Confined Modified Realizability

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We present a refinement of the bounded modified realizability which provides both *upper* and *lower* bounds for witnesses. Our interpretation is based on a generalisation of Howard/Bezem's notion of strong majorizability. We show how the bounded modified realizability coincides with (a weak version of) our interpretation in the case when least elements exist (e.g. natural numbers). The new interpretation, however, permits the extraction of more accurate bounds, and provides an ideal setting for dealing *directly* with data types whose natural ordering is not well-founded.

1 Introduction

In 1945 Stephen Kleene [6] introduced the notion of *realizability* with the purpose of making intuitionistic reasoning (informally explained by the Brouwer-Heything-Kolmogorov interpretation) precise. In short, the idea of realizability is to witness (or "realize") existential quantifiers and disjunctions, and carry this information from premisses to conclusion in implications. Realizability has currently several different variants and applications [2, 8, 11]. In this introductory section, we only stress the variations of realizability that form a natural course into our work, namely modified realizability, monotone realizability and bounded modified realizability.

Modified realizability, as its name evinces, is a variation of Kleene's original realizability notion. It was introduced by Georg Kreisel [9] in 1962 in the context of finite type arithmetic. More recently, Ulrich Kohlenbach has shown the importance of working with *bounds* instead of precise witnesses (see [7,8] for more information on *monotone realizability*). In the monotone realizability, the search for bounds is part of the proof interpretation, but not of the way formulas are interpretated (monotone realizability still interprets the formulas using modified realizability). In 2006 Fernando Ferreira and Ana Nunes, following the *bounded functional interpretation* [4], introduced a new notion of realizability - *bounded modified realizability* [3] - based on an assignment of formulas (when doing the interpretation) that disregards precise witnesses and focuses only on *upper bounds*.

The version of realizability we present in this paper, which we call *confined modified realizability* and is strongly inspired by the bounded modified realizability, provides "intervals" that bound the witnesses. The novelty is that the interpretation of formulas stores not only *upper* but also *lower bounds*. In the natural number setting, bounded modified realizability can be seen as a kind of confined modified realizability in which the lower bounds are always zero (0). In this environment we strongly believe that the novel interpretation can be useful for augmenting the precision when searching for witnesses.

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But in the case of data types without a least element (e.g. integers and rationals) being able to directly carry over information about lower bounds might be of interest on its own right[1].

The present paper is structured as follows. In the rest of the introduction we describe the logical system used for the confined realizability interpretation. The interpretation itself is presented in Section 2. We argue that the interpretation can be defined via two different majorizability relations on intervals, one is presented in Section 1.2 the other in Section 1.3. The relation between the confined and the bounded realizability interpretations is studied in Section 3. Finally, in Section 5 we discuss some fields where the confined realizability interpretation might be particularly useful.

1.1 Intuitionistic logic in all finite types

In this section, we briefly introduce the language and the basic system over which we define the new interpretation; and we present some definitions and auxiliary results needed in the subsequent sessions (for related work see [3]).

In the context of all finite types with a base type 0, let $\mathcal{L}^\omega$ be a language with a denumerable set of variables for each type, a constant c^0 of base type (to ensure that each type has at least one element), the usual combinators Π and Σ of types $\rho \to (\tau \to \rho)$ and $(\delta \to (\rho \to \tau)) \to ((\delta \to \rho) \to (\delta \to \tau))$ respectively and a binary relation symbols $=_0$ (infixing between terms of type 0).

The theory IL^ω is intuitionistic logic in all finite types. For completeness of list the logical rules below (using Gentzen's sequent calculus):

1. $A \vdash A$ 2. $\dfrac{\Gamma, A, B, \Delta \ \vdash\ C}{\Gamma, B, A, \Delta \ \vdash\ C}$ (perm.) 3. $\dfrac{\Gamma, A, A \ \vdash\ B}{\Gamma, A \ \vdash\ B}$ (cont.)

4. $\dfrac{\Gamma \ \vdash\ B}{\Gamma, A \ \vdash\ B} \qquad \dfrac{\Gamma \ \vdash\ \bot}{\Gamma \ \vdash\ A}$ (weak.) 5. $\dfrac{\Gamma \ \vdash\ A \qquad \Gamma, A \ \vdash\ B}{\Gamma \ \vdash\ B}$ (cut)

6. $\dfrac{\Gamma, A, B \ \vdash\ C}{\Gamma, A \wedge B \ \vdash\ C} \qquad \dfrac{\Gamma \ \vdash\ A \qquad \Gamma \ \vdash\ B}{\Gamma \ \vdash\ A \wedge B}$ (conj.)

7. $\dfrac{\Gamma, A \ \vdash\ C \qquad \Gamma, B \ \vdash\ C}{\Gamma, A \vee B \ \vdash\ C} \qquad \dfrac{\Gamma \ \vdash\ A}{\Gamma \ \vdash\ A \vee B} \qquad \dfrac{\Gamma \ \vdash\ B}{\Gamma \ \vdash\ A \vee B}$ (disj.)

8. $\dfrac{\Gamma \ \vdash\ A \qquad \Gamma, B \ \vdash\ C}{\Gamma, A \to B \ \vdash\ C} \qquad \dfrac{\Gamma, A \ \vdash\ B}{\Gamma \ \vdash\ A \to B}$ (impl.)

9. $\dfrac{\Gamma, A(t) \ \vdash\ B}{\Gamma, \forall x A(x) \ \vdash\ B} \qquad \dfrac{\Gamma \ \vdash\ A(y)}{\Gamma \ \vdash\ \forall x A(x)}$ ($\forall$)

10. $\dfrac{\Gamma, A(y) \ \vdash\ B}{\Gamma, \exists x A(x) \ \vdash\ B} \qquad \dfrac{\Gamma \ \vdash\ A(t)}{\Gamma \ \vdash\ \exists x A(x)}$ ($\exists$)

where A, B, C are formulas of $\mathcal{L}^\omega$, Γ, Δ are sets of formulas of $\mathcal{L}^\omega$, t is a term and y in the right $\forall$-rule and in the left $\exists$-rule is a variable that does not occur in Γ neither in Γ, B respectively. Equality is treated as in [3], i.e. we assume a neutral treatment of equality since full extensionality does not seem to be interpretable via the confined modified realizability:

11. $x =_0 x$

[1] Another field where this notion of confined modified realizability can be useful is in *complexity theory*. When trying to understand the precise complexity of a program or algorithm the usual routine is to produce lower and upper bounds on the running time of the program. Applications of proof mining to computer science, however, are still very limited.

12. $x =_0 y \wedge \varphi[x/w] \rightarrow \varphi[y/w]$

where φ is an atomic formula with a variable w of type 0; and the usual axioms regulating the combinators Π and Σ:

13. $\varphi[\Pi(x,y)/w] \leftrightarrow \varphi[x/w]$

14. $\varphi[\Sigma(x,y,z)/w] \leftrightarrow \varphi[xz(yz)/w]$

with φ an atomic formula with a variable w and x, y and z variables of appropriate types.

1.2 The theory $\mathrm{IL}^\omega_\leq$

Let $\mathrm{IL}^\omega_\leq$ be the extension of IL^ω with a new relation symbol $\leq_0$ and new function symbols mi and ma of type $0 \rightarrow (0 \rightarrow 0)$ satisfying:

15. $x \leq_0 x$

16. $x \leq_0 y \wedge y \leq_0 z \rightarrow x \leq_0 z$

17. $x \leq_0 \mathrm{ma}(x,y) \wedge y \leq_0 \mathrm{ma}(x,y)$

18. $x \leq_0 x' \wedge y \leq_0 y' \rightarrow \mathrm{ma}(x,y) \leq_0 \mathrm{ma}(x',y')$

19. $\mathrm{mi}(x,y) \leq_0 x \wedge \mathrm{mi}(x,y) \leq_0 y$

20. $x \leq_0 x' \wedge y \leq_0 y' \rightarrow \mathrm{mi}(x,y) \leq_0 \mathrm{mi}(x',y')$.

We use the designation mi and ma instead of $\min$ and $\max$ because the axioms above do not ensure that the functions return the least and the greatest of the two elements involved.

In $\mathcal{L}^\omega_\leq$, we use the abbreviation $\forall[x;y]$ for $\forall x \forall y$ and $\exists[x;y]$ for $\exists x \exists y$ and we can define, by induction on the types, the following quaternary relation denoted by $\subseteq$ ($\overset{\rho}{\subseteq}$ means relation $\subseteq$ for type ρ and we use the infix notation $[x;y] \subseteq [z;w]$ for $\subseteq(x,y,z,w)$):

$$\begin{aligned}
[x;y] \overset{0}{\subseteq} [z;w] \quad &:= \quad (z \leq_0 x \wedge x \leq_0 w) \wedge (z \leq_0 y \wedge y \leq_0 w) \\
[x;y] \overset{\rho\rightarrow\sigma}{\subseteq} [z;w] \quad &:= \quad \forall[h,k]\forall[j;l]([h;k] \overset{\rho}{\subseteq} [j;l] \rightarrow [xh;yk] \overset{\sigma}{\subseteq} [zj;wl] \wedge [zh;wk] \overset{\sigma}{\subseteq} [zj;wl]).
\end{aligned}$$

We define $x \in [y;z] \ := \ [x;x] \subseteq [y;z]$ and consider, as primitive in $\mathcal{L}^\omega_\leq$, the two *bounded quantifiers* $\forall x \in [t;q]A(x)$ and $\exists x \in [t;q]A(x)$, with t and q terms where x does not occur, having the theory $\mathrm{IL}^\omega_\leq$ the following axiom schema:

21. $\forall x \in [r;s]A(x) \leftrightarrow \forall x(x \in [r;s] \rightarrow A(x))$

22. $\exists x \in [r;s]A(x) \leftrightarrow \exists x(x \in [r;s] \wedge A(x))$.

