Functional Interpretations of Linear and Intuitionistic Logic

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Abstract. This paper surveys several computational interpretations of classical linear logic based on two-player one-move games. The moves of the games are higher-order functionals in the language of finite types. All interpretations discussed treat the exponential-free fragment of linear logic in a common way. They only differ in how much advantage one of the players has in the exponential games. We discuss how the several choices for the interpretation of the modalities correspond to various well-known functional interpretations of intuitionistic logic, including Gödel's Dialectica interpretation and Kreisel's modified realizability.

1 Introduction

This article surveys several interpretations [4, 17, 19, 20] of classical linear logic based on one-move two-player (Eloise and Abelard) games. As we will see, these are related to functional interpretations of intuitionistic logic such as Gödel's Dialectica interpretations [12] and Kreisel's modified realizability [14].

The intuition behind the interpretation is that each formula A defines an adjudication relation between arguments pro (Eloise's move) and against (Abelard's move) the truth of A. Note that we do not ask the moves of Eloise and Abelard to be proofs (or disproofs) of A. The moves only need to be arguments, which can be thought of as incomplete proofs. In this way, even if A is an open problem, whose proof or disproof has yet to be discovered, the game A is still well-defined. If the formula is provable, and hence true, Eloise should have no problem winning the game. On the other hand, if the negation of A is provable, and hence A is false, Abelard should be able to extract a winning move from the refutation of A.

Thinking of the moves as incomplete proofs, it is clear that we must require both players to make their moves *simultaneously*. If one player is allowed to see what the other has chosen for his/her move, that player could simply look for the gap in the opponent's move and provide a counterexample for that. So, the player which is allowed to play second would in fact have a winning strategy, even without having a complete (dis)proof of A. As a simple example, think of the ancient "Odd or Even" game. Since neither player has a winning strategy, it is crucial that both players make their moves simultaneously.

The fact that we work with one-move games is not a restriction when the moves can be higher-order. Consider the game of Chess, for instance. It can also be viewed as a one-move game where each of the two players writes down their strategy as a function mapping board configurations to moves. The game then consists of the two players handing in their strategies, which are then simulated against each other.

The interpretation of linear negation, logical connectives, quantifiers and exponentials corresponds to constructions for building new games out of previously built ones. Given the symmetry of the interpretation, the game corresponding to the linear negation of A is simply the game A with the roles of the two players swapped. In this way, linear double negation would bring us back to the original game, which should be the case since linear negation is involutive. As we will see, the game constructions corresponding to the logical connectives and quantifiers are canonical.

In the case of the exponentials, however, the situation is quite different. It is well know that the rules for the exponentials do not uniquely determine these modalities. This is reflected in the flexibility of interpreting the corresponding modal games. Nevertheless, all interpretations involve a break of symmetry in the game, giving an advantage to one of the players. How much advantage is given separates the different interpretations. In all cases, the advantage is given in the form of one of the players being allowed to look at the opponents move, and make a set of possible moves, rather than a single move. The simplest interpretation of the exponential games allows this set to contain all possible moves, which is equivalent to not making a move at all and winning the game in case a winning move exists.

The paper is organised as follows. The basic interpretation of the exponential-free fragment of classical linear logic is presented in Section 2, and soundness of the interpretation is proved. The interpretation is characterised in Section 2.1. A simple form of branching quantifier is used for the characterisation. In Section 3, we discuss the various possibilities for the interpretation of the exponentials.

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1.1 Linear Logic

We work with an extension of classical linear logic to the language of all finite types. The set of *finite types* $\mathcal{T}$ is inductively defined as follows: $i, b \in \mathcal{T}$; and if $\rho, \sigma \in \mathcal{T}$ then $\rho \to \sigma \in \mathcal{T}$. For simplicity, we deal with only two basic finite types i (e.g. $\mathbb{N}$) and b (Booleans).

We assume that the terms of LL^ω contain all typed λ-terms, i.e. variables x^ρ for each finite type ρ; λ-abstractions $(\lambda x^\rho . t^\sigma)^{\rho \to \sigma}$; term applications $(t^{\rho \to \sigma} s^\rho)^\sigma$, and conditional $(z)(t_0, t_1)$. The conditional λ-term reduces to either t_0 or t_1 depending on whether the boolean variable z reduces to true or false, respectively. The atomic formulas of LL^ω are $A_{\mathsf{at}}, B_{\mathsf{at}}, \ldots$ and $A_{\mathsf{at}}^\perp, B_{\mathsf{at}}^\perp, \ldots$. For simplicity, the standard propositional constants $0, 1, \perp, \top$ of linear logic have been omitted, since the interpretation of atomic formulas is trivial (see Definition 3).

$A_{\mathsf{at}}, A_{\mathsf{at}}^{\perp}$ (id)	$\dfrac{\Gamma, A \quad \Delta, A^{\perp}}{\Gamma, \Delta}$ (cut)	$\dfrac{\Gamma}{\pi\{\Gamma\}}$ (per)
$\dfrac{\Gamma[\gamma_0], A \quad \Gamma[\gamma_1], B}{\Gamma[(z)(\gamma_0, \gamma_1)], A \diamond_z B}\,(\diamond_z)$	$\dfrac{\Gamma, A}{\Gamma, A \diamond_{\mathrm{t}} B}\,(\diamond_{\mathrm{t}})$	$\dfrac{\Gamma, B}{\Gamma, A \diamond_{\mathrm{f}} B}\,(\diamond_{\mathrm{f}})$

$\dfrac{\Gamma, A, B}{\Gamma, A \parr B}\,(\parr)$	$\dfrac{\Gamma, A \quad \Delta, B}{\Gamma, \Delta, A \otimes B}\,(\otimes)$	$\dfrac{\Gamma, A}{\Gamma, \forall z A}\,(\forall)$	$\dfrac{\Gamma, A[t/z]}{\Gamma, \exists z A}\,(\exists)$
$\dfrac{?\Gamma, A}{?\Gamma,\ !A}\,(!)$	$\dfrac{\Gamma, A}{\Gamma, ?A}\,(?)$	$\dfrac{\Gamma, ?A, ?A}{\Gamma, ?A}$ (con)	$\dfrac{\Gamma}{\Gamma, ?A}$ (wkn)

Table 1. Classical linear logic LL^{ω}

Formulas are built out of atomic formulas $A_{\mathsf{at}}, B_{\mathsf{at}}, \ldots$ and $A_{\mathsf{at}}^{\perp}, B_{\mathsf{at}}^{\perp}, \ldots$ via the connectives $A \parr B$ (par), $A \otimes B$ (tensor), $A \diamond_z B$ (if-then-else), quantifiers $\exists x A$ and $\forall x A$, and modalities $?A$ and $!A$. The *linear negation* $A^{\perp}$ of an arbitrary formula A is an abbreviation as follows:

$$
\begin{array}{lcllcl}
(A_{\mathsf{at}})^{\perp} & \equiv & A_{\mathsf{at}}^{\perp} & (A_{\mathsf{at}}^{\perp})^{\perp} & \equiv & A_{\mathsf{at}} \\
(\exists z A)^{\perp} & \equiv & \forall z A^{\perp} & (\forall z A)^{\perp} & \equiv & \exists z A^{\perp} \\
(A \parr B)^{\perp} & \equiv & A^{\perp} \otimes B^{\perp} & (A \otimes B)^{\perp} & \equiv & A^{\perp} \parr B^{\perp} \\
(?A)^{\perp} & \equiv & !A^{\perp} & (!A)^{\perp} & \equiv & ?A^{\perp} \\
(A \diamond_z B)^{\perp} & \equiv & A^{\perp} \diamond_z B^{\perp}. & & &
\end{array}
$$

So, $(A^{\perp})^{\perp}$ is syntactically equal to A. As usual, we write $A \multimap B$ as an abbreviation for $A^{\perp} \parr B$. We will denote by pLL^{ω} (pure LL^{ω}) the fragment of LL^{ω} without the exponentials, and by $\mathsf{LL}_{\mathsf{qf}}^{\omega}$ and $\mathsf{pLL}_{\mathsf{qf}}^{\omega}$ the corresponding quantifier-free fragments.

The formal system for classical linear logic that we will use in this paper is presented in Table 1. The contexts Γ and Δ are sequences of formulas (possibly with repetitions). The structural rules of linear logic (first row) do not contain the usual rules of weakening and contraction. These are added separately, in a controlled manner via the use of modalities (bottom row). We also have the usual side condition in the rule $(\forall)$ that the variable z must not appear free in Γ.

Note that we are deviating from the standard formulation of linear logic, in the sense that we use the if-then-else logical constructor $A \diamond_z B$ instead of standard additive conjunction and disjunction[1]. The logical rules for $A \diamond_z B$ are shown in Table 1. The standard additives can be defined as

$$A \wedge B :\equiv \forall z^b (A \diamond_z B)$$
$$A \vee B :\equiv \exists z^b (A \diamond_z B)$$

with the help of quantification over booleans. For more information on linear logic see Girard's original papers [10, 11].

