}(\boldsymbol{x}_1; \boldsymbol{y}_1) \wedge A \to B_{\mathrm{Bq}}(\boldsymbol{s} \boldsymbol{a}_{AB} \boldsymbol{x}_1; \boldsymbol{y}_2)).$$

A witness for the interpretation of the conclusion of the rule can be taken as $\lambda \boldsymbol{a}_B \,.\, \boldsymbol{s} \boldsymbol{a}_{AB}(\boldsymbol{q} \boldsymbol{a}_A)$ (if it is not closed, we close it by replacing its variables by $\mathbf{0}$).

$\underline{A_b \wedge u \trianglelefteq v \to tu \trianglelefteq qv \wedge qu \trianglelefteq qv \Rightarrow A_b \to t \trianglelefteq q}$. Here we use the fact that if A_b is a bounded formula (that is, all its quantifications are bounded), then $\mathsf{IL}^{\omega}_{\trianglelefteq} \vdash (A_b)_{\mathrm{Bq}}(\boldsymbol{x}; \boldsymbol{y}) \leftrightarrow A_b$, where $(A_b)_{\mathrm{Bq}}(\boldsymbol{x}; \boldsymbol{y})$ is a bounded formula and $\boldsymbol{x}$ and $\boldsymbol{y}$ are empty tuples that may be omitted, and so we can apply the rule with $(A_b)_{\mathrm{Bq}}$. By induction hypothesis we have empty witnesses for the interpretation of the premise of the rule, that is,

$$\mathsf{IL}^{\omega} \vdash \tilde{\forall} \boldsymbol{a}, b, c \forall \boldsymbol{z}, u, v \trianglelefteq \boldsymbol{a}, b, c((A_b)_{\mathrm{Bq}} \wedge u \trianglelefteq v \wedge A_b \wedge u \trianglelefteq v \to tu \trianglelefteq qv \wedge qu \trianglelefteq qv).$$

We rewrite it as $\mathsf{IL}^{\omega} \vdash (z \trianglelefteq a \wedge A_b) \wedge u \trianglelefteq v \to tu \trianglelefteq qv \wedge qu \trianglelefteq qv$ using $\mathsf{IL}^{\omega} \vdash u \trianglelefteq v \to v \trianglelefteq v$, so by the rule we get $\mathsf{IL}^{\omega} \vdash (z \trianglelefteq a \wedge A_b) \to t \trianglelefteq q$. Then we have empty witnesses for the interpretation for the conclusion of the rule, that is,

$$\mathsf{IL}^{\omega} \vdash \tilde{\forall} \boldsymbol{a} \forall \boldsymbol{z} \trianglelefteq \boldsymbol{a}((A_b)_{\mathrm{Bq}} \to t \trianglelefteq q).$$

The other cases are treated similarly. □

Proposition 7.6 *For all formulas A of $\mathsf{IL}^{\omega}_{\trianglelefteq}$ we have*

$$\mathsf{IL}^{\omega}_{\trianglelefteq} \vdash \tilde{\forall} \boldsymbol{x}, \boldsymbol{y} (A_{\mathrm{Bt}}(\boldsymbol{x}; \boldsymbol{y}) \leftrightarrow A_{\mathrm{Bq}}(\boldsymbol{x}; \boldsymbol{y}) \wedge A).$$

P r o o f. Easy induction on the structure of formulas. We use $\mathsf{IL}^{\omega}_{\trianglelefteq} \vdash \tilde{\forall} y (\tilde{\forall} x \trianglelefteq y (A \wedge B) \leftrightarrow \tilde{\forall} x \trianglelefteq y A \wedge B)$, where x is not free in B.