Abbreviations. When x and y are such that $[x;y] \subseteq [x;y]$ we call $[x;y]$ a *monotone interval*. For the rest of the article we will make use of the following abbreviations for

- *monotone quantifications*

  $\tilde{\forall}[a;b]A \equiv \forall[a;b]([a;b] \subseteq [a;b] \rightarrow A)$

  $\tilde{\exists}[a;b]A \equiv \exists[a;b]([a;b] \subseteq [a;b] \wedge A)$

- *bounded interval quantifications*

  $\forall[x;y]\subseteq[z;w]A\equiv\forall[x;y]([x;y]\subseteq[z;w]\to A)$

  $\exists[x;y]\subseteq[z;w]A\equiv\exists[x;y]([x;y]\subseteq[z;w]\wedge A)$.

Also, we write $\boldsymbol{x}$ for a tuple of variables $x_1,\ldots,x_n$, so that $[\boldsymbol{x};\boldsymbol{y}]\subseteq[\boldsymbol{t};\boldsymbol{q}]$ stands for $[x_1;y_1]\subseteq[t_1;q_1]\wedge\ldots\wedge[x_n;y_n]\subseteq[t_n;q_n]$ and $\forall[\boldsymbol{x};\boldsymbol{y}]\subseteq[\boldsymbol{t};\boldsymbol{q}]A$ stands for $\forall[x_1;y_1],\ldots,[x_n;y_n]([\boldsymbol{x},\boldsymbol{y}]\subseteq[\boldsymbol{t};\boldsymbol{q}]\to A)$ (similarly for $\exists$).

Proposition 1.1 *If $[t;q]$ and $[r;s]$ are monotone intervals of types $\rho\to\sigma$ and ρ respectively then $[tr;qs]$ is a monotone interval of type σ.*

Lemma 1.2 *Assuming $[x;y]\subseteq[a;b]$ we have:*

(a) $[a;b]\subseteq[a;b]$

(b) $[a;b]\subseteq[c;d]\to[x;y]\subseteq[c;d]$

(c) $[v;w]\subseteq[a;b]\to[x;w]\subseteq[a;b]$

(d) $x\in[a;b]\wedge y\in[a;b]$.

Proof. The first clause follows immediately by definition, the other assertions can be proved by induction on the type of the relation $\subseteq$. □

In the following lemma we present technical properties involving sequent calculus that will be needed in the next section. Concerning notation, a *double line* is used when in the derivation we are possibly applying more than one rule.

Lemma 1.3 *The following derivations are valid:*

(a) *When $G(x,y)$ implies that $[x;y]$ is monotone we have*

$$\frac{\vdash\ \tilde\forall[x;y](G(x,y)\to A(x,y))}{G(x,y)\ \vdash\ A(x,y)}$$

(b) $$\frac{A(x,y)\ \vdash\ B(x,y)}{\vdash\ \tilde\forall[x;y](A(x,y)\to B(x,y))}$$

(c) *If $[t;q]$ is a monotone interval then* $\dfrac{\Gamma,A(t,q)\ \vdash\ B}{\Gamma,\tilde\forall[x;y]A(x,y)\ \vdash\ B}$.

We define, by induction on the types, the functionals mi_ρ and ma_ρ of type $\rho\to(\rho\to\rho)$:

$$\begin{cases}\mathrm{mi}_0(n,m) & := & \mathrm{mi}(n,m)\\ \mathrm{mi}_{\rho\to\sigma}(x,y) & := & \lambda u^\rho.\mathrm{mi}_\sigma(xu,yu),\end{cases}$$

$$\begin{cases}\mathrm{ma}_0(n,m) & := & \mathrm{ma}(n,m)\\ \mathrm{ma}_{\rho\to\sigma}(x,y) & := & \lambda u^\rho.\mathrm{ma}_\sigma(xu,yu).\end{cases}$$

We define $[a;b]\cup[c;d]$ as being $[\mathrm{mi}(a,c);\mathrm{ma}(b,d)]$.

Proposition 1.4 *The following are useful properties of* mi *and* ma

(a) $[x;y]\subseteq[a;b]\wedge[v;w]\subseteq[c;d]\to[x;y]\cup[v;w]\subseteq[a;b]\cup[c;d]$

(b) $[\mathrm{mi};\mathrm{ma}]\subseteq[\mathrm{mi};\mathrm{ma}]$.

Proof. (a) The proof is done by induction on the types. For the type zero the result follows easily from the axioms regulating mi and ma and the transitivity of the relation $\leq_0$. For higher types, assume $[x;y] \overset{\rho\to\tau}{\subseteq} [a;b]$ and $[v;w] \overset{\rho\to\tau}{\subseteq} [c;d]$. By the definition of the relation $\subseteq$ we have that whenever $[h;k] \overset{\rho}{\subseteq} [j;l]$ then

$$[xh;yk] \overset{\tau}{\subseteq} [aj;bl] \wedge [ah;bk] \overset{\tau}{\subseteq} [aj;bl]$$
$$[vh;wk] \overset{\tau}{\subseteq} [cj;dl] \wedge [ch;dk] \overset{\tau}{\subseteq} [cj;dl].$$

By induction hypothesis this implies

$$[\mathrm{mi}(xh,vh);\mathrm{ma}(yk,wk)] \overset{\tau}{\subseteq} [\mathrm{mi}(aj,cj);\mathrm{ma}(bl,dl)]$$

and

$$[\mathrm{mi}(ah,ch);\mathrm{ma}(bk,dk)] \overset{\tau}{\subseteq} [\mathrm{mi}(aj,cj);\mathrm{ma}(bl,dl)].$$

(b) Consider that mi and ma are of type $\rho\to(\rho\to\rho)$. Given $[h;k]\subseteq[j;l]$ and $[h';k']\subseteq[j';l']$ of type ρ, we need to prove that

- $[\mathrm{mi}(h,h');\mathrm{ma}(k,k')]\subseteq[\mathrm{mi}(j,j');\mathrm{ma}(l,l')]$ and
- $[\mathrm{mi}(j,h');\mathrm{ma}(l,k')]\subseteq[\mathrm{mi}(j,j');\mathrm{ma}(l,l')]$.

The first assertion follows immediately from (a) and the second assertion follows easily from Lemma 1.2 (a) and (a). ☐

Proposition 1.5 *The following holds of* $\cup$ *and* $\subseteq$

$$([a;b]\subseteq[a;b])\wedge([c;d]\subseteq[c;d])\to([a;b]\subseteq[a;b]\cup[c;d])\wedge([c;d]\subseteq[a;b]\cup[c;d]).$$

Proof. The proof is done by induction on the types. Type zero follows immediately by transitivity. For other types, take $[h;k]\subseteq[j;l]$. We want to prove the following 3 assertions:

1) $[ah;bk]\subseteq[\mathrm{mi}(aj,cj);\mathrm{ma}(bl,dl)]$

2) $[\mathrm{mi}(ah,ch);\mathrm{ma}(bk,dk)]\subseteq[\mathrm{mi}(aj,cj);\mathrm{ma}(bl,dl)]$

3) $[ch;dk]\subseteq[\mathrm{mi}(aj,cj);\mathrm{ma}(bl,dl)]$.

By hypothesis $[ah;bk]\subseteq[aj;bl]$ and $[ch;dk]\subseteq[cj;dl]$. So, by Lemma 1.2 (a) we know that $[aj;bl]\subseteq[aj;bl]$ and $[cj;dl]\subseteq[cj;dl]$. By induction hypothesis

- $[aj;bl]\subseteq[\mathrm{mi}(aj,cj);\mathrm{ma}(bl,dl)]$ and
- $[cj;dl]\subseteq[\mathrm{mi}(aj,cj);\mathrm{ma}(bl,dl)]$.

Thus, applying Lemma 1.2 (b), we derive assertions 1) and 3). Assertion 2) follows by hypothesis $[ah;bk]\subseteq[aj;bl]$ and $[ch;dk]\subseteq[cj;dl]$ and Proposition 1.4 (a). ☐

Definition 1.6 ($\tilde{\exists}$-free formula) A formula of $\mathcal{L}^\omega_\leq$ is called a $\tilde{\exists}$-*free formula* if it is built from atomic formulas by means of conjunctions, disjunctions, implications, bounded quantifications and universal monotone quantifications.

Definition 1.7 (Confined theory) A theory $\mathrm{T}^\omega_\leq$ in $\mathcal{L}^\omega_\leq$ is called a *confined theory* if it extends $\mathrm{IL}^\omega_\leq$ and, for every constant c^ρ, there are closed terms t^ρ and q^ρ such that $\mathrm{T}^\omega_\leq \vdash c\in[t;q]$.

It is easy to verify that $\mathrm{IL}^{\omega}_{\lesssim}$ is a confined theory since $\Pi \in [\Pi;\Pi]$, $\Sigma \in [\Sigma;\Sigma]$, $\mathrm{mi}, \mathrm{ma} \in [\mathrm{mi};\mathrm{ma}]$ and $c^0 \in [c^0;c^0]$. In Section 4.3 we will also show that HA^{ω} is a confined theory, by showing how Gödel's primitive recursor can be confined.