1.2 Intuitionistic Logic

Table 2 describes a formal system for intuitionistic logic in all finite types, which also uses the if-then-else connective, rather than the usual conjunction and disjunction. In the case of intuitionistic logic, the context Γ is a set of formulas (repetitions and order are not relevant). We will be making use of the following variation of Girard's embedding of intuitionistic logic into linear logic with conditionals.

Definition 1 ([10]). *For any formula A of intuitionistic logic its linear translation A^l is defined inductively as*

$$\begin{aligned} A_{\mathsf{at}}^l &:\equiv A_{\mathsf{at}} \\ (A \diamond_z B)^l &:\equiv A^l \diamond_z B^l \\ (A \to B)^l &:\equiv\ !A^l \multimap B^l \\ (\forall x A)^l &:\equiv \forall x A^l \\ (\exists x A)^l &:\equiv \exists x !A^l. \end{aligned}$$

The translation above is such that $A_0, \ldots, A_n \vdash B$ is derivable in IL^ω if and only if $(!A_0^l)^\perp, \ldots, (!A_n^l)^\perp, B^l$ is derivable in LL^ω. We will also consider the following forgetful translation of intuitionistic logic into linear logic.

Definition 2. *For any linear logic formula A in the image of the translation $(\cdot)^l$ its intuitionistic translation A^i is defined inductively as*

$$\begin{aligned} A_{\mathsf{at}}^i &:\equiv A_{\mathsf{at}} \\ (A \diamond_z B)^i &:\equiv A^i \diamond_z B^i \\ (A \multimap B)^i &:\equiv A^i \to B^i \\ (!A)^i &:\equiv A^i \\ (\forall x A)^i &:\equiv \forall x A^i \\ (\exists x A)^i &:\equiv \exists x A^i. \end{aligned}$$

[1] See Girard's comments in [10] (p13) and [11] (p73) on the relation between the additive connectives and the if-then-else construct.

$$\{\ldots, A_{\mathsf{at}}, \ldots\} \vdash A_{\mathsf{at}} \quad \text{(id)} \qquad \frac{\Gamma \vdash A \quad \Gamma, A \vdash B}{\Gamma \vdash B}\,\text{(cut)}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \diamond_{\mathrm{t}} B}\,(\diamond^r_{\mathrm{t}}) \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \diamond_{\mathrm{f}} B}\,(\diamond^r_{\mathrm{f}}) \quad \frac{\Gamma, A \vdash C}{\Gamma, A \diamond_{\mathrm{t}} B \vdash C}\,(\diamond^l_{\mathrm{t}}) \quad \frac{\Gamma, B \vdash C}{\Gamma, A \diamond_{\mathrm{f}} B \vdash C}\,(\diamond^l_{\mathrm{f}})$$

$$\frac{\Gamma[\boldsymbol{\gamma}_0] \vdash A \quad \Gamma[\boldsymbol{\gamma}_1] \vdash B}{\Gamma[(z)(\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1)] \vdash A \diamond_z B}\,(\diamond_z) \quad \frac{\Gamma[\boldsymbol{\gamma}_0], A \vdash C[\boldsymbol{c}_0] \quad \Gamma[\boldsymbol{\gamma}_1], B \vdash C[\boldsymbol{c}_1]}{\Gamma[(z)(\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1)], A \diamond_z B \vdash C[(z)(\boldsymbol{c}_0, \boldsymbol{c}_1)]}\,(\diamond_z)$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\,(\to^r) \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \forall z A}\,(\forall^r) \qquad \frac{\Gamma \vdash A[t/z]}{\Gamma \vdash \exists z A}\,(\exists^r)$$

$$\frac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}\,(\to^l) \qquad \frac{\Gamma, A \vdash B}{\Gamma, \forall z A \vdash B}\,(\forall^l) \qquad \frac{\Gamma, A \vdash B}{\Gamma, \exists z A \vdash B}\,(\exists^l)$$

Table 2. Intuitionistic logic IL^ω

The translation $(\cdot)^i$ works as an inverse of $(\cdot)^l$, i.e. $A \equiv (A^l)^i$, for any formula A of intuitionistic logic, where $\equiv$ denotes syntactic equality.

For the rest of the article we use bold face variables $\boldsymbol{f}, \boldsymbol{g}, \ldots, \boldsymbol{x}, \boldsymbol{y}, \ldots$ for tuples of variables, and bold face terms $\boldsymbol{a}, \boldsymbol{b}, \ldots, \boldsymbol{\gamma}, \boldsymbol{\delta}, \ldots$ for tuples of terms. Given sequence of terms $\boldsymbol{a}$ and $\boldsymbol{b}$, by $\boldsymbol{a}(\boldsymbol{b})$, we mean the sequence of terms $a_0(\boldsymbol{b}), \ldots, a_n(\boldsymbol{b})$. Similarly for $\boldsymbol{a}[\boldsymbol{b}/\boldsymbol{x}]$.

2 Basic Interpretation

In this section we will describe the interpretation of the pure fragment of classical linear logic. The interpretation of the exponentials is treated in Section 3. To each formula A of the exponential-free fragment of linear logic we associate a quantifier-free formula $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$, where $\boldsymbol{x}, \boldsymbol{y}$ are fresh-variables not appearing in A. Intuitively, the interpretation of a formula A is a two-player (Eloise and Abelard) one-move game, where $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ is the adjudication relation of the game. Eloise and Abelard simultaneously make moves $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively, and Eloise wins if and only if $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ holds. For instance, in the game "Odd of Even" the adjudication relation is "$x + y$ is odd" (assuming Eloise is playing Odd). We want that Eloise has a winning move whenever A is provable in LL^ω. Moreover, the linear logic proof of A will provide Eloise's winning move $\boldsymbol{a}$ and a verification of this fact, i.e. a proof of $\forall \boldsymbol{y} |A|^{\boldsymbol{a}}_{\boldsymbol{y}}$. The interpretation of formulas of linear logic as adjudication relations is defined inductively on the structure of the formulas as follows.[2]

Definition 3 (Basic interpretation [19, 20]). *Assume we have already defined $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ and $|B|^{\boldsymbol{v}}_{\boldsymbol{w}}$, we define*

$$
\begin{aligned}
|A \mathbin{⅋} B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{y},\boldsymbol{w}} &:\equiv |A|^{\boldsymbol{f}\boldsymbol{w}}_{\boldsymbol{y}} \mathbin{⅋} |B|^{\boldsymbol{g}\boldsymbol{y}}_{\boldsymbol{w}} \\
|A \otimes B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{f},\boldsymbol{g}} &:\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{v}} \otimes |B|^{\boldsymbol{v}}_{\boldsymbol{g}\boldsymbol{x}} \\
|A \diamond_z B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} &:\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \diamond_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
|\forall z^\rho A|^{\boldsymbol{f}}_{\boldsymbol{y},z} &:\equiv |A|^{\boldsymbol{f}z}_{\boldsymbol{y}} \\
|\exists z^\rho A|^{\boldsymbol{x},z}_{\boldsymbol{f}} &:\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{f}z}.
\end{aligned}
$$

The interpretation of atomic formulas are the atomic formulas themselves, i.e.

$$
\begin{aligned}
|A_{\mathsf{at}}| &:\equiv A_{\mathsf{at}} \\
|A^{\perp}_{\mathsf{at}}| &:\equiv A^{\perp}_{\mathsf{at}}.
\end{aligned}
$$

Notice that for atomic formulas the tuples of witnesses and challenges are both empty. It is easy to see that $|A^{\perp}|^{\boldsymbol{y}}_{\boldsymbol{x}} \equiv (|A|^{\boldsymbol{x}}_{\boldsymbol{y}})^{\perp}$.

Let us briefly motivate this choice of interpretation. Assume we have already defined the games A and B, i.e. we have adjudication relations $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ and $|B|^{\boldsymbol{v}}_{\boldsymbol{w}}$. Consider, for instance, the adjudication relation for the game $A \mathbin{⅋} B$. In this case, we are giving Eloise a certain advantage, since her move in game A can be depend on Abelard's move in game B, and her move in game B can depend on Abelard's move in game A. The dependence on Eloise's move is formalised by allowing her move in the game $A \mathbin{⅋} B$ to be a pair of functionals $\boldsymbol{f}, \boldsymbol{g}$. The reason for this cross-dependence is that she might not have a winning move for the game A nor for the game $A^{\perp}$, and yet we expect her to easily win the game $A \mathbin{⅋} A^{\perp}$. The cross-dependence allows a pair of simple copy-cat moves ($\boldsymbol{f}, \boldsymbol{g}$ being identity functions) to be her winning move. A symmetric situation occurs in the game $A \otimes B$, only that now Abelard has the advantage and can easily win the game $A \otimes A^{\perp}$, as expected.