$\underline{A \vee B}$.

$$\begin{aligned} (A \vee B)_{\mathrm{Bt}}(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{y}, \boldsymbol{w}) &\equiv \tilde{\forall} \tilde{\boldsymbol{y}} \trianglelefteq \boldsymbol{y} A_{\mathrm{Bt}}(\boldsymbol{x}; \tilde{\boldsymbol{y}}) \vee \tilde{\forall} \tilde{\boldsymbol{w}} \trianglelefteq \boldsymbol{w} B_{\mathrm{Bt}}(\boldsymbol{v}; \tilde{\boldsymbol{w}}) \\ &\overset{\text{(IH)}}{\leftrightarrow} \tilde{\forall} \tilde{\boldsymbol{y}} \trianglelefteq \boldsymbol{y} (A_{\mathrm{Bq}}(\boldsymbol{x}; \tilde{\boldsymbol{y}}) \wedge A) \vee \tilde{\forall} \tilde{\boldsymbol{w}} \trianglelefteq \boldsymbol{w} (B_{\mathrm{Bq}}(\boldsymbol{v}; \tilde{\boldsymbol{w}}) \wedge B) \\ &\leftrightarrow (\tilde{\forall} \tilde{\boldsymbol{y}} \trianglelefteq \boldsymbol{y} A_{\mathrm{Bq}}(\boldsymbol{x}; \tilde{\boldsymbol{y}}) \wedge A) \vee (\tilde{\forall} \tilde{\boldsymbol{w}} \trianglelefteq \boldsymbol{w} B_{\mathrm{Bq}}(\boldsymbol{v}; \tilde{\boldsymbol{w}}) \wedge B) \\ &\leftrightarrow ((\tilde{\forall} \tilde{\boldsymbol{y}} \trianglelefteq \boldsymbol{y} A_{\mathrm{Bq}}(\boldsymbol{x}; \tilde{\boldsymbol{y}}) \wedge A) \vee (\tilde{\forall} \tilde{\boldsymbol{w}} \trianglelefteq \boldsymbol{w} B_{\mathrm{Bq}}(\boldsymbol{v}; \tilde{\boldsymbol{w}}) \wedge B)) \wedge (A \vee B) \\ &\equiv (A \vee B)_{\mathrm{Bq}}(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{y}, \boldsymbol{w}) \wedge (A \vee B). \end{aligned}$$

$\underline{\exists z A}$.

$$\begin{aligned}(\exists z A)_{\mathrm{Bt}}(u, \boldsymbol{x}; \boldsymbol{y}) &\equiv \exists z \trianglelefteq u \tilde{\forall} \tilde{\boldsymbol{y}} \trianglelefteq \boldsymbol{y} A_{\mathrm{Bt}}(\boldsymbol{x}; \tilde{\boldsymbol{y}}) \\ &\overset{\text{(IH)}}{\leftrightarrow} \exists z \trianglelefteq u \tilde{\forall} \tilde{\boldsymbol{y}} \trianglelefteq \boldsymbol{y}(A_{\mathrm{Bq}}(\boldsymbol{x}; \tilde{\boldsymbol{y}}) \wedge A) \\ &\leftrightarrow \exists z \trianglelefteq u(\tilde{\forall} \tilde{\boldsymbol{y}} \trianglelefteq \boldsymbol{y} A_{\mathrm{Bq}}(\boldsymbol{x}; \tilde{\boldsymbol{y}}) \wedge A) \\ &\leftrightarrow \exists z \trianglelefteq u(\tilde{\forall} \tilde{\boldsymbol{y}} \trianglelefteq \boldsymbol{y} A_{\mathrm{Bq}}(\boldsymbol{x}; \tilde{\boldsymbol{y}}) \wedge A) \wedge \exists z A \\ &\equiv (\exists z A)_{\mathrm{Bq}}(u, \boldsymbol{x}; \boldsymbol{y}) \wedge \exists z A.\end{aligned}$$