If $\mathrm{T}^{\omega}_{\lesssim}$ is a confined theory and t, q are closed terms of $\mathcal{L}^{\omega}_{\lesssim}$ then there are closed terms $\tilde{t}$, $\tilde{q}$ such that $\mathrm{T}^{\omega}_{\lesssim} \vdash [t;q] \subseteq [\tilde{t};\tilde{q}]$. We say that $[\tilde{t}(\boldsymbol{x});\tilde{q}(\boldsymbol{x})]$ confines $[t(\boldsymbol{x});q(\boldsymbol{x})]$ ($t, q, \tilde{t}, \tilde{q}$ with the free variables as shown) if $\mathrm{T}^{\omega}_{\lesssim} \vdash [\lambda \boldsymbol{x}.t(\boldsymbol{x});\lambda \boldsymbol{x}.q(\boldsymbol{x})] \subseteq [\lambda \boldsymbol{x}.\tilde{t}(\boldsymbol{x});\lambda \boldsymbol{x}.\tilde{q}(\boldsymbol{x})]$. If $\mathrm{T}^{\omega}_{\lesssim}$ is a confined theory and $[t(\boldsymbol{x});q(\boldsymbol{x})]$ is an interval with t and q open terms, then there are $\tilde{t}(\boldsymbol{x})$ and $\tilde{q}(\boldsymbol{x})$ such that $[\tilde{t}(\boldsymbol{x});\tilde{q}(\boldsymbol{x})]$ confines $[t(\boldsymbol{x});q(\boldsymbol{x})]$. The following lemma follows trivially.

Lemma 1.8 *Given $t(x)$ a term in $\mathcal{L}^{\omega}_{\lesssim}$ with a free variable x and monotone $[a;b]$ such that $x \in [a;b]$ then $t(x) \in [\tilde{t}(a);\tilde{q}(b)]$ with $[\tilde{t};\tilde{q}]$ confining t.*

1.3 Alternative definition of $x \in [a;b]$

In this section we show that we could have chosen to define $x \in [a;b]$ directly, rather than via the definition of $[x;y] \subseteq [a;b]$.

Definition 1.9 For each type ρ we denote by $x \in^*_{\rho} [a;b]$ the ternary relation inductively defined (on the structure of the type ρ) by:

$$\begin{array}{lcl} x \in^*_0 [a;b] & := & a \leq_0 x \wedge x \leq_0 b \\ x \in^*_{\rho\to\sigma} [a;b] & := & \forall [c^{\rho};d^{\rho}] \forall y \in^*_{\rho} [c;d] (xy \in^*_{\sigma} [ac;bd] \wedge ay \in^*_{\sigma} [ac;bd] \wedge by \in^*_{\sigma} [ac;bd]). \end{array}$$

Lemma 1.10 $\mathrm{IL}^{\omega}_{\lesssim} \vdash [x;y] \subseteq [a;b] \leftrightarrow (x \in^* [a;b] \wedge y \in^* [a;b])$.

Proof. The proof is done by induction on the types. For type zero the result is trivial. For other types, we have the following chain of equivalences:

$$\begin{array}{rcl} [x;y] \subseteq [a;b] & \equiv & \forall [c;d] \forall [v;w] \subseteq [c;d] ([xv;yw] \subseteq [ac;bd] \wedge [av;bw] \subseteq [ac;bd]) \\ & \overset{(\dagger)}{\leftrightarrow} & \forall [c;d] \forall z \in^* [c;d] ([xz;yz] \subseteq [ac;bd] \wedge [az;bz] \subseteq [ac;bd]) \\ & \overset{(\mathrm{IH})}{\leftrightarrow} & \forall [c;d] \forall z \in^* [c;d] \\ & & \quad (xz \in^* [ac;bd] \wedge yz \in^* [ac;bd] \wedge az \in^* [ac;bd] \wedge bz \in^* [ac;bd]) \\ & \equiv & (x \in^* [a;b] \wedge y \in^* [a;b]). \end{array}$$

Let us prove ($\dagger$). For the left to right implication, just take $v := w$ and $z := w$ and use the induction hypothesis. For the right to left implication, fix $[c;d]$ and $[v;w]$ such that $[v;w] \subseteq [c;d]$. By induction hypothesis we have $v \in^* [c;d]$ and $w \in^* [c;d]$. So, by hypothesis we know that

$$[xv;yv] \subseteq [ac;bd] \quad [av;bv] \subseteq [ac;bd] \quad [xw;yw] \subseteq [ac;bd] \quad [aw;bw] \subseteq [ac;bd].$$

Applying Lemma 1.2 (c) twice, we obtain $[xv;yw] \subseteq [ac;bd]$ and $[av;bw] \subseteq [ac;bd]$. $\square$

From the above Lemma we can immediately conclude that $\in^*$ and $\in$ are equivalent notions:

Corollary 1.11 $\mathrm{IL}^{\omega}_{\lesssim} \vdash x \in [y;z] \leftrightarrow x \in^* [y;z]$.

The above corollary shows that we can defined the confined modified realizability departing from Definition 1.9 instead of defining the relation $\subseteq$. In that case $[x;y] \subseteq [z;w]$ would be an abbreviation for $x \in^* [z;w] \wedge y \in^* [z;w]$.

2 The confined modified realizability

In this section, we define the new interpretation *confined modified realizability* within the theory $\mathrm{IL}^{\omega}_{\lesssim}$ and we prove a soundness theorem. Instead of the more traditional way of expressing realizability by '$[\boldsymbol{a},\boldsymbol{b}]$ realize A' we write it in the form $(A)_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}]$ (see the definition below).

Definition 2.1 (Confined modified realizability) To each formula A of the language $\mathcal{L}^{\omega}_{\leq}$, we assign a formula $(A)_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}]$ of the same language according to the following clauses:

$$1. \quad (P)_{\mathsf{cr}}[;] \;:=\; P \qquad \text{(for } P \text{ atomic)}$$

If we have already interpretations for A and B given by $(A)_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}]$ and $(B)_{\mathsf{cr}}[\boldsymbol{c};\boldsymbol{d}]$ respectively then, we define:

$$\begin{array}{lll}
2. & (A\wedge B)_{\mathsf{cr}}[\boldsymbol{a},\boldsymbol{c};\boldsymbol{b},\boldsymbol{d}] & := & (A)_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}]\wedge(B)_{\mathsf{cr}}[\boldsymbol{c};\boldsymbol{d}]\\
3. & (A\vee B)_{\mathsf{cr}}[\boldsymbol{a},\boldsymbol{c};\boldsymbol{b},\boldsymbol{d}] & := & (A)_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}]\vee(B)_{\mathsf{cr}}[\boldsymbol{c};\boldsymbol{d}]\\
4. & (A\to B)_{\mathsf{cr}}[\boldsymbol{f};\boldsymbol{g}] & := & \tilde{\forall}[\boldsymbol{a};\boldsymbol{b}]((A)_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}]\to(B)_{\mathsf{cr}}[\boldsymbol{fa};\boldsymbol{gb}])\\
5. & (\forall z A(z))_{\mathsf{cr}}[\boldsymbol{f};\boldsymbol{g}] & := & \tilde{\forall}[a;b]\forall z\in[a;b](A(z))_{\mathsf{cr}}[\boldsymbol{f}a;\boldsymbol{g}b]\\
6. & (\exists z A(z))_{\mathsf{cr}}[\boldsymbol{a},c;\boldsymbol{b},d] & := & \exists z\in[c;d](A(z))_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}]\\
7. & (\forall z\in[t;q]A(z))_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}] & := & \forall z\in[t;q](A(z))_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}]\\
8. & (\exists z\in[t;q]A(z))_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}] & := & \exists z\in[t;q](A(z))_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}].
\end{array}$$

The tuples $[\boldsymbol{a};\boldsymbol{b}]$ in $(A)_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}]$ should be seen as the realizers for A. More specifically we should see $\boldsymbol{a}$ as the lower bounds and $\boldsymbol{b}$ as the upper bounds for the witnesses. Notice that for atomic formulas the tuple of realizers is empty. Being negation $\neg A$ a particular case of implication $A\to\perp$, its interpretation is $\tilde{\forall}[\boldsymbol{a};\boldsymbol{b}]\neg(A)_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}]$, having an empty tuple of realizers. This implies that this notion of realizability (as the previous versions mentioned in the introduction) is not suitable for dealing directly with negated formulas.

In the same way that bounds are required to be "monotone non-decreasing" in the bounded interpretations, in the confined interpretation both upper and lower bounds are also required to be "monotone non-decreasing". This might seem strange at first, since a "dualisation" of upper bounds could suggest that lower bounds should be "monotone non-increasing". The fact that both types of bounds have to have the same asymptotic behaviour becomes natural once one notices that the same situation arises in complexity theory, where only *proper complexity functions* are used [10].

Lemma 2.2 (Monotonicity) $\mathrm{T}^{\omega}_{\leq}\vdash(A(\boldsymbol{z}))_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}]\wedge[\boldsymbol{a};\boldsymbol{b}]\subseteq[\boldsymbol{c};\boldsymbol{d}]\to(A(\boldsymbol{z}))_{\mathsf{cr}}[\boldsymbol{c};\boldsymbol{d}]$.