Given that $A \multimap B$ is an abbreviation for $A^{\perp} \mathbin{⅋} B$, in particular we have that the adjudication relation for the game $A \multimap B$ is

$$
|A \multimap B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} \;\equiv\; |A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{w}} \multimap |B|^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{w}}.
$$

The game $A \diamond_z B$ is simply a flagged disjoint union of the games A and B. More precisely, the game $A \diamond_z B$ is either the game A or the game B, depending on the boolean flag z. Since the moves in the games A and B might be of different types, we ask the players to make moves in both games, although only one of their moves will be actually used.

[2] We will make use the language of linear logic itself to describe the adjudication relations. If one wishes, a further embedding of linear logic into classical logic would give a semantics for linear logic. Due to the fact that the embedding of linear logic into classical logic is not faithful, however, the semantics will be sound but not complete.

Finally, the quantifier games can be viewed as a family of games parametrised by z^ρ. In the case of the game $\forall z A(z)$ for instance, Abelard chooses which game in the family he wants to play (by choosing z) while Eloise is allowed to make a conditional move (in the form of a functional $\boldsymbol{f}$) which produces her move in the game $A(z)$ for each given z. Again, a symmetric situation occurs in the game $\exists z A(z)$: Eloise chooses one of the games and Abelard has to be prepared for any possible choice of Eloise.

The following theorem formalises the intuition that Eloise's winning move in the game $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ can be extracted from a proof of A in classical linear logic (exponentials treated in Section 3).

Theorem 1 (Soundness). *Let $A_0, \ldots, A_n$ be formulas of pLL^ω, with $\boldsymbol{z}$ as the only free-variables. If*

$$A_0(\boldsymbol{z}), \ldots, A_n(\boldsymbol{z})$$

is provable in pLL^ω then terms $\boldsymbol{a}_0, \ldots, \boldsymbol{a}_n$ can be extracted from this proof such that

$$|A_0(\boldsymbol{z})|^{\boldsymbol{a}_0}_{\boldsymbol{y}_0}, \ldots, |A_n(\boldsymbol{z})|^{\boldsymbol{a}_n}_{\boldsymbol{y}_n}$$

is also provable in $\mathsf{pLL}^\omega_{\mathsf{qf}}$, where $\mathsf{FV}(\boldsymbol{a}_i) \in \{\boldsymbol{z}, \boldsymbol{y}_0, \ldots, \boldsymbol{y}_n\} \backslash \{\boldsymbol{y}_i\}$.

Proof. See [17]. $\square$

Remark 1 (Semantics). Note that the interpretation described above gives rise to a semantics for pure linear logic: Simply replace linear logic connectives by classical connectives in the interpreted formulas:

$$
\begin{array}{ll}
|A \parr B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{y},\boldsymbol{w}} & :\equiv |A|^{\boldsymbol{f}\boldsymbol{w}}_{\boldsymbol{y}} \text{ or } |B|^{\boldsymbol{g}\boldsymbol{y}}_{\boldsymbol{w}} \\
|A \otimes B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{f},\boldsymbol{g}} & :\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{v}} \text{ and } |B|^{\boldsymbol{v}}_{\boldsymbol{g}\boldsymbol{x}} \\
|A \diamond_z B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} & :\equiv \text{if } z \text{ then } |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \text{ else } |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
|\forall z^\rho A|^{\boldsymbol{f}}_{\boldsymbol{y},z} & :\equiv |A|^{\boldsymbol{f}z}_{\boldsymbol{y}} \\
|\exists z^\rho A|^{\boldsymbol{x},z}_{\boldsymbol{f}} & :\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{f}z}.
\end{array}
$$

A formula A is said to be "true" if Eloise has a winning move for the game $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$, for any given assignment of one-move two-player games to atomic formulas.

2.1 Characterisation of Basic Interpretation

In this section we investigate the characterisation of the interpretation given above. More precisely, we ask the question: for which extension of pure linear logic it is the case that if there are terms $\boldsymbol{a}_0, \ldots, \boldsymbol{a}_n$ such that

$$|A_0(\boldsymbol{z})|^{\boldsymbol{a}_0}_{\boldsymbol{y}_0}, \ldots, |A_n(\boldsymbol{z})|^{\boldsymbol{a}_n}_{\boldsymbol{y}_n}$$

is provable then the sequent $A_0, \ldots, A_n$ is also provable? In other words, we have seen how provability in pure linear logic gives rise to a winning move for Eloise. What can we say about the converse? How do we turn a winning move of Eloise for the symmetric game $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ into a proof of A?

We can only answer these questions once we understand precisely how provability of the formula A relates to winning moves for Eloise in the game $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$. Since the provability of A gives a winning move for Eloise in the corresponding game, we would be tempted to think that a formula A is interpreted as the existence of a winning move for Eloise, i.e. that A is equivalent to $\exists \boldsymbol{x} \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$. If that were the case, then $A^{\perp}$ would be equivalent to $\exists \boldsymbol{y} \forall \boldsymbol{x} (|A|^{\boldsymbol{x}}_{\boldsymbol{y}})^{\perp}$, since $|A^{\perp}|^{\boldsymbol{y}}_{\boldsymbol{x}} \equiv (|A|^{\boldsymbol{x}}_{\boldsymbol{y}})^{\perp}$. Hence, the trivial theorem $A \parr A^{\perp}$ would be equivalent to $\exists \boldsymbol{x} \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \parr (\forall \boldsymbol{y} \exists \boldsymbol{x} |A|^{\boldsymbol{x}}_{\boldsymbol{y}})^{\perp}$, which is not always true.

The problem can be solved if we take seriously the fact that a formula A is interpreted as a *symmetric* game $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ between the two players, where the players must make their moves simultaneously. That can be done using a simple form of branching quantifier to ensure that no player has an advantage over the other. Therefore, assume that for all sequences of variables of finite type $\boldsymbol{x}$ and $\boldsymbol{y}$, we can form a new formula $\exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A$, and let us refer to these as *simultaneous quantifiers*. In the same way that a formula $\exists \boldsymbol{x} \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ can be interpreted as a game where Eloise makes a move $\boldsymbol{x}$ and then Abelard chooses his move $\boldsymbol{y}$, the formula $\exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ corresponds to the game where both players choose their moves simultaneously. With the help of this simple branching quantifier we can, for instance, describe the "Odd or Even" game in terms of the formula $\exists\!\!\!\forall^{n}_{m} (n + m$ is even$)$. The simultaneous quantifier can be viewed as a simplification of Henkin's (branching) quantifier [5, 13], in which no alternation of quantifiers is allowed on the two branches[3].

It was shown by Ehrenfeucht (cf. [13]) that Henkin's quantifier can be used to define the quantifier "there exists infinitely many". It follows from a result of Mostowski [15] that the classical predicate calculus extended with Henkin's quantifier is not axiomatisable[4]. Since linear logic does not have a standard truth semantics, it is impossible to talk about a standard axiomatisation of the simultaneous quantifier over pLL^{ω}. Nevertheless, as we will see, the following derivation rule is sound and complete with respect to the game semantics (cf. Remark 1) outlined above:

$$\frac{A_0(\boldsymbol{a}_0, \boldsymbol{y}_0), \ldots, A_n(\boldsymbol{a}_n, \boldsymbol{y}_n)}{\exists\!\!\!\forall^{\boldsymbol{x}_0}_{\boldsymbol{y}_0} A_0(\boldsymbol{x}_0, \boldsymbol{y}_0), \ldots, \exists\!\!\!\forall^{\boldsymbol{x}_n}_{\boldsymbol{y}_n} A_n(\boldsymbol{x}_n, \boldsymbol{y}_n)} \; (\exists\!\!\!\forall)$$

with the side-condition: $\boldsymbol{y}_i$ may only appear free in the terms $\boldsymbol{a}_j$, for $j \neq i$. In particular, we will have that each $\boldsymbol{y}_i$ will not be free in the conclusion of the rule. Note that $\boldsymbol{x}_i, \boldsymbol{y}_i$ are sequences of variables, and $\boldsymbol{a}_i$ are sequences of terms.

The standard quantifier rules can be obtained from this single rule. The rule ($\forall$) can be obtained in the case when only the tuple $\boldsymbol{y}_n$ is non-empty. The rule ($\exists$) can be obtained in the case when only the tuple $\boldsymbol{x}_n$ is non-empty. Hence, for the rest of this section we will consider that standard quantifiers $\forall x A$ and $\exists x A$ are in fact abbreviations for $\exists\!\!\!\forall_x A$ and $\exists\!\!\!\forall^x A$, respectively.

The most interesting characteristic of this simultaneous quantifier is with respect to linear negation, which is defined as

[3] See Bradfield [6] as well, where this simple form of branching quantifier is also used.

[4] Mostowski uses that the ring of integers is not axiomatisable, and that a non-densely ordered ring is isomorphic to the ring of integers if and only if for each positive x there are finitely many elements between 0 and x.