The other cases are treated similarly. □

Theorem 7.7 (Soundness for bounded functional interpretation with truth) *Let A be an arbitrary formula of $\mathsf{IL}^{\omega}_{\trianglelefteq}$ and let $\boldsymbol{z}$ be all its free variables. If $\mathsf{IL}^{\omega}_{\trianglelefteq} \vdash A$, then we can extract from a proof of A closed monotone terms $\boldsymbol{t}$ of $\mathsf{IL}^{\omega}_{\trianglelefteq}$ such that $\mathsf{IL}^{\omega}_{\trianglelefteq} \vdash \tilde{\forall} \boldsymbol{a} \forall \boldsymbol{z} \trianglelefteq \boldsymbol{a} \tilde{\forall} \boldsymbol{y} A_{\mathrm{Bq}}(\boldsymbol{t}\boldsymbol{a}; \boldsymbol{y})$.*

P r o o f. Follows from Theorem 7.5 and Proposition 7.6. □

8 Conclusion

The second author has recently investigated [5, 14–16] how different functional interpretations of intuitionistic logic can be analysed via functional interpretations of linear logic. In this article we have made use of this analysis in order to show how the q- and truth variants of modified realizability follow from a single modified realizability with truth of linear logic. In particular, we observed that q-variants correspond to the standard embedding $(\cdot)^*$ of intuitionistic logic into linear logic, whereas truth-variants arise from Girard's alternative embedding $(\cdot)^{\circ}$. This led us to two heuristics that succeeded in obtained q- and truth variants for bounded modified realizability, the Diller-Nahm interpretation, and the bounded functional interpretation; as well as to explain the relation between Kleene's and Aczel's slash translations.

Recently, the analysis of functional interpretations via linear logic has also given rise to a hybrid functional interpretation [8, 17], where multiple interpretations can be applied to a single proof. It seems that also the truth variants of these interpretations can be easily incorporated into the hybrid interpretation. One, however, no longer gets a linear order between the different modalities (of the multi-modal linear logic), but rather a lattice, as sketched in Figure 3, where $!_{\mathrm{k}}$, $!_{\mathrm{d}}$ and $!_{\mathrm{g}}$ stand for Kreisel, Diller-Nahm and Gödel modalities, and $!_{t*}$ their respective truth variant (it seems that there cannot be a Gödel truth interpretation, due to the contraction problem/decidability issue). These multiple modalities are given different interpretations as follows:

$$\begin{aligned}\{!_{\mathrm{k}} A\}^{\boldsymbol{x}} &:\equiv\ !\forall \boldsymbol{y}\{A\}^{\boldsymbol{x}}_{\boldsymbol{y}} && \text{(Kreisel's modified realizability)} \\ \{!_{\mathrm{d}} A\}^{\boldsymbol{x}}_{\boldsymbol{y}} &:\equiv\ !\forall \boldsymbol{y}' \in \boldsymbol{y}\{A\}^{\boldsymbol{x}}_{\boldsymbol{y}'} && \text{(Diller-Nahm interpretation)} \\ \{!_{\mathrm{g}} A\}^{\boldsymbol{x}}_{\boldsymbol{y}} &:\equiv\ !\{A\}^{\boldsymbol{x}}_{\boldsymbol{y}} && \text{(Gödel's dialectica interpretation)} \\ \{!_{\mathrm{tk}} A\}^{\boldsymbol{x}} &:\equiv\ !\forall \boldsymbol{y}\{A\}^{\boldsymbol{x}}_{\boldsymbol{y}} \otimes !A && \text{(Kreisel's modified realizability with truth)} \\ \{!_{\mathrm{td}} A\}^{\boldsymbol{x}}_{\boldsymbol{y}} &:\equiv\ !\forall \boldsymbol{y}' \in \boldsymbol{y}\{A\}^{\boldsymbol{x}}_{\boldsymbol{y}'} \otimes !A && \text{(Diller-Nahm interpretation with truth)}\end{aligned}$$

In Figure 3 we write $!_X$ above $!_Y$ if the interpretation of $!_X A$ implies the interpretation of $!_Y A$. As such, we could say that modified realizability with truth and Gödel's dialectica interpretation are the two "extreme" interpretations.

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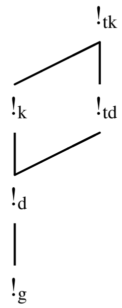

Fig. 3 Partial order between multiple modalities (interpretations)

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