Proof. The result can be verified by inspection on the clauses of Definition 2.1. Note that from Lemma 1.2 (b) we know that $[x;y]\subseteq[a;b]\wedge[a;b]\subseteq[c;d]\to[x;y]\subseteq[c;d]$. Thus, in particular, if $z\in[a;b]\wedge[a;b]\subseteq[c;d]$ then $z\in[c;d]$. And we also know that if we have a monotone interval $[a;b]$ of appropriate type and $[f;g]\subseteq[f^*;g^*]$ then $[fa;gb]\subseteq[f^*a;g^*b]$. ☐

Proposition 2.3 *If A is a $\tilde{\exists}$-free formula then $(A)_{\mathsf{cr}}[;]\leftrightarrow A$.*

Proof. The proof is done by induction on the complexity of A. If A is an atomic formula the result is trivial. If $A:=B\wedge C$, by induction hypothesis we know that $(B)_{\mathsf{cr}}[;]\leftrightarrow B$ and $(C)_{\mathsf{cr}}[;]\leftrightarrow C$, so $(A)_{\mathsf{cr}}[;]:=(B)_{\mathsf{cr}}[;]\wedge(C)_{\mathsf{cr}}[;]\leftrightarrow B\wedge C$. The cases $A:=B\vee C$, $A:=B\to C$, $A:=\forall x\in[t;q]B(x)$ and $A:=\exists x\in[t;q]B(x)$ are also completely straightforward. When $A:=\tilde{\forall}[x;y]B(x,y)$, notice that it is an abbreviation for $A:=\forall x\forall y([x;y]\subseteq[x;y]\to B(x,y))$. So, $(A)_{\mathsf{cr}}[;]$ is $\tilde{\forall}[a;b]\forall x\in[a;b]\tilde{\forall}[c;d]\forall y\in[c;d]([x;y]\subseteq[x;y]\to(B(x,y))_{\mathsf{cr}}[;])$ which is equivalent to $\forall x\forall y([x;y]\subseteq[x;y]\to B(x,y))$. ☐

Note that $[x;y]\overset{\rho\to\sigma}{\subseteq}[z;w]$ can be written as

$$\tilde{\forall}[j;l]\forall[h;k]\subseteq[j;l]([xh;yk]\subseteq[zj;wl]\wedge[zh;wk]\subseteq[zj;wl]),$$

being a $\tilde{\exists}$-free formula. Hence we have that $([x;y]\subseteq[z;w])_{\mathsf{cr}}[;]\leftrightarrow[x;y]\subseteq[z;w]$ and $(x\in[z;w])_{\mathsf{cr}}[;]\leftrightarrow x\in[z;w]$.

Theorem 2.4 (Soundness of confined modified realizability) *Let $\Gamma(\boldsymbol{z})$ be a set of formulas of $\mathcal{L}^{\omega}_{\leq}$, $A(\boldsymbol{z})$ a formula of the same language and $\mathrm{T}^{\omega}_{\leq}$ a confined theory. Consider that all the free variables of Γ and A are among the variables in the tuple $\boldsymbol{z}$. If*

$$\Gamma(z) \vdash_{\mathrm{IL}^\omega_\leq} A(z),$$

then there are closed monotone intervals $[t;q]$ *of appropriate types such that for all monotone* $[a;b]$ *and* $[c;d]$ *and for all* $z \in [c;d]$

$$(\Gamma(z))_{\mathsf{cr}}[a;b] \vdash_{\mathrm{T}^\omega_\leq} (A(z))_{\mathsf{cr}}[tca;qdb].$$

Proof. In order to simplify notation, we will not explicitly include the tuple z unless the free variables matter. The proof is done by induction on the number of lines of the derivation of $\Gamma \vdash A$. Note that the axioms from 11. to 20. are universal being, therefore, realized by the empty tuple. For the logical rules we show how bounding intervals for the premises of the rule can be converted into bounding intervals for the conclusion. We consider only the less trivial cases. Concerning notation, a *dashed* line in a proof means that we are only rewritten the sequent using definitions or abbreviations already established.

3. Contraction rule:

$$\dfrac{\dfrac{\dfrac{(A)_{\mathsf{cr}}[u;v],(A)_{\mathsf{cr}}[z;w] \;\vdash\; (B)_{\mathsf{cr}}[t(u,z);q(v,w)]}{(A)_{\mathsf{cr}}[u;v],(A)_{\mathsf{cr}}[u;v] \;\vdash\; (B)_{\mathsf{cr}}[t(u,u);q(v,v)]}\,([u/z],[v/w])}{(A)_{\mathsf{cr}}[u;v] \;\vdash\; (B)_{\mathsf{cr}}[t(u,u);q(v,v)]}\,(\text{cont.})}{(A)_{\mathsf{cr}}[u;v] \;\vdash\; (B)_{\mathsf{cr}}[r(u);s(v)]}$$

with $r := \lambda u.t(u,u)$ and $s := \lambda v.q(v,v)$.

5. Cut rule:

$$\dfrac{\dfrac{\vdash\; (A)_{\mathsf{cr}}[t;q] \qquad \dfrac{(A)_{\mathsf{cr}}[z;w] \;\vdash\; (B)_{\mathsf{cr}}[f(z);g(w)]}{(A)_{\mathsf{cr}}[t;q] \;\vdash\; (B)_{\mathsf{cr}}[f(t);g(q)]}\,([t/z],[q/w])}{\vdash\; (B)_{\mathsf{cr}}[f(t);g(q)]}\,(\text{cut})}{\vdash\; (B)_{\mathsf{cr}}[r;s]}$$

with $r := f(t)$ and $s := g(q)$. Since $[t;q]$ and $[f;g]$ are, by induction hypothesis, monotone intervals, easily we can verify that $[r;s]$ are also monotone (see Proposition 1.1).

7. Disjunction (left):

$$\dfrac{\dfrac{\dfrac{\dfrac{(A)_{\mathsf{cr}}[z;w] \;\vdash\; (C)_{\mathsf{cr}}[tz;qw]}{(A)_{\mathsf{cr}}[z;w] \;\vdash\; (C)_{\mathsf{cr}}([tz;qw]\cup[fu;gv])} \qquad \dfrac{(B)_{\mathsf{cr}}[u;v] \;\vdash\; (C)_{\mathsf{cr}}[fu;gv]}{(B)_{\mathsf{cr}}[u;v] \;\vdash\; (C)_{\mathsf{cr}}([tz;qw]\cup[fu;gv])}}{(A)_{\mathsf{cr}}[z;w] \vee (B)_{\mathsf{cr}}[u;v] \;\vdash\; (C)_{\mathsf{cr}}([tz;qw]\cup[fu;gv])}\,(\text{disj.})}{(A\vee B)_{\mathsf{cr}}[z,u;w,v] \;\vdash\; (C)_{\mathsf{cr}}([tz;qw]\cup[fu;gv])}}{(A\vee B)_{\mathsf{cr}}[z,u;w,v] \;\vdash\; (C)_{\mathsf{cr}}[r(z,u);s(w,v)]}$$

with $r := \lambda z,u.\mathrm{mi}(t(z),f(u))$ and $s := \lambda w,v.\mathrm{ma}(q(w),g(v))$. The derivations hidden in the double lines result from cut, applying Lemma 2.2 and Proposition 1.5. For the monotonicity of $[r;s]$ see Proposition 1.4.

8. Implication (right):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\Gamma)_{\mathsf{cr}}[x;y],(A)_{\mathsf{cr}}[z;w] \;\vdash\; (B)_{\mathsf{cr}}[txz;qyw]}{(\Gamma)_{\mathsf{cr}}[x;y],[z;w]\subseteq[z;w],(A)_{\mathsf{cr}}[z;w] \;\vdash\; (B)_{\mathsf{cr}}[txz;qyw]}\,(\text{weak.+perm.})}{(\Gamma)_{\mathsf{cr}}[x;y],[z;w]\subseteq[z;w] \;\vdash\; (A)_{\mathsf{cr}}[z;w]\to(B)_{\mathsf{cr}}[txz;qyw]}\,(\text{impl.})}{(\Gamma)_{\mathsf{cr}}[x;y] \;\vdash\; ([z;w]\subseteq[z;w])\to((A)_{\mathsf{cr}}[z;w]\to(B)_{\mathsf{cr}}[txz;qyw])}\,(\text{impl.})}{(\Gamma)_{\mathsf{cr}}[x;y] \;\vdash\; \forall z,w(([z;w]\subseteq[z;w])\to((A)_{\mathsf{cr}}[z;w]\to(B)_{\mathsf{cr}}[txz;qyw]))}\,(\forall)}{(\Gamma)_{\mathsf{cr}}[x;y] \;\vdash\; \tilde{\forall}[z;w]((A)_{\mathsf{cr}}[z;w]\to(B)_{\mathsf{cr}}[txz;qyw])}}{(\Gamma)_{\mathsf{cr}}[x;y] \;\vdash\; (A\to B)_{\mathsf{cr}}[rx;sy]}$$

with $\boldsymbol{r} := \lambda\boldsymbol{x}.(\lambda\boldsymbol{z}.\boldsymbol{txz})$ and $\boldsymbol{s} := \lambda\boldsymbol{x}.(\lambda\boldsymbol{z}.\boldsymbol{qxz})$.

9. Universal quantification (left):

$$\begin{array}{c}
(\Gamma)_{\mathsf{cr}}[\boldsymbol{x};\boldsymbol{y}], (A(t))_{\mathsf{cr}}[\boldsymbol{z};\boldsymbol{w}] \ \vdash \ (B)_{\mathsf{cr}}[\boldsymbol{f}(\boldsymbol{x},\boldsymbol{z});\boldsymbol{g}(\boldsymbol{y},\boldsymbol{w})] \\
\hline
(\Gamma)_{\mathsf{cr}}[\boldsymbol{x};\boldsymbol{y}], (A(t))_{\mathsf{cr}}[\boldsymbol{u}\tilde{t};\boldsymbol{v}\tilde{q}] \ \vdash \ (B)_{\mathsf{cr}}[\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}\tilde{t});\boldsymbol{g}(\boldsymbol{y},\boldsymbol{v}\tilde{q})] \\
\hline\hline
(\Gamma)_{\mathsf{cr}}[\boldsymbol{x};\boldsymbol{y}], \forall x \in [\tilde{t};\tilde{q}](A(x))_{\mathsf{cr}}[\boldsymbol{u}\tilde{t};\boldsymbol{v}\tilde{q}] \ \vdash \ (B)_{\mathsf{cr}}[\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}\tilde{t});\boldsymbol{g}(\boldsymbol{y},\boldsymbol{v}\tilde{q})] \\
\hline\hline
(\Gamma)_{\mathsf{cr}}[\boldsymbol{x};\boldsymbol{y}], \tilde{\forall}[a;b]\forall x \in [a;b](A(x))_{\mathsf{cr}}[\boldsymbol{u}a;\boldsymbol{v}b] \ \vdash \ (B)_{\mathsf{cr}}[\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}\tilde{t});\boldsymbol{g}(\boldsymbol{y},\boldsymbol{v}\tilde{q})] \\
\hdashline
(\Gamma)_{\mathsf{cr}}[\boldsymbol{x};\boldsymbol{y}], (\forall x A(x))_{\mathsf{cr}}[\boldsymbol{u};\boldsymbol{v}] \ \vdash \ (B)_{\mathsf{cr}}[\boldsymbol{r}(\boldsymbol{x},\boldsymbol{u});\boldsymbol{s}(\boldsymbol{y},\boldsymbol{v})]
\end{array}
\begin{array}{l}
([\boldsymbol{v}\tilde{q}/\boldsymbol{w}], [\boldsymbol{u}\tilde{t}/\boldsymbol{z}]) \\
(\forall_{\in} + \text{cut}) \\
(\text{L } 1.3)
\end{array}$$

with $[\tilde{t};\tilde{q}]$ terms that confine t (possible free variables can be easily treated - see Lemma 1.8), $[\boldsymbol{u};\boldsymbol{v}]$ monotone intervals of appropriate type, $\boldsymbol{r} := \lambda\boldsymbol{x},\boldsymbol{u}.\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}\tilde{t})$ and $\boldsymbol{s} := \lambda\boldsymbol{y},\boldsymbol{v}.\boldsymbol{g}(\boldsymbol{y},\boldsymbol{v}\tilde{q})$.