Symbol	System
pLL^ω	Pure linear logic
sLL^ω	Pure linear logic with simultaneous quantifier
sLL^*	$\mathsf{sLL}^\omega + (\mathsf{AC}_s) + (\mathsf{AC}_p)$

Table 3. Systems of linear logic

$$(\exists\!\!\!/\,^{\boldsymbol{x}}_{\boldsymbol{y}} A)^\perp \equiv \exists\!\!\!/\,^{\boldsymbol{y}}_{\boldsymbol{x}} A^\perp$$

and corresponds precisely to the switch of roles between the players. Let us refer to the extension of pLL^ω with the new simultaneous quantifier by sLL^ω.

Theorem 2 ([18]). *Consider the extension to the system* sLL^ω *of the interpretation given in Definition 3, where the simultaneous quantifier is interpreted as*

$$|\exists\!\!\!/\,^{\boldsymbol{v}}_{\boldsymbol{w}} A(\boldsymbol{v}, \boldsymbol{w})|^{\boldsymbol{f}, \boldsymbol{v}}_{\boldsymbol{g}, \boldsymbol{w}} :\equiv |A(\boldsymbol{v}, \boldsymbol{w})|^{\boldsymbol{f}\boldsymbol{w}}_{\boldsymbol{g}\boldsymbol{v}}.$$

Theorem 1 holds for the extended system sLL^ω*, where the verifying system is still* $\mathsf{pLL}^\omega_{\mathsf{qf}}$.

In fact, since the simultaneous quantifiers are eliminated, we obtain an interpretation of sLL^ω into $\mathsf{pLL}^\omega_{\mathsf{qf}}$. In particular, this implies that the rule suggested above is sound. Let us proceed now to define a further extension of sLL^ω which is complete with respect to the interpretation of Section 2. Consider the following principles for the simultaneous quantifier

$$\begin{array}{lcl} \mathsf{AC}_s & : & \forall z \exists\!\!\!/\,^{\boldsymbol{x}}_{\boldsymbol{y}} A_{\mathsf{qf}}(\boldsymbol{x}, \boldsymbol{y}, z) \multimap \exists\!\!\!/\,^{\boldsymbol{f}}_{\boldsymbol{y}, z} A_{\mathsf{qf}}(\boldsymbol{f} z, \boldsymbol{y}, z) \\ \mathsf{AC}_p & : & \exists\!\!\!/\,^{\boldsymbol{x}}_{\boldsymbol{y}} A_{\mathsf{qf}}(\boldsymbol{x}) \parr \exists\!\!\!/\,^{\boldsymbol{v}}_{\boldsymbol{w}} B_{\mathsf{qf}}(\boldsymbol{v}) \multimap \exists\!\!\!/\,^{\boldsymbol{f}, \boldsymbol{g}}_{\boldsymbol{y}, \boldsymbol{w}} (A_{\mathsf{qf}}(\boldsymbol{f}\boldsymbol{w}) \parr B_{\mathsf{qf}}(\boldsymbol{g}\boldsymbol{y})) \end{array}$$

for *quantifier-free* formula A_{qf} and B_{qf}. We refer to these as the *sequential choice* AC_s and *parallel choice* AC_p. For those familiar with the usual functional interpretations of intuitionistic logic, the principle AC_s corresponds to the standard (intentional) axiom of choice, while AC_p is a generalisation of the independence of premise principle (case when tuples $\boldsymbol{y}$ and $\boldsymbol{w}$ are empty). It is an easy exercise to check that the converse of the two implications above can be derived in sLL^ω.

Let us denote by sLL^* the extension of sLL^ω with these two extra schemata AC_s and AC_p. These extra principles are all one needs to show (over sLL^ω) the equivalence between A and its interpretation $\exists\!\!\!/\,^{\boldsymbol{x}}_{\boldsymbol{y}} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$. One then obtains the following characterisation theorem.

Theorem 3. *Let A be a formula in the language of* sLL^ω*. Then A is derivable in* sLL^* *if and only if $|A|^{\boldsymbol{t}}_{\boldsymbol{y}}$ is derivable in* $\mathsf{pLL}^\omega_{\mathsf{qf}}$*, for some sequence of terms $\boldsymbol{t}$.*

3 Interpretations of the Exponentials

The exponential-free fragment of LL^ω, despite its nice properties, bears little relation to the standard logical systems of classical and intuitionistic logic. In order to recover the full strength of classical logic, we need to add back contraction and weakening. These are recovered in linear logic in a controlled manner, with the help of modalities (exponentials) $?A$ and $!A$. The exponentials are dual to each other, i.e.

$$(?A)^\perp \equiv\, !A^\perp \qquad\qquad (!A)^\perp \equiv\, ?A^\perp.$$

Girard points out in several places (cf. [11] (p84)) that these modalities, contrary to the other connectives, are not canonical. More precisely, if we add new modalities $?'A$ and $!'A$ with the same rules as shown in Table 1, we will not be able to derive the equivalences $?A \leftrightarrow\, ?'A$ and $!A \leftrightarrow\, !'A$. This is reflected in the flexibility (discussed below) with which we can interpret these modalities.

In terms of games, we will see that the exponentials correspond to a break of symmetry between the two players, allowing one player to see the opponent's move before making his/her move. Besides the advantage of allowing one of the players to play second, the exponential can be interpreted in such a way that the favoured player can play a *set of moves*, rather than a single move. If $\boldsymbol{\rho}$ is the type of the move in question, let us write $\boldsymbol{\rho}^*$ for the type corresponding to the sets of moves of type $\boldsymbol{\rho}$ (i.e. $\boldsymbol{\rho}^* \subseteq \mathcal{P}(\boldsymbol{\rho})$). The choice of how big we allow that set of moves to be determines the interpretation. As we will see, for instance, we can choose the set of moves to be the whole type ($\boldsymbol{\rho}^* \equiv \{\boldsymbol{\rho}\}$), finite sets ($\boldsymbol{\rho}^* \equiv \mathcal{P}_{\mathsf{fin}}(\boldsymbol{\rho})$), sets with common majorant (see Section 3.5), or singleton sets ($\boldsymbol{\rho}^* \cong \boldsymbol{\rho}$).

Let us start by analysing which are the allowed sets of subsets $\boldsymbol{\rho}^*$ that give rise to proper interpretations (cf. [16]). We do that by considering an abstract interpretation where the choice of sets is left open, and only certain conditions are put on these sets. In the remaining subsections we will look at particular choices which are related to well-known functional interpretations of intuitionistic logic (see Figure 1). For each finite type $\boldsymbol{\rho}$ let $\boldsymbol{\rho}^*$ be a new abstract type. Moreover, for each formula A, let $\forall \boldsymbol{x}^{\boldsymbol{\rho}} \sqsubset \boldsymbol{a}^{\boldsymbol{\rho}^*} A$ and $\exists \boldsymbol{x}^{\boldsymbol{\rho}} \sqsubset \boldsymbol{a}^{\boldsymbol{\rho}^*} A$ be formula abbreviations such that

$$(\exists \boldsymbol{x} \sqsubset \boldsymbol{a}\, A)^\perp \equiv \forall \boldsymbol{x} \sqsubset \boldsymbol{a}\, A^\perp \qquad\qquad (\forall \boldsymbol{x} \sqsubset \boldsymbol{a}\, A)^\perp \equiv \exists \boldsymbol{x} \sqsubset \boldsymbol{a}\, A^\perp.$$

A formula A is called $\sqsubset$-fixed if it does not contain unbounded quantifiers and all bounded quantifiers $\forall \boldsymbol{x} \sqsubset \boldsymbol{a}\, A$ and $\exists \boldsymbol{x} \sqsubset \boldsymbol{a}\, A$ are immediately preceded by a ! and ?, respectively. For each $\sqsubset$-fixed formula A, assume we have sequence of terms $\boldsymbol{\epsilon}, \boldsymbol{\eta}$ and $\boldsymbol{\mu}$ such that the following sequents are derivable:

(D) $!\forall \boldsymbol{y} \sqsubset \boldsymbol{\eta x}\, A(\boldsymbol{y}) \vdash A(\boldsymbol{x})$

(C) $!\forall \boldsymbol{y} \sqsubset \boldsymbol{\epsilon y}_0 \boldsymbol{y}_1\, A(\boldsymbol{y}) \vdash\, !\forall \boldsymbol{y} \sqsubset \boldsymbol{y}_i\, A(\boldsymbol{y}) \qquad (i \in \{0, 1\})$

(P) $!\forall \boldsymbol{y} \sqsubset \boldsymbol{\mu h w}\, A(\boldsymbol{y}) \vdash\, !\forall \boldsymbol{x} \sqsubset \boldsymbol{w}\; !\forall \boldsymbol{y} \sqsubset \boldsymbol{h x}\, A(\boldsymbol{y})$

The provability sign in the conditions stands for provability in the system under which the functional interpretation will be verified, which might be an extension of LL^ω.

$$\begin{tikzcd}
\mathsf{IL}^\omega \arrow[r, "\text{derived inter.}"'] \arrow[d, "(\cdot)^l"'] & \mathsf{IL}^\omega \\
\mathsf{LL}^\omega \arrow[r, "|\cdot|"] & \mathsf{LL}^\omega \arrow[u, "(\cdot)^i"']
\end{tikzcd}$$

Fig. 1. Deriving functional interpretations of intuitionistic logic

Intuitively, the first condition says that for any type $\boldsymbol{\rho}$ we must have an injection $\boldsymbol{\eta}$ into $\boldsymbol{\rho}^*$. Condition (C) says that for any two sets in $\boldsymbol{\rho}^*$ there is a bigger set which includes both. Finally, condition (P) corresponds to a collection principle which says that any collection of sets $\boldsymbol{hx}$ parametrised by a bounded $\boldsymbol{x} \sqsubset \boldsymbol{w}$ can be uniformly bounded.