9. Universal quantification (right):

$$\begin{array}{c}
\vdash \ \tilde{\forall}[a;b]\forall y \in [a;b]((\Gamma)_{\mathsf{cr}}[\boldsymbol{z};\boldsymbol{w}] \to (A(y))_{\mathsf{cr}}[\boldsymbol{t}a\boldsymbol{z};\boldsymbol{q}b\boldsymbol{w}] \\
\hline\hline
\vdash \ (\Gamma)_{\mathsf{cr}}[\boldsymbol{z};\boldsymbol{w}] \to \tilde{\forall}[a;b]\forall y \in [a;b](A(y))_{\mathsf{cr}}[\boldsymbol{t}a\boldsymbol{z};\boldsymbol{q}b\boldsymbol{w}] \\
\hdashline
\vdash \ (\Gamma)_{\mathsf{cr}}[\boldsymbol{z};\boldsymbol{w}] \to (\forall x A(x))_{\mathsf{cr}}[\lambda a.\boldsymbol{t}a\boldsymbol{z};\lambda b.\boldsymbol{q}b\boldsymbol{w}] \\
\hdashline
\vdash \ (\Gamma)_{\mathsf{cr}}[\boldsymbol{z};\boldsymbol{w}] \to (\forall x A(x))_{\mathsf{cr}}[\boldsymbol{rz};\boldsymbol{sw}]
\end{array}$$

where $\boldsymbol{r} := \lambda\boldsymbol{z}.(\lambda a.\boldsymbol{t}a\boldsymbol{z})$ and $\boldsymbol{s} := \lambda\boldsymbol{w}.(\lambda b.\boldsymbol{q}b\boldsymbol{w})$. Note that y does not occur free in Γ.

10. Existential quantification (left):

$$\begin{array}{c}
\vdash \ \forall y \in [a;b]((\Gamma)_{\mathsf{cr}}[\boldsymbol{z};\boldsymbol{w}] \wedge (A(y))_{\mathsf{cr}}[\boldsymbol{u};\boldsymbol{v}] \to (B)_{\mathsf{cr}}[\boldsymbol{t}(a,\boldsymbol{z},\boldsymbol{u});\boldsymbol{q}(b,\boldsymbol{w},\boldsymbol{v})]) \\
\hline\hline
\vdash \ (\Gamma)_{\mathsf{cr}}[\boldsymbol{z};\boldsymbol{w}] \wedge \exists y \in [a;b](A(y))_{\mathsf{cr}}[\boldsymbol{u};\boldsymbol{v}] \to (B)_{\mathsf{cr}}[\boldsymbol{t}(a,\boldsymbol{z},\boldsymbol{u});\boldsymbol{q}(b,\boldsymbol{w},\boldsymbol{v})] \\
\hdashline
\vdash \ (\Gamma)_{\mathsf{cr}}[\boldsymbol{z};\boldsymbol{w}] \wedge (\exists x A(x))_{\mathsf{cr}}[\boldsymbol{u},a;\boldsymbol{v},b] \to (B)_{\mathsf{cr}}[\boldsymbol{t}(a,\boldsymbol{z},\boldsymbol{u});\boldsymbol{q}(b,\boldsymbol{w},\boldsymbol{v})] \\
\hdashline
\vdash \ (\Gamma)_{\mathsf{cr}}[\boldsymbol{z};\boldsymbol{w}] \wedge (\exists x A(x))_{\mathsf{cr}}[\boldsymbol{u},a;\boldsymbol{v},b] \to (B)_{\mathsf{cr}}[\boldsymbol{r}(\boldsymbol{z},\boldsymbol{u},a);\boldsymbol{s}(\boldsymbol{w},\boldsymbol{v},b)]
\end{array}$$

with $\boldsymbol{r} := \lambda\boldsymbol{z},\boldsymbol{u},a.\boldsymbol{t}(a,\boldsymbol{z},\boldsymbol{u})$ and $\boldsymbol{s} := \lambda\boldsymbol{w},\boldsymbol{v},b.\boldsymbol{q}(b,\boldsymbol{w},\boldsymbol{v})$.

10. Existential quantification (right):

$$\begin{array}{c}
(\Gamma)_{\mathsf{cr}}[\boldsymbol{z};\boldsymbol{w}] \ \vdash \ (A(t))_{\mathsf{cr}}[\boldsymbol{fz};\boldsymbol{gw}] \\
\hline\hline
(\Gamma)_{\mathsf{cr}}[\boldsymbol{z};\boldsymbol{w}] \ \vdash \ \exists x \in [\tilde{t};\tilde{q}](A(x))_{\mathsf{cr}}[\boldsymbol{fz};\boldsymbol{gw}] \\
\hdashline
(\Gamma)_{\mathsf{cr}}[\boldsymbol{z};\boldsymbol{w}] \ \vdash \ (\exists x A(x))_{\mathsf{cr}}[\boldsymbol{fz},\tilde{t};\boldsymbol{gw},\tilde{q}] \\
\hdashline
(\Gamma)_{\mathsf{cr}}[\boldsymbol{z};\boldsymbol{w}] \ \vdash \ (\exists x A(x))_{\mathsf{cr}}[\boldsymbol{h}(\boldsymbol{z}),r(\boldsymbol{z});\boldsymbol{l}(\boldsymbol{w}),s(\boldsymbol{w})]
\end{array}
\ (\exists_{\in} + \text{cut})$$

with $[\tilde{t};\tilde{q}]$ terms that confine t, $\boldsymbol{h} := \lambda\boldsymbol{z}.\boldsymbol{fz}$, $r := \lambda\boldsymbol{z}.\tilde{t}$, $\boldsymbol{l} := \lambda\boldsymbol{w}.\boldsymbol{gw}$ and $s := \lambda\boldsymbol{w}.\tilde{q}$.

22. Bounded existential quantification. Consider, for instance, the implication:

$$\exists x(x \in [r;s] \wedge A(x)) \to \exists x \in [r;s]A(x).$$

The interpretation of the premise is

$$(\exists x(x \in [r;s] \wedge A(x)))_{\mathsf{cr}}[\boldsymbol{a},c;\boldsymbol{b},d] \equiv \exists x \in [c;d](x \in [r;s] \wedge (A(x))_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}])$$

while the interpretation of the conclusion gives

$$(\exists x \in [r;s]A(x))_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}] \equiv \exists x \in [r;s](A(x))_{\mathsf{cr}}[\boldsymbol{a};\boldsymbol{b}]$$

It is clear that the former implies the latter. The converse implication and the axiom for bounded universal quantification can be similarly verified. That concludes the proof. □

3 Relation to Bounded Modified Realizability

We now show that the *bounded modified realizability* (cf. [3]) can be obtained as a particular case of our *confined modified realizability*. When extending $\mathrm{T}^{\omega}_{\leq}$ to arithmetic, the element 0^0 is the least element of type 0, i.e. $0 \leq_0 x$ for all element x of type 0. Let the zero functionals be inductively defined as $0^{\rho\to\sigma} \equiv \lambda u^{\rho}.0^{\sigma}$. It follows that those are also least elements of type $\rho \to \sigma$ with respect to the partial order $x \leq_{\rho\to\sigma} y \equiv \forall z^{\rho}(xz \leq_{\sigma} yz)$.

Since this section deals with *bounded modified realizability*, we call to mind some relevant definitions in that context. For a more complete survey of the general framework see [3]. The language $\mathcal{L}^{\omega}_{br}$ used to introduce bounded modified realizability is similar to the language $\mathcal{L}^{\omega}_{\leq}$ defined in Section 1.2, being the constant 'mi' absent and replacing the bounded quantifications $\forall x \in [t;q]$ and $\exists x \in [t;q]$ by primitive quantifications of the form $\forall x \leq^* t$ and $\exists x \leq^* t$ ruled by the axioms $\forall x \leq^* t\, A \leftrightarrow \forall x(x \leq^* t \to A)$ and $\exists x \leq^* t\, A \leftrightarrow \exists x(x \leq^* t \wedge A)$ respectively. By $\leq^*$ we denote Bezem's strong majorizability relation [1], defined by induction on the types as:

$$
\begin{aligned}
x \leq^*_0 y \quad &:= \quad x \leq_0 y \\
x \leq^*_{\rho\to\sigma} y \quad &:= \quad \forall k^{\rho} \forall h \leq^*_{\rho} k\, (xh \leq^*_{\sigma} yk \wedge yh \leq^*_{\sigma} yk).
\end{aligned}
$$

As in Section 1.2, we use the following abbreviations:

- $\tilde{\forall} x A \;\equiv\; \forall x(x \leq^* x \to A)$
- $\tilde{\exists} x A \;\equiv\; \exists x(x \leq^* x \wedge A)$.

The bounded modified realizability associates to each formula A of the language $\mathcal{L}^{\omega}_{br}$ a formula $(A)_{\mathsf{br}}(\boldsymbol{a})$ of the same language in the following way:

1. $(P)_{\mathsf{br}}() \quad := \quad P \qquad$ (for P atomic).

If we have already interpretations for A and B given by $(A)_{\mathsf{br}}(\boldsymbol{a})$ and $(B)_{\mathsf{br}}(\boldsymbol{b})$ respectively then, we define:

2. $(A \wedge B)_{\mathsf{br}}(\boldsymbol{a}, \boldsymbol{b}) \quad := \quad (A)_{\mathsf{br}}(\boldsymbol{a}) \wedge (B)_{\mathsf{br}}(\boldsymbol{b})$
3. $(A \vee B)_{\mathsf{br}}(\boldsymbol{a}, \boldsymbol{b}) \quad := \quad (A)_{\mathsf{br}}(\boldsymbol{a}) \vee (B)_{\mathsf{br}}(\boldsymbol{b})$
4. $(A \to B)_{\mathsf{br}}(\boldsymbol{f}) \quad := \quad \tilde{\forall} \boldsymbol{a}((A)_{\mathsf{br}}(\boldsymbol{a}) \to (B)_{\mathsf{br}}(\boldsymbol{f}\boldsymbol{a}))$
5. $(\forall z A(z))_{\mathsf{br}}(\boldsymbol{f}) \quad := \quad \tilde{\forall} a \forall z \leq^* a (A(z))_{\mathsf{br}}(\boldsymbol{f} a)$
6. $(\exists z A(z))_{\mathsf{br}}(\boldsymbol{a}, b) \quad := \quad \exists z \leq^* b (A(z))_{\mathsf{br}}(\boldsymbol{a})$
7. $(\forall z \leq^* t\, A(z))_{\mathsf{br}}(\boldsymbol{a}) \quad := \quad \forall z \leq^* t\, (A(z))_{\mathsf{br}}(\boldsymbol{a})$
8. $(\exists z \leq^* t\, A(z))_{\mathsf{br}}(\boldsymbol{a}) \quad := \quad \exists z \leq^* t\, (A(z))_{\mathsf{br}}(\boldsymbol{a})$.

Proposition 3.1 *The following are provable in* $\mathrm{T}^{\omega}_{\leq}$

(a) $x \in [a;b] \to x \in [0;b]$

(b) $x \leq^* a \leftrightarrow x \in [0;a]$.

Proof. (a) The proof is done by induction on the types. For type zero the implication follows immediately by definition of $\in$ and of least element. For types different from zero, consider $[v;w] \subseteq [c;d]$ of appropriate types. We want to prove that $[xv;xw] \subseteq [0;bd]$ and $[0;bw] \subseteq [0;bd]$. By our assumption we have $[xv;xw] \subseteq [ac;bd]$ and $[av;bw] \subseteq [ac;bd]$. Lemma 1.2 (d), the induction hypothesis, Lemma 1.2 (c) and the fact that $0 \in [0,bd]$ imply $[xv;xw] \subseteq [0;bd]$ and $[0;bw] \subseteq [0;bd]$. (b) We reason again by induction on the types. For type zero que result is immediate since we are in a theory with 0 as a least element. Let us study the equivalence in higher types:

$$
\begin{aligned}
x \in [0;a] &\overset{(C1.11)}{\leftrightarrow} \forall [c;d] \forall z \in [c;d](xz \in [0;ad] \wedge 0 \in [0;ad] \wedge az \in [0;ad]) \\
&\leftrightarrow \forall [c;d] \forall z \in [c;d](xz \in [0;ad] \wedge az \in [0;ad]) \\
&\overset{(IH)}{\leftrightarrow} \forall [c;d] \forall z \in [c;d](xz \leq^* ad \wedge az \leq^* ad) \\
&\overset{(a)}{\leftrightarrow} \forall d \forall z \in [0;d](xz \leq^* ad \wedge az \leq^* ad) \\
&\overset{(IH)}{\leftrightarrow} \forall d \forall z \leq^* d\, (xz \leq^* ad \wedge az \leq^* ad).
\end{aligned}
$$

That concludes the proof. □

Note that every expression in the language $\mathcal{L}^{\omega}_{br}$ can be expressed in the language$\mathcal{L}^{\omega}_{\leq}$.

Proposition 3.2 *Let A be a formula of the language $\mathcal{L}^{\omega}_{br}$. Then* $\mathrm{T}^{\omega}_{\leq} \vdash (A)_{\mathsf{cr}}[\mathbf{0};\boldsymbol{q}] \leftrightarrow (A)_{\mathsf{br}}(\boldsymbol{q})$.

Proof. The proof is done by induction on the complexity of the formula A. If A is an atomic formula, the result is trivial.

If $A \equiv B \wedge C$

$$
\begin{aligned}
(B \wedge C)_{\mathsf{cr}}[\mathbf{0};\boldsymbol{q}] &\leftrightarrow (B \wedge C)_{\mathsf{cr}}[\mathbf{0},\mathbf{0};\boldsymbol{q}_0,\boldsymbol{q}_1] \\
&:\leftrightarrow (B)_{\mathsf{cr}}[\mathbf{0};\boldsymbol{q}_0] \wedge (C)_{\mathsf{cr}}[\mathbf{0};\boldsymbol{q}_1] \\
&\overset{(IH)}{\leftrightarrow} (B)_{\mathsf{br}}(\boldsymbol{q}_0) \wedge (C)_{\mathsf{br}}(\boldsymbol{q}_1) \\
&:\leftrightarrow (B \wedge C)_{\mathsf{br}}(\boldsymbol{q}_0,\boldsymbol{q}_1) \\
&\leftrightarrow (B \wedge C)_{\mathsf{br}}(\boldsymbol{q}).
\end{aligned}
$$

The case $A \equiv B \vee C$ can be studied in a similar way.

If $A \equiv B \to C$.

$$
\begin{aligned}
(B \to C)_{\mathsf{cr}}[\mathbf{0};\boldsymbol{q}] &:\leftrightarrow \tilde{\forall}[\boldsymbol{x};\boldsymbol{y}]((B)_{\mathsf{cr}}[\boldsymbol{x};\boldsymbol{y}] \to (C)_{\mathsf{cr}}[\mathbf{0};\boldsymbol{q}\boldsymbol{y}]) \\
&\leftrightarrow \forall \boldsymbol{x} \forall \boldsymbol{y}(\boldsymbol{x} \in [\boldsymbol{x};\boldsymbol{y}] \wedge \boldsymbol{y} \in [\boldsymbol{x};\boldsymbol{y}] \to ((B)_{\mathsf{cr}}[\boldsymbol{x};\boldsymbol{y}] \to (C)_{\mathsf{cr}}[\mathbf{0};\boldsymbol{q}\boldsymbol{y}])) \\
&\overset{(\dagger)}{\leftrightarrow} \forall \boldsymbol{y}(\boldsymbol{y} \leq^* \boldsymbol{y} \to ((B)_{\mathsf{cr}}[\mathbf{0};\boldsymbol{y}] \to (C)_{\mathsf{cr}}[\mathbf{0};\boldsymbol{q}\boldsymbol{y}])) \\
&\overset{(IH)}{\leftrightarrow} \forall \boldsymbol{y}(\boldsymbol{y} \leq^* \boldsymbol{y} \to ((B)_{\mathsf{br}}(\boldsymbol{y}) \to (C)_{\mathsf{br}}(\boldsymbol{q}\boldsymbol{y}))) \\
&\leftrightarrow \tilde{\forall} \boldsymbol{y}((B)_{\mathsf{br}}(\boldsymbol{y}) \to (C)_{\mathsf{br}}(\boldsymbol{q}\boldsymbol{y})) \\
&\leftrightarrow (B \to C)_{\mathsf{br}}(\boldsymbol{q}).
\end{aligned}
$$

Let us prove the equivalence (†). For the left to right implication fix $\boldsymbol{y}$ such that $\boldsymbol{y} \leq^* \boldsymbol{y} \wedge (B)_{\mathsf{cr}}[\mathbf{0};\boldsymbol{y}]$. From $\boldsymbol{y} \leq^* \boldsymbol{y}$, applying Proposition 3.1, we know that $\boldsymbol{y} \in [\mathbf{0};\boldsymbol{y}]$, so $\mathbf{0} \in [\mathbf{0};\boldsymbol{y}]$. From the antecedent, taking $\boldsymbol{x}$ as being $\mathbf{0}$, we obtain $(C)_{\mathsf{cr}}[\mathbf{0};\boldsymbol{q}\boldsymbol{y}]$. For the right to left implication fix $\boldsymbol{x},\boldsymbol{y}$ such that $\boldsymbol{x} \in [\boldsymbol{x};\boldsymbol{y}] \wedge \boldsymbol{y} \in [\boldsymbol{x};\boldsymbol{y}] \wedge (B)_{\mathsf{cr}}[\boldsymbol{x};\boldsymbol{y}]$. Again by Proposition 3.1, we know that $\boldsymbol{y} \in [\mathbf{0};\boldsymbol{y}]$ and $\boldsymbol{y} \leq^* \boldsymbol{y}$. Similarly $\boldsymbol{x} \in [\mathbf{0};\boldsymbol{y}]$, so $[\boldsymbol{x};\boldsymbol{y}] \subseteq [\mathbf{0};\boldsymbol{y}]$. Since we have $(B)_{\mathsf{cr}}[\boldsymbol{x};\boldsymbol{y}]$, applying Lemma 2.2, we obtain $(B)_{\mathsf{cr}}[\mathbf{0};\boldsymbol{y}]$. By hypothesis we conclude $(C)_{\mathsf{cr}}[\mathbf{0};\boldsymbol{q}\boldsymbol{y}]$.