We can then show that, for any formula abbreviation satisfying conditions ($\mathsf{D}, \mathsf{C}, \mathsf{P}$), a functional interpretation of classical linear logic can be obtained by defining the interpretation of the exponentials as

$$|!A|^{\boldsymbol{x}}_{\boldsymbol{f}} :\equiv !\forall \boldsymbol{y} \sqsubset \boldsymbol{fx}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$$
$$|?A|^{\boldsymbol{f}}_{\boldsymbol{y}} :\equiv ?\exists \boldsymbol{x} \sqsubset \boldsymbol{fy}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.$$

The conditions (D, P) are used to ensure the soundness of the dereliction and promotion rules, respectively. The condition (C) is used for the soundness of the contraction rule. The rule of weakening only needs that $\boldsymbol{\rho}^*$ is not empty, so that that $\forall \boldsymbol{x} \sqsubset \boldsymbol{a}\, A$ is a proper formula for some $\boldsymbol{a}$.

3.1 Interpretation 1: Kreisel's Modified Realizability

The first alternative for the interpretation of the exponentials we consider is one in which the game $?A$ gives maximal advantage to Eloise, and game $!A$ gives maximal advantage to Abelard. The maximal advantage corresponds to the player in question not needing to make any move, with their best possible move being played for them. This corresponds to allowing the move of the player to be the whole set of possible moves. More precisely, the interpretation is defined as:

Definition 4. *Extend the interpretation given in Definition 3 as*

$$|!A|^{\boldsymbol{x}} :\equiv !\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$$
$$|?A|_{\boldsymbol{y}} :\equiv ?\exists \boldsymbol{x} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.$$

It is easy to see that Theorem 1 still holds when Definition 3 is extended in this way. Note, however, that once exponentials are treated as in Definition 4, the relation $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ is no longer quantifier-free. Nevertheless, it is the case that formulas in the image of the interpretation (we call these *fixed formulas*) are also in the kernel of the interpretation.

More, precisely, if A is in the kernel of the interpretation then $|A| \equiv A$. The completeness result of Section 2.1 needs to be calibrated, as the schemata AC_s and AC_p need to be taken for all fixed-formulas (and not just quantifier-free formulas). Moreover, we need an extra principle

$$\mathsf{TA} \quad : \quad !\exists\!\!\!/^{\boldsymbol{x}}_{\boldsymbol{y}} A \multimap \exists \boldsymbol{x} ! \forall \boldsymbol{y} A$$

called *trump advantage*, for fixed-formulas[5] A, in order to obtain the equivalences involving exponentials, i.e. equivalence between $!A$ and its interpretation $\exists \boldsymbol{x} ! \forall \boldsymbol{y} A$. The principle TA in particular implies that the modality ! commutes with the existential quantifier, i.e. $!\exists x A \leftrightarrow \exists x ! A$.

We have shown [17] that when combined with the embedding of intuitionistic logic into linear logic, this choice for the interpretation of the exponentials corresponds to Kreisel's modified realizability interpretation [14] of intuitionistic logic. This result is rephrased in the theorem below. For an introduction to modified realizability see chapter III of [22] or the book chapter [23].

Theorem 4 (Kreisel's modified realizability). *Let $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ be as in Definition 3 and 4. For formulas A of intuitionistic logic let us define*

$$\boldsymbol{x} \,\mathsf{mr}\, A \quad :\equiv \quad (|!A^l|^{\boldsymbol{x}})^i,$$

where $(\cdot)^l$ and $(\cdot)^i$ denote the embeddings described in Section 1.2. The following equivalences hold intuitionistically:

$$\begin{array}{rcccl}
\boldsymbol{x}, \boldsymbol{v} & \mathsf{mr} & (A \wedge B) & \Leftrightarrow & (\boldsymbol{x} \,\mathsf{mr}\, A) \wedge (\boldsymbol{v} \,\mathsf{mr}\, B) \\
\boldsymbol{x}, \boldsymbol{v}, z & \mathsf{mr} & (A \vee B) & \Leftrightarrow & (\boldsymbol{x} \,\mathsf{mr}\, A) \diamond_z (\boldsymbol{v} \,\mathsf{mr}\, B) \\
\boldsymbol{f} & \mathsf{mr} & (A \to B) & \Leftrightarrow & \forall \boldsymbol{x}((\boldsymbol{x} \,\mathsf{mr}\, A) \to (\boldsymbol{f}\boldsymbol{x} \,\mathsf{mr}\, B)) \\
\boldsymbol{x}, z & \mathsf{mr} & \exists z A & \Leftrightarrow & \boldsymbol{x} \,\mathsf{mr}\, A \\
\boldsymbol{f} & \mathsf{mr} & \forall z A & \Leftrightarrow & \forall z(\boldsymbol{f} z \,\mathsf{mr}\, A).
\end{array}$$

3.2 Interpretation 2: Gödel's Dialectica Interpretation

The most restricted interpretation we consider is the one where the favoured player has to choose a singleton set in the exponential game. Therefore, the only head-start will be to be able to see the opponents move. Based on the opponent's move the player will then have to make a single move. This leads to an extension of the interpretation given in Definition 3 with the interpretation of the exponentials as:

Definition 5. *Extend Definition 3 as*

$$|!A|^{\boldsymbol{x}}_{\boldsymbol{f}} :\equiv \, !|A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{x}}$$
$$|?A|^{\boldsymbol{f}}_{\boldsymbol{y}} :\equiv \, ?|A|^{\boldsymbol{f}\boldsymbol{y}}_{\boldsymbol{y}}$$

[5] Although we only need the principle TA for fixed-formulas in order to obtain the characterisation, modified realizability actually interprets this principle for arbitrary formulas A.

where we are identifying the singleton sets (i.e. $\boldsymbol{f}\boldsymbol{x}$ and $\boldsymbol{f}\boldsymbol{y}$) with their unique element.

Note that in this case the target of the interpretation is again a quantifier-free calculus (as in the basic interpretation of Section 2). For the soundness, however, we must assume that quantifier-free formulas are decidable in order to satisfy the contraction rule, since we must choose one among two singleton sets of potential witnesses. The soundness of the weakening rule, and the rules (!) and (?) is trivial. Besides being sound for the principles $\mathsf{AC}_s, \mathsf{AC}_p$ (Section 2.1) and the principle TA (Section 3.1) the Dialectica interpretation of LL^ω will also be sound for the following principle

$$\mathsf{MP}_D \quad : \quad \forall \boldsymbol{x} ! A \multimap ! \forall \boldsymbol{x} A$$

for quantifier-free formulas A. This is the linear logic counterpart of the semi intuitionistic Markov principle. In fact, these are all the extra principles needed to show the equivalence between A and its Dialectica interpretation $\exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ (see [17, 18]).

This interpretation corresponds to Gödel's Dialectica interpretation [1, 12] of intuitionistic logic, used in connection to a partial realisation of Hilbert's consistency program: the consistency of classical first-order arithmetic relative to the consistency of the quantifier-free calculus T. This correspondence is formalised in the following theorem.

Theorem 5 (Gödel's Dialectica interpretation). *Let $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ be as in Definition 3 and 5. For formulas A of intuitionistic logic let us define*

$$A_d(\boldsymbol{x}; \boldsymbol{y}) \quad :\equiv \quad (|A^l|^{\boldsymbol{x}}_{\boldsymbol{y}})^i.$$

The following equivalences hold intuitionistically:

$$\begin{array}{lcl}
(A \wedge B)_d(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{y}, \boldsymbol{w}, z) & \Leftrightarrow & A_d(\boldsymbol{x}z; \boldsymbol{y}) \diamond_z B_d(\boldsymbol{v}z; \boldsymbol{w}) \\
(A \vee B)_d(\boldsymbol{x}, \boldsymbol{v}, z; \boldsymbol{f}, \boldsymbol{g}) & \Leftrightarrow & A_d(\boldsymbol{x}; \boldsymbol{f}\boldsymbol{x}\boldsymbol{v}z) \diamond_z B_d(\boldsymbol{v}; \boldsymbol{g}\boldsymbol{x}\boldsymbol{v}z) \\
(A \to B)_d(\boldsymbol{f}, \boldsymbol{g}; \boldsymbol{x}, \boldsymbol{w}) & \Leftrightarrow & A_d(\boldsymbol{x}; \boldsymbol{g}\boldsymbol{x}\boldsymbol{w}) \to B_d(\boldsymbol{f}\boldsymbol{x}, \boldsymbol{w}) \\
(\forall z A)_d(\boldsymbol{f}; \boldsymbol{y}, z) & \Leftrightarrow & A_d(\boldsymbol{f}z; \boldsymbol{y}) \\
(\exists z A)_d(\boldsymbol{x}, z; \boldsymbol{f}) & \Leftrightarrow & A_d(\boldsymbol{x}; \boldsymbol{f}\boldsymbol{x}z).
\end{array}$$

This is not exactly how Gödel defined his Dialectica interpretation [12], but it is equivalent. In the case of conjunction and disjunction, the extra boolean information z given to the functionals is irrelevant, since each functional will only be applied when the boolean is either true or false. The equivalence between the two different interpretations of disjunction and existential quantifiers is discussed in the following theorem.

Theorem 6. *Let A_D be the interpretation of A as in Gödel's original definition of the Dialectica interpretation. Then $A^D \equiv \exists \boldsymbol{v} \forall \boldsymbol{w} A_D(\boldsymbol{v}; \boldsymbol{w})$ is intuitionistically equivalent to $\exists \boldsymbol{x} \forall \boldsymbol{y} A_d(\boldsymbol{x}; \boldsymbol{y})$.*

Proof. Using the characterisation principles for the Dialectica interpretation of intuitionistic logic we can show that

$$\exists \boldsymbol{v} \forall \boldsymbol{w} A_D(\boldsymbol{v}; \boldsymbol{w}) \leftrightarrow A \leftrightarrow \exists \boldsymbol{x} \forall \boldsymbol{y} A_d(\boldsymbol{x}; \boldsymbol{y}).$$

We first look at the proof of the implication

$(*)\ \exists \boldsymbol{v} \forall \boldsymbol{w} A_D(\boldsymbol{v}; \boldsymbol{w}) \to \exists \boldsymbol{x} \forall \boldsymbol{y} A_d(\boldsymbol{x}; \boldsymbol{y})$.

By the Dialectica interpretation we can eliminate the characterisation principles from this proof and at the same time produce terms $\boldsymbol{t}, \boldsymbol{s}$ such that

$$\forall \boldsymbol{v}, \boldsymbol{y}(A_D(\boldsymbol{v}; \boldsymbol{s}\boldsymbol{v}\boldsymbol{y}) \to A_d(\boldsymbol{t}\boldsymbol{v}; \boldsymbol{y}))$$

is derivable in pure intuitionistic logic. But this in particular implies that $(*)$ is provable in pure intuitionistic logic. The implication from right to left can be proved similarly. $\Box$

3.3 Interpretation 3: Diller-Nahm Interpretation

We have seen two extreme interpretations of the exponential games. One in which the favoured player can try any of his possible moves (Section 3.1) and the other where he/she chooses a single move (Section 3.2). Another possibility for the interpretation is to give the player in question a restricted advantage by allowing the player to see the opponent's move and then select a *non-empty finite set of moves*[6]. If any of these is a good move the player wins. This leads to the following interpretation of the exponentials:

Definition 6. *Extend Definition 3 as*

$$|!A|^{\boldsymbol{x}}_{\boldsymbol{f}} :\equiv\ !\forall \boldsymbol{y} \in \boldsymbol{f}\boldsymbol{x}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$$
$$|?A|^{\boldsymbol{f}}_{\boldsymbol{y}} :\equiv\ ?\exists \boldsymbol{x} \in \boldsymbol{f}\boldsymbol{y}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$$

where $\boldsymbol{f}\boldsymbol{x}$ *and* $\boldsymbol{f}\boldsymbol{y}$ *are finite sets.*

Again, this extension of Definition 3 makes the Soundness Theorem 1 valid for full classical linear logic. It is clear that in this case enough term construction needs to be added to the verifying system in order to deal with finite sets of arbitrary type. This choice for the treatment of the exponentials corresponds to a variant of Gödel's Dialectica interpretation due to Diller and Nahm [7], as formalised in the following theorem.

Theorem 7 (Diller-Nahm interpretation). *Let* $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ *be as in Definition 3 and 6. For formulas* A *of intuitionistic logic let us define*

$$A_{dn}(\boldsymbol{x}; \boldsymbol{y}) \quad :\equiv \quad (|A^l|^{\boldsymbol{x}}_{\boldsymbol{y}})^i.$$

The following equivalences hold intuitionistically:

$$\begin{array}{lcl}
(A \wedge B)_{dn}(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{y}, \boldsymbol{w}, z) & \Leftrightarrow & A_{dn}(\boldsymbol{x}z; \boldsymbol{y}) \diamond_z B_{dn}(\boldsymbol{v}z; \boldsymbol{w}) \\
(A \vee B)_{dn}(\boldsymbol{x}, \boldsymbol{v}, z; \boldsymbol{f}, \boldsymbol{g}) & \Leftrightarrow & \forall \boldsymbol{y} \in \boldsymbol{f}\boldsymbol{x}\boldsymbol{v}z\, A_{dn}(\boldsymbol{x}; \boldsymbol{y}) \diamond_z \forall \boldsymbol{w} \in \boldsymbol{g}\boldsymbol{x}\boldsymbol{v}z\, B_{dn}(\boldsymbol{v}; \boldsymbol{w}) \\
(A \to B)_{dn}(\boldsymbol{f}, \boldsymbol{g}; \boldsymbol{x}, \boldsymbol{w}) & \Leftrightarrow & \forall \boldsymbol{y} \in \boldsymbol{g}\boldsymbol{x}\boldsymbol{w}\, A_{dn}(\boldsymbol{x}; \boldsymbol{y}) \to B_{dn}(\boldsymbol{f}\boldsymbol{x}, \boldsymbol{w}) \\
(\forall z A)_{dn}(\boldsymbol{f}; \boldsymbol{y}, z) & \Leftrightarrow & A_{dn}(\boldsymbol{f}z; \boldsymbol{y}) \\
(\exists z A)_{dn}(\boldsymbol{x}, z; \boldsymbol{f}) & \Leftrightarrow & \forall \boldsymbol{y} \in \boldsymbol{f}\boldsymbol{x}z\, A_{dn}(\boldsymbol{x}; \boldsymbol{y}).
\end{array}$$

[6] Allowing the set of moves to be empty corresponds to allowing the player to choose to "surrender" (see [3]).

As in the case of Gödel's Dialectica, the interpretation derived above is intuitionistically (but not syntactically) equivalent to the one introduced by Diller-Nahm [7].

3.4 Interpretation 4: Stein's Interpretation

In Interpretation 1 we considered the case where the favoured player can choose the set of all possible moves, whereas in Interpretation 3 only finite sets were allowed. The whole set $\boldsymbol{\rho}$ can be identified with the identity map of type $\boldsymbol{\rho} \to \boldsymbol{\rho}$, while finite sets of elements of type $\boldsymbol{\rho}$ can be viewed as partial functions $\mathbb{N} \to \boldsymbol{\rho}$ with a finite support. In both cases we have a family of objects of type $\boldsymbol{\rho}$ where the indexing set is either $\boldsymbol{\rho}$ or a finite subset of $\mathbb{N}$.

A hybrid interpretation between these two options can also be given for each natural number n, where n controls the type level from which we should use option 3 (Diller-Nahm), and up to which level we should choose option 1 (modified realizability). In other words, the kind of subsets of $\boldsymbol{\rho}$ we allow are those which can be indexed by the pure type n, i.e. elements of $n \to \boldsymbol{\rho}$. Note that if the type level of $\boldsymbol{\rho}$ is less or equal to n then the whole set $\boldsymbol{\rho}$ is also an allowed move. Only when the type level of $\boldsymbol{\rho}$ is bigger than n we have a restricted move for the favoured player. The slogan is "only higher-type objects are witnessed".

Given a tuple of variables $\boldsymbol{x}$, we will denote by $\underline{\boldsymbol{x}}$ the sub-tuple containing the variables in $\boldsymbol{x}$ which have type level $\geq n$, whereas $\overline{\boldsymbol{x}}$ denotes the sub-tuple of the variables in $\boldsymbol{x}$ which have type level $< n$. In the following we identify $n \in \mathbb{N}$ with the pure type of type level n. Let us write $\forall \boldsymbol{y} \in \mathsf{rng}(\boldsymbol{b}^{n \to \boldsymbol{\rho}}) A[\boldsymbol{y}]$ and $\exists \boldsymbol{y} \in \mathsf{rng}(\boldsymbol{b}^{n \to \boldsymbol{\rho}}) A[\boldsymbol{y}]$ as abbreviations for $\forall i^n A[\boldsymbol{b}i]$ and $\exists i^n A[\boldsymbol{b}i]$, respectively.