If $A \equiv \forall z B(z)$

$$
\begin{aligned}
(\forall z B(z))_{\mathsf{cr}}[\mathbf{0};\boldsymbol{q}] &:\leftrightarrow \tilde{\forall}[a;b] \forall z \in [a;b](B(z))_{\mathsf{cr}}[\mathbf{0};\boldsymbol{q}b] \\
&\leftrightarrow \forall a,b(a \in [a;b] \wedge b \in [a;b] \to \forall z(z \in [a;b] \to (B(z))_{\mathsf{cr}}[\mathbf{0};\boldsymbol{q}b])) \\
&\overset{(\ddagger)}{\leftrightarrow} \forall b(b \leq^* b \to \forall z(z \leq^* b \to (B(z))_{\mathsf{cr}}[\mathbf{0};\boldsymbol{q}b])) \\
&\overset{(IH)}{\leftrightarrow} \forall b(b \leq^* b \to \forall z(z \leq^* b \to (B(z))_{\mathsf{br}}(\boldsymbol{q}b))) \\
&\leftrightarrow \tilde{\forall} b \forall z \leq^* b(B(z))_{\mathsf{br}}(\boldsymbol{q}b) \\
&:\leftrightarrow (\forall z B(z))_{\mathsf{br}}(\boldsymbol{q}).
\end{aligned}
$$

The equivalence (‡) can be proved using Proposition 3.1. For the left to right implication fix b and z such that $b \leq^* b \wedge z \leq^* b$. We can deduce $b \in [0;b]$, $0 \in [0;b]$ and $z \in [0;b]$. Thus, taking $a = 0$ in the antecedent we obtain $(B(z))_{\mathsf{cr}}[\mathbf{0};\boldsymbol{q}b]$. For the right to left implication fix a, b, z such that $a \in [a;b] \wedge b \in [a;b] \wedge z \in [a;b]$. We can deduce that $a \in [0;b]$, $b \in [0;b]$, $b \leq^* b$, $z \in [0;b]$ and $z \leq^* b$. Applying the antecedent we immediately obtain $(B(z))_{\mathsf{cr}}[\mathbf{0};\boldsymbol{q}b]$.

If $A \equiv \exists z B(z)$.

$$\begin{aligned}
(\exists z B(z))_{\mathsf{cr}}[\mathbf{0};\boldsymbol{q}] \quad &\leftrightarrow \quad (\exists z B(z))_{\mathsf{cr}}[\mathbf{0},0;\boldsymbol{q}_0,q_1] \\
&:\leftrightarrow \quad \exists z \in [0;q_1](B(z))_{\mathsf{cr}}[\mathbf{0};\boldsymbol{q}_0] \\
&\overset{\text{(IH)}}{\leftrightarrow} \quad \exists z(z \in [0;q_1] \wedge (B(z))_{\mathsf{br}}(\boldsymbol{q}_0)) \\
&\overset{\text{(P3.1)}}{\leftrightarrow} \quad \exists z(z \leq^* q_1 \wedge (B(z))_{\mathsf{br}}(\boldsymbol{q}_0)) \\
&\leftrightarrow \quad \exists z \leq^* q_1(B(z))_{\mathsf{br}}(\boldsymbol{q}_0) \\
&:\leftrightarrow \quad (\exists z B(z))_{\mathsf{br}}(\boldsymbol{q}_0, q_1) \\
&\leftrightarrow \quad (\exists z B(z))_{\mathsf{br}}(\boldsymbol{q}).
\end{aligned}$$

If $A \equiv \forall x \leq^* t B(x)$. Notice that

$$A \leftrightarrow \forall x(x \leq^* t \to B(x)) \overset{\text{(P3.1)}}{\leftrightarrow} \forall x(x \in [0;t] \to B(x)) \leftrightarrow \forall x \in [0;t]B(x).$$

Thus,

$$\begin{aligned}
(\forall x \in [0;t]B(x))_{\mathsf{cr}}[\mathbf{0};\boldsymbol{q}] \quad &:\leftrightarrow \quad \forall x \in [0;t](B(x))_{\mathsf{cr}}[\mathbf{0};\boldsymbol{q}] \\
&\overset{\text{(IH)}}{\leftrightarrow} \quad \forall x(x \in [0;t] \to (B(x))_{\mathsf{br}}(\boldsymbol{q})) \\
&\overset{\text{(P3.1)}}{\leftrightarrow} \quad \forall x(x \leq^* t \to (B(x))_{\mathsf{br}}(\boldsymbol{q})) \\
&\leftrightarrow \quad \forall x \leq^* t(B(x))_{\mathsf{br}}(\boldsymbol{q}) \\
&\leftrightarrow \quad (\forall x \leq^* tB(x))_{\mathsf{br}}(\boldsymbol{q}).
\end{aligned}$$

The case $A \equiv \exists x \leq^* tB(x)$ can be studied using an entirely similar strategy, also noticing that A can be expressed in $\mathcal{L}^\omega_\leq$ by $\exists x \in [0;t]B(x)$. □

4 HA^ω as a confined theory

In this section we prove that Heyting arithmetic in all finite types HA^ω is an example of a confined theory. The main challenge has been to find a non-trivial lower bound to Gödel's primitive recursor. Take $\min$ and $\max$ the functionals mi and ma with an extra assumption on type 0, namely that $\min$ and $\max$ give exactly the minimum and the maximum of two numbers.

We are going to prove that Gödel's primitive recursors R, defined by:

$$\mathrm{R}fg0 = g^\rho$$

$$\mathrm{R}fg(n+1) = f^{0\to(\rho\to\rho)}n(\mathrm{R}fgn)$$

verify $\mathrm{R} \in [\mathrm{R}^-;\mathrm{R}^+]$, where the functionals R^- and R^+ are defined as

$$\mathrm{R}^-fg0 = \min(g, f00)$$

$$\mathrm{R}^-fg(n+1) = \min(\mathrm{R}^-fgn, fn\mathrm{R}^-fgn)$$

and

$$\mathrm{R}^+fg0 = \max(g, f0g)$$

$$\mathrm{R}^+fg(n+1) = \max(\mathrm{R}^+fgn, fn\mathrm{R}^+fgn)$$

respectively.

Proposition 4.1 *We start with the following auxiliary results:*

(a) $a \in [x; y] \wedge b \in [x; y] \to \max(a, b) \in [x; y]$

(b) $a \in [x; y] \wedge b \in [0; y] \to \max(a, b) \in [x; y]$

(c) $\max(a, b) \in [x; y] \to \max(b, a) \in [x; y]$

(d) $a \in [x; \max(y, z)] \to a \in [x; \max(z, y)]$

(e) $a^\rho \in [x; y] \wedge b^\rho \in [0; b] \to a \in [x; \max(y, b)]$

Proof. All results follow by induction on types, using the definition of $a \in [x; y]$.

□

Proposition 4.2 *We also have the following similar properties concerning* min*:*

(a) $a \in [x; y] \wedge b \in [x; y] \to \min(a, b) \in [x; y]$

(b) $a \in [x; y] \wedge b \in [x; z] \wedge y \in [x; z] \to \min(a, b) \in [x; y]$

(c) $\min(a, b) \in [x; y] \to \min(b, a) \in [x; y]$

(d) $a \in [\min(x, y); z] \to a \in [\min(y, x); z]$

(e) *If* $a^\rho \in [x; y]$ *and there is c such that* $b^\rho \in [b; c]$ *then* $a \in [\min(x, b); y]$.

Proof. Similarly, the results follow by a simple induction on the types.

□

We can now prove the assertion already mentioned:

Proposition 4.3 $\mathrm{R} \in [\mathrm{R}^-; \mathrm{R}^+]$.

Proof. For arbitrary $r^{0\to\rho\to\rho} \in [s; t]$, $h^\rho \in [l; p]$ and $n^0 \in [k; m]$, we want to prove that

$$\mathrm{R}rhn, \mathrm{R}^-sln, \mathrm{R}^+tpn, \mathrm{R}^-shn, \mathrm{R}^+thn, \mathrm{R}^-rhn, \mathrm{R}^+rhn \in [\mathrm{R}^-slk; \mathrm{R}^+tpm].$$

The prove is done by induction on n.

• Case $n = 0$: Since $n = 0$, we know that $k = 0$. We reason by induction on m. Suppose $m = 0$. We want to prove that h, $\min(l, s00)$, $\max(p, t0p)$, $\min(h, s00)$, $\max(h, t0h)$, $\min(h, r00)$, $\max(h, r0h) \in [\min(l, s00); \max(p, t0p)]$. Since $t \in [s; t]$ we know that $t \in [0; t]$. Similarly, we have that $p \in [0; p]$. Since we also know that $0 \in [0; 0]$ we obtain $t0p \in [0; t0p]$. Applying Proposition 4.1 (e), this last assertion together with $h \in [l; p]$ implies $h \in [l; \max(p, t0p)]$. In an analogous way we can check that $s00 \in [s00; t0p]$, so, by Proposition 4.2 (e), we have $h \in [\min(l, s00); \max(p, t0p)]$. All the other assertions can directly be obtained from the following facts:

- $\min, \max \in [\min; \max]$,
- $l, p \in [l; p]$,
- $s00, t0p, t0h, r00, r0h \in [s00; t0p]$.