Definition 7. *For any fixed pure type n, extend Definition 3 as*

$$|!A|^{\boldsymbol{x}}_{\boldsymbol{f}} :\equiv\ !\forall \underline{\boldsymbol{y}} \in \mathsf{rng}(\boldsymbol{f}\boldsymbol{x}) \forall \overline{\boldsymbol{y}}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$$
$$|?A|^{\boldsymbol{f}}_{\boldsymbol{y}} :\equiv\ ?\exists \underline{\boldsymbol{x}} \in \mathsf{rng}(\boldsymbol{f}\boldsymbol{y}) \exists \overline{\boldsymbol{x}}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.$$

Note that if $n = 0$ the interpretation above corresponds to Definition 6 where "finite sets" are replaced by "countable sets", whereas in the limit ($n = \infty$) this coincides with that given in Definition 4. The interpretation of Definition 7 corresponds to Stein's interpretation [21], and again leads to a sound interpretation of full classical linear logic.

Theorem 8 (Stein's interpretation). *Let $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ be as in Definition 3 and 7. For formulas A of intuitionistic logic let us define*

$$A_s(\boldsymbol{x}; \boldsymbol{y}) \ :\equiv\ (|A^l|^{\boldsymbol{x}}_{\boldsymbol{y}})^i.$$

The following equivalences hold intuitionistically:

$$\begin{array}{lcl}
(A \wedge B)_s(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{y}, \boldsymbol{w}, z) & \Leftrightarrow & A_s(\boldsymbol{x}z; \boldsymbol{y}) \diamond_z B_s(\boldsymbol{v}z; \boldsymbol{w}) \\
(A \vee B)_s(\boldsymbol{x}, \boldsymbol{v}, z; \boldsymbol{f}, \boldsymbol{g}) & \Leftrightarrow & \forall \underline{\boldsymbol{y}} \in \mathsf{rng}(\boldsymbol{f}\boldsymbol{x}\boldsymbol{v}z) \forall \overline{\boldsymbol{y}} \forall \underline{\boldsymbol{w}} \in \mathsf{rng}(\boldsymbol{g}\boldsymbol{x}\boldsymbol{v}z) \forall \overline{\boldsymbol{w}} \\
 & & \quad (A_s(\boldsymbol{x}; \boldsymbol{y}) \diamond_z B_s(\boldsymbol{v}; \boldsymbol{w})) \\
(A \to B)_s(\boldsymbol{f}, \boldsymbol{g}; \boldsymbol{x}, \boldsymbol{w}) & \Leftrightarrow & \forall \underline{\boldsymbol{y}} \in \mathsf{rng}(\boldsymbol{g}\boldsymbol{x}\boldsymbol{w}) \forall \overline{\boldsymbol{y}}\, A_s(\boldsymbol{x}; \boldsymbol{y}) \to B_s(\boldsymbol{f}\boldsymbol{x}, \boldsymbol{w}) \\
(\forall z A)_s(\boldsymbol{f}; \boldsymbol{y}, z) & \Leftrightarrow & A_s(\boldsymbol{f}z; \boldsymbol{y}) \\
(\exists z A)_s(\boldsymbol{x}, z; \boldsymbol{f}) & \Leftrightarrow & \forall \underline{\boldsymbol{y}} \in \mathsf{rng}(\boldsymbol{f}\boldsymbol{x}z) \forall \overline{\boldsymbol{y}}\, A_s(\boldsymbol{x}; \boldsymbol{y}).
\end{array}$$

3.5 Interpretation 5: Bounded Functional Interpretation

The interpretations presented in the previous sections are straightforward instantiations of the parametrised interpretation, in the sense that the choice of $\boldsymbol{\rho}^*$ suggests natural terms $\boldsymbol{\eta}, \boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ satisfying conditions $(\mathsf{D}, \mathsf{C}, \mathsf{P})$. Consider, however, choosing ρ^* as the subsets of ρ which are defined by a majorant (in the sense of Bezem [2]), i.e. let

$$\rho^* :\equiv \{\{x \ : \ x \leq^* y\} \ : \ y^\rho \text{ monotone}\},$$

where $x \leq^* y$ denotes Howard-Bezem's strong majorizabilty relation between functionals (cf. [2]), and y being monotone (self-majorizable) is defined as $y \leq^* y$. Let us abbreviate quantifications over monotone objects as $\tilde{\forall} a A$ and $\tilde{\exists} a A$ (cf. [8, 9]).

Unfortunately, there are a couple of problems with this choice of ρ^*. First of all, in general, we do not have functionals $\boldsymbol{\eta}$ producing for each $\boldsymbol{x}$ a functional which majorizes $\boldsymbol{x}$. We get into similar problems when trying to satisfy conditions (C) and (P). A possible solution is to ensure that all functionals involved are monotone, since in this case conditions $(\mathsf{D}, \mathsf{C}, \mathsf{P})$ can be easily satisfied: take $\boldsymbol{\eta}, \boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ to be the identity functional, the pointwise maximum and functional application, respectively. We can enforce that all objects we are dealing with are monotone by working with "majorants of witnesses" instead of "actual witnesses". This involves changing the interpretation of the quantifiers as:

$$|\exists z^\rho A|^{\boldsymbol{x}, a}_{\boldsymbol{f}} :\equiv \exists z \leq^* a \, |A|^{\boldsymbol{x}}_{\boldsymbol{f} a}$$
$$|\forall z^\rho A|^{\boldsymbol{f}}_{\boldsymbol{y}, a} :\equiv \forall z \leq^* a \, |A|^{\boldsymbol{f} a}_{\boldsymbol{y}}.$$

In this way, we can restrict quantifications over a to monotone quantifications, without losing generality. If we also add to the language bounded quantifiers, and interpret them as[7]

$$|\exists z^\rho \leq^* t \, A|^{\boldsymbol{x}}_{\boldsymbol{y}} :\equiv \exists z \leq^* t \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$$
$$|\forall z^\rho \leq^* t \, A|^{\boldsymbol{x}}_{\boldsymbol{y}} :\equiv \forall z \leq^* t \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$$

the interpretation of quantifiers above corresponds to a combination of the standard interpretation (Section 2) with a *prior relativisation* of the quantifiers to Bezem's model $\mathcal{M}$ of strongly majorizable functionals [2]. More precisely: For each formula A of LL^ω let $[A]$ be obtained inductively as

$$\begin{array}{lcl}
[A_{\mathsf{at}}] & :\equiv & A_{\mathsf{at}}, \quad \text{for atomic formulas} \\
[A \star B] & :\equiv & [A] \star [B], \quad \text{for } \star \in \{ \otimes, \bindnasrepma, \Diamond_z \} \\
[\star A] & :\equiv & \star [A], \quad \text{for } \star \in \{!, ?\} \\
[\forall x A(x)] & :\equiv & \tilde{\forall} b \forall x \leq^* b \, [A(x)] \\
[\exists x A(x)] & :\equiv & \tilde{\exists} b \exists x \leq^* b \, [A(x)].
\end{array}$$

[7] We are assuming that the majorizability relation and bounded quantifiers have been added to the language as in [8].

The formula $[A]$ can be viewed as a relativisation of the quantifiers in A to the model $\mathcal{M}$, since $\tilde{\forall} b \, \forall x \leq^* b \, A(x)$, for instance, is equivalent to $\forall x (\tilde{\exists} b (x \leq^* b) \to A(x))$.

Although this solves the problem of interpreting the modalities, we have now changed the way free-variables are dealt with (through the different treatment of quantifiers). This leads to a second problem, namely the interpretation of the additive connectives, since the interpretation presented in Section 2 relies on the if-then-else term constructor $(z)(t_0, t_1)$, which is not monotone[8] on the argument z.

In summary, this fifth interpretation of the modalities is sound given a prior relativisation of quantifiers to the model of strongly majorizable functionals. This relativisation, however, conflicts with the interpretation of the additives given above. We conclude this section with two results. First, we show that the instantiation ρ^* above is still sound for multiplicative-exponential linear logic (MELL^ω) plus bounded quantifiers, when the interpretation is combined with the relativisation $[\cdot]$. Second, we show that the (unsound) formula interpretation of the additives in the linear logic context still corresponds to (an equivalent formulation of) the bounded functional interpretation of intuitionistic logic.

Theorem 9. *Extend Definition 3 as*

$$
\begin{array}{ll}
|!A|^{\boldsymbol{x}}_{\boldsymbol{f}} & :\equiv \, !\tilde{\forall} \boldsymbol{y} \leq^* \boldsymbol{f}\boldsymbol{x} \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \\
|?A|^{\boldsymbol{f}}_{\boldsymbol{y}} & :\equiv \, ?\tilde{\exists} \boldsymbol{x} \leq^* \boldsymbol{f}\boldsymbol{y} \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \\
|\exists z^\rho \leq^* t \, A|^{\boldsymbol{x}}_{\boldsymbol{y}} & :\equiv \exists z \leq^* t \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \\
|\forall z^\rho \leq^* t \, A|^{\boldsymbol{x}}_{\boldsymbol{y}} & :\equiv \forall z \leq^* t \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}
\end{array}
$$

and drop the if-then-else constructor (i.e. the additives). If

$$A_0(\boldsymbol{z}), \ldots, A_n(\boldsymbol{z})$$

is provable in MELL^ω *then from this proof* monotone *terms* $\boldsymbol{a}_0, \ldots, \boldsymbol{a}_n$ *can be extracted such that*

$$\boldsymbol{z} \leq^* \boldsymbol{z}^*, \boldsymbol{y}_i \leq^* \boldsymbol{y}_i \vdash |[A_0(\boldsymbol{z})]|^{\boldsymbol{a}_0}_{\boldsymbol{y}_0}, \ldots, |[A_n(\boldsymbol{z})]|^{\boldsymbol{a}_n}_{\boldsymbol{y}_n}$$

is also provable in MELL^ω*, where* $\mathsf{FV}(\boldsymbol{a}_i) \in \{\boldsymbol{z}^*, \boldsymbol{y}_0, \ldots, \boldsymbol{y}_n\} \backslash \{\boldsymbol{y}_i\}$.

Proof. We consider only the case of existential quantifier

$$
\dfrac{\dfrac{\dfrac{\dfrac{\boldsymbol{y} \leq^* \boldsymbol{y}, \boldsymbol{z} \leq^* \boldsymbol{z}^* \vdash |\Gamma|^{\gamma[\boldsymbol{y}]}_{\boldsymbol{w}}, |[A(t[\boldsymbol{z}])]|^{\boldsymbol{a}}_{\boldsymbol{y}} \qquad \boldsymbol{z} \leq^* \boldsymbol{z}^* \vdash t[\boldsymbol{z}] \leq^* t^*[\boldsymbol{z}^*]}{\boldsymbol{y} \leq^* \boldsymbol{y}, \boldsymbol{z} \leq^* \boldsymbol{z}^* \vdash |\Gamma|^{\gamma[\boldsymbol{y}]}_{\boldsymbol{w}}, \exists x \leq^* t^*[\boldsymbol{z}^*] |[A(x)]|^{\boldsymbol{a}}_{\boldsymbol{y}}}}{\boldsymbol{f}(t^*[\boldsymbol{z}^*]) \leq^* \boldsymbol{f}(t^*[\boldsymbol{z}^*]), \boldsymbol{z} \leq^* \boldsymbol{z}^* \vdash |\Gamma|^{\gamma[\boldsymbol{f}(t^*[\boldsymbol{z}^*])]}_{\boldsymbol{w}}, \exists x \leq^* t^*[\boldsymbol{z}^*] |[A(x)]|^{\boldsymbol{a}}_{\boldsymbol{f}(t^*[\boldsymbol{z}^*])}}}{\boldsymbol{z} \leq^* \boldsymbol{z}^*, \boldsymbol{f} \leq^* \boldsymbol{f} \vdash |\Gamma|^{\gamma[\boldsymbol{f}(t^*[\boldsymbol{z}^*])]}_{\boldsymbol{w}}, \exists x \leq^* t^*[\boldsymbol{z}^*] |[A(x)]|^{\boldsymbol{a}}_{\boldsymbol{f}(t^*[\boldsymbol{z}^*])}}}{\boldsymbol{z} \leq^* \boldsymbol{z}^*, \boldsymbol{f} \leq^* \boldsymbol{f} \vdash |\Gamma|^{\gamma[\boldsymbol{f}(t^*[\boldsymbol{z}^*])]}_{\boldsymbol{w}}, |[\exists x A(x)]|^{\boldsymbol{a}, t^*[\boldsymbol{z}^*]}_{\boldsymbol{f}}}
$$

[8] Although $(z)(t_0, t_1)$ is not monotone, it can be easily majorized by the pointwise-maximum functional $\max\{t_0, t_1\}$. This is the solution used in the context of intuitionistic logic, where the interpreted formulas are monotone on the witnessing variable. In linear logic this monotonicity property does not hold.

The other cases are treated similarly. $\Box$

Next, we show that by keeping the additives in the bounded functional interpretation of linear logic, through the relativisation of quantifiers, we obtain an equivalent formulation of the bounded functional interpretation of intuitionistic logic.

Theorem 10 (Bounded functional interpretation). *Let $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ be as in Definition 3 and Theorem 9 (keeping the additives and viewing boolean quantifications as bounded quantifications). For formulas A of intuitionistic logic let us define*

$$A_B(\boldsymbol{x};\boldsymbol{y}) \;:\equiv\; (|[A^l]|^{\boldsymbol{x}}_{\boldsymbol{y}})^i.$$

The following equivalences hold intuitionistically:

$$\begin{array}{lcl}
(A \wedge B)_B(\boldsymbol{x},\boldsymbol{v};\boldsymbol{y},\boldsymbol{w}) & \Leftrightarrow & A_B(\boldsymbol{x};\boldsymbol{y}) \wedge B_B(\boldsymbol{v};\boldsymbol{w}) \\
(A \vee B)_B(\boldsymbol{x},\boldsymbol{v};\boldsymbol{f},\boldsymbol{g}) & \Leftrightarrow & \tilde{\forall}\boldsymbol{y} \leq^* \boldsymbol{f}\boldsymbol{x}\boldsymbol{v} A_B(\boldsymbol{x};\boldsymbol{y}) \vee \tilde{\forall}\boldsymbol{w} \leq^* \boldsymbol{g}\boldsymbol{x}\boldsymbol{v} B_B(\boldsymbol{v};\boldsymbol{w}) \\
(A \to B)_B(\boldsymbol{f},\boldsymbol{g};\boldsymbol{x},\boldsymbol{w}) & \Leftrightarrow & \tilde{\forall}\boldsymbol{y} \leq^* \boldsymbol{g}\boldsymbol{x}\boldsymbol{w}\, A_B(\boldsymbol{x};\boldsymbol{y}) \to B_B(\boldsymbol{f}\boldsymbol{x},\boldsymbol{w}) \\
(\forall z A)_B(\boldsymbol{f};\boldsymbol{y},a) & \Leftrightarrow & \forall z \leq^* a\, A_B(\boldsymbol{f}a;\boldsymbol{y}) \\
(\exists z A)_B(\boldsymbol{x},a;\boldsymbol{f}) & \Leftrightarrow & \exists z \leq^* a\, \tilde{\forall}\boldsymbol{y} \leq^* \boldsymbol{f}\boldsymbol{x}a\, A_B(\boldsymbol{x};\boldsymbol{y}).
\end{array}$$

Proof. By induction on the structure of A. Consider, for instance, the case of existential quantifier

$$\begin{array}{lcl}
(\exists z A)_B(\boldsymbol{x},a;\boldsymbol{f}) & \equiv & (|[(\exists z A)^l]|^{\boldsymbol{x},a}_{\boldsymbol{f}})^i \\
& \stackrel{\text{(D1)}}{\equiv} & (|[\exists z ! A^l]|^{\boldsymbol{x},a}_{\boldsymbol{f}})^i \\
& \equiv & (|\exists a \exists z \leq^* a\, ![A^l]|^{\boldsymbol{x},a}_{\boldsymbol{f}})^i \\
& \stackrel{\text{(D3,T9)}}{\equiv} & (\exists z \leq^* a\, |![A^l]|^{\boldsymbol{x}}_{\boldsymbol{f}a})^i \\
& \stackrel{\text{(T9)}}{\equiv} & (\exists z \leq^* a ! \tilde{\forall}\boldsymbol{y} \leq^* \boldsymbol{f}\boldsymbol{x}a |[A^l]|^{\boldsymbol{x}}_{\boldsymbol{y}})^i \\
& \stackrel{\text{(D2)}}{\equiv} & \exists z \leq^* a\, \tilde{\forall}\boldsymbol{y} \leq^* \boldsymbol{f}\boldsymbol{x}a (|[A^l]|^{\boldsymbol{x}}_{\boldsymbol{y}})^i \\
& \stackrel{\text{(IH)}}{\equiv} & \exists z \leq^* a\, \tilde{\forall}\boldsymbol{y} \leq^* \boldsymbol{f}\boldsymbol{x}a\, A_B(\boldsymbol{x};\boldsymbol{y}).
\end{array}$$

The other cases are treated similarly. $\Box$

Although A_B does not syntactically coincide with the bounded functional interpretation of intuitionistic logic (because of the different treatment of $\vee$ and $\exists$), it is easy to see that (cf. Theorem 6)

$$\begin{array}{rl}
\tilde{\exists}\boldsymbol{x}, a \tilde{\forall}\boldsymbol{f} (\exists z A)_B(\boldsymbol{x},a;\boldsymbol{f}) & \equiv \tilde{\exists}\boldsymbol{x}, a \tilde{\forall}\boldsymbol{f} \exists z \leq^* a \tilde{\forall}\boldsymbol{y} \leq^* \boldsymbol{f}\boldsymbol{x}a\, A_B(\boldsymbol{x};\boldsymbol{y}) \\
& \Leftrightarrow \tilde{\exists}\boldsymbol{x}, a \tilde{\forall}\boldsymbol{c} \exists z \leq^* a \tilde{\forall}\boldsymbol{y} \leq^* \boldsymbol{c}\, A_B(\boldsymbol{x};\boldsymbol{y}).
\end{array}$$

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