Suppose the result is valid for m and let us prove it for $m+1$. If $n \in [k; m+1]$ we want to show that $\mathrm{R}rhn, \mathrm{R}^-sln, \ldots, \mathrm{R}^+rhn \in [\mathrm{R}^-slk; \max(\mathrm{R}^+tpm, tm\mathrm{R}^+tpm)]$. From $n \in [k; m+1]$, knowing that $n, k = 0$, we have that $n \in [k; m]$. So, by induction hypothesis

$$(\dagger) \quad \mathrm{R}rhn, \ldots, \mathrm{R}^+rhn \in [\mathrm{R}^-slk; \mathrm{R}^+tpm].$$

From $\mathrm{R}^+tpm \in [0; \mathrm{R}^+tpm]$ we can easily see that $tm\mathrm{R}^+tpm \in [0; tm\mathrm{R}^+tpm]$. So, from (†), applying Proposition 4.1 (e), we obtain $\mathrm{R}rhn, \ldots, \mathrm{R}^+rhn \in [\mathrm{R}^-slk; \max(\mathrm{R}^+tpm, tm\mathrm{R}^+tpm)]$.

• Induction step: Suppose the result valid for n, i.e. for all k, m, if $n \in [k; m]$ then $\mathrm{R}rhn$, R^-sln, R^+tpn, R^-shn, R^+thn, R^-rhn, $\mathrm{R}^+rhn \in [\mathrm{R}^-slk; \mathrm{R}^+tpm]$. Let us prove the result for $n+1$, i.e. for all k, m, if $n+1 \in [k; m]$ then $\mathrm{R}rh(n+1)$, $\mathrm{R}^-sl(n+1)$, $\mathrm{R}^+tp(n+1)$, $\mathrm{R}^-sh(n+1)$, $\mathrm{R}^+th(n+1)$, $\mathrm{R}^-rh(n+1)$, $\mathrm{R}^+rh(n+1) \in [\mathrm{R}^-slk; \mathrm{R}^+tpm]$. Obviously $m \neq 0$, which implies $\mathrm{R}^+tpm = \max(\mathrm{R}^+tp(m-1), t(m-1)\mathrm{R}^+tp(m-1))$. We study two cases: (1) $k=0$ and (2) $k \neq 0$.

(1) If $k = 0$ then $\mathrm{R}^-slk := \min(l, s00)$ and $n+1 \in [k; m] \to n \in [k; m-1]$. Let us prove that $\mathrm{R}rh(n+1) := rn\mathrm{R}rhn \in [\min(l, s00); \max(\mathrm{R}^+tp(m-1), t(m-1)\mathrm{R}^+tp(m-1))]$. Since $n \in [k; m-1]$, by induction hypothesis we know that $\mathrm{R}rhn \in [\mathrm{R}^-slk; \mathrm{R}^+tp(m-1)]$, so $\mathrm{R}rhn \in [0; \mathrm{R}^+tp(m-1)]$. The last assertion, together with $r \in [s; t]$ and $n \in [0; m-1]$ implies $rn\mathrm{R}rhn \in [s00; t(m-1)\mathrm{R}^+tp(m-1)]$. Applying propositions 4.1 and 4.2 (e), since $l \in [l; p]$ and $\mathrm{R}^+tp(m-1) \in [0; \mathrm{R}^+tp(m-1)]$, we have $rn\mathrm{R}rhn \in [\min(s00, l); \max(t(m-1)\mathrm{R}^+tp(m-1), \mathrm{R}^+tp(m-1))]$. By assertions (d) of the same propositions we obtain $rn\mathrm{R}rhn \in [\min(l, s00); \max(\mathrm{R}^+tp(m-1), t(m-1)\mathrm{R}^+tp(m-1))]$.

In order to prove that $\mathrm{R}^-sl(n+1) := \min(\mathrm{R}^-sln, sn\mathrm{R}^-sln) \in [\min(l, s00); \max(\mathrm{R}^+tp(m-1), t(m-1)\mathrm{R}^+tp(m-1))]$ we start noticing that, by induction hypothesis, we know that $\mathrm{R}^-sln \in [\min(l, s00), \mathrm{R}^+tp(m-1)]$. Since $t(m-1)\mathrm{R}^+tp(m-1) \in [0; t(m-1)\mathrm{R}^+tp(m-1)]$ we have $\mathrm{R}^-sln \in [\min(l, s00), \max(\mathrm{R}^+tp(m-1), t(m-1)\mathrm{R}^+tp(m-1))]$. From $sn\mathrm{R}^-sln \in [s00; t(m-1)\mathrm{R}^+tp(m-1)]$, knowing that $l \in [l; p]$ and $\mathrm{R}^+tp(m-1) \in [0; \mathrm{R}^+tp(m-1)]$, we have $sn\mathrm{R}^-sln \in [\min(l, s00); \max(\mathrm{R}^+tp(m-1), t(m-1)\mathrm{R}^+tp(m-1))]$. So, by Proposition 4.2 (a), $\min(\mathrm{R}^-sln, sn\mathrm{R}^-sln) \in [\min(l, s00); \max(\mathrm{R}^+tp(m-1), t(m-1)\mathrm{R}^+tp(m-1))]$.

$\mathrm{R}^-sh(n+1)$ and $\mathrm{R}^-rh(n+1)$ can be checked in a similar way.

Let us prove that $\mathrm{R}^+tp(n+1) := \max(\mathrm{R}^+tpn, tn\mathrm{R}^+tpn) \in [\mathrm{R}^-slk; \max(\mathrm{R}^+tp(m-1), t(m-1)\mathrm{R}^+tp(m-1))]$.

Since $n \in [k; m-1]$, by induction hypothesis, we have that $\mathrm{R}^+tpn \in [\mathrm{R}^-slk; \mathrm{R}^+tp(m-1)]$. Noticing that $t(m-1)\mathrm{R}^+tp(m-1) \in [0; t(m-1)\mathrm{R}^+tp(m-1)]$, we have $\mathrm{R}^+tpn \in [\mathrm{R}^-slk; \max(\mathrm{R}^+tp(m-1), t(m-1)\mathrm{R}^+tp(m-1))]$. And from $tn\mathrm{R}^+tpn \in [0; t(m-1)\mathrm{R}^+tp(m-1)]$ and $\mathrm{R}^+tp(m-1) \in [0; \mathrm{R}^+tp(m-1)]$ we know that $tn\mathrm{R}^+tpn \in [0; \max(\mathrm{R}^+tp(m-1), t(m-1)\mathrm{R}^+tp(m-1))]$. So, by Proposition 4.1 (b), we conclude that $\max(\mathrm{R}^+tpn, tn\mathrm{R}^+tpn) \in [\mathrm{R}^-slk; \max(\mathrm{R}^+tp(m-1), t(m-1)\mathrm{R}^+tp(m-1))]$.

$\mathrm{R}^+th(n+1)$ and $\mathrm{R}^+rh(n+1)$ can be studied with an analogous strategy.

(2) If $k \neq 0$ then $\mathrm{R}^-slk := \min(\mathrm{R}^-sl(k-1), s(k-1)\mathrm{R}^-sl(k-1))$ and $n+1 \in [k; m] \to n \in [k-1; m-1]$. Let us prove that $\mathrm{R}rh(n+1) := rn\mathrm{R}rhn \in [\min(\mathrm{R}^-sl(k-1), s(k-1)\mathrm{R}^-sl(k-1)); \max(\mathrm{R}^+tp(m-1), t(m-1)\mathrm{R}^+tp(m-1))]$.

Since $n \in [k-1; m-1]$, by induction hypothesis, we have $\mathrm{R}rhn \in [\mathrm{R}^-sl(k-1); \mathrm{R}^+tp(m-1)]$. The last assertion together with $r \in [s; t]$ and $n \in [k-1; m-1]$ allows us to establish $rn\mathrm{R}rhn \in [s(k-1)\mathrm{R}^-sl(k-1); t(m-1)\mathrm{R}^+tp(m-1)]$. Since we also know that $\mathrm{R}^-sl(k-1) \in [\mathrm{R}^-sl(k-1); \mathrm{R}^+tp(m-1)]$ and $\mathrm{R}^+tp(m-1) \in [0; \mathrm{R}^+tp(m-1)]$ we can conclude that $rn\mathrm{R}rhn \in [\min(\mathrm{R}^-sl(k-1), s(k-1)\mathrm{R}^-sl(k-1)); \max(\mathrm{R}^+tp(m-1), t(m-1)\mathrm{R}^+tp(m-1))]$.

Next we prove that $\mathrm{R}^-sl(n+1) := \min(\mathrm{R}^-sln, sn\mathrm{R}^-sln) \in [\min(\mathrm{R}^-sl(k-1), s(k-1)\mathrm{R}^-sl(k-1)); \max(\mathrm{R}^+tp(m-1), t(m-1)\mathrm{R}^+tp(m-1))]$.

By induction hypothesis, we have $\mathrm{R}^-sln \in [\mathrm{R}^-sl(k-1); \mathrm{R}^+tp(m-1)]$. Also knowing that $s(k-1)\mathrm{R}^-sl(k-1) \in [s(k-1)\mathrm{R}^-sl(k-1); t(m-1)\mathrm{R}^+tp(m-1)]$ and $t(m-1)\mathrm{R}^+tp(m-1) \in [0; t(m-1)\mathrm{R}^+tp(m-1)]$, we obtain $\mathrm{R}^-sln \in [\min(\mathrm{R}^-sl(k-1), s(k-1)\mathrm{R}^-sl(k-1)); \max(\mathrm{R}^+tp(m-1), t(m-1)\mathrm{R}^+tp(m-1))]$. Since $sn\mathrm{R}^-sln \in [s(k-1)\mathrm{R}^-sl(k-1); t(m-1)\mathrm{R}^+tp(m-1)]$, easily we can observe that also $sn\mathrm{R}^-sln \in [\min(\mathrm{R}^-sl(k-1), s(k-1)\mathrm{R}^-sl(k-1)); \max(\mathrm{R}^+tp(m-1), t(m-1)\mathrm{R}^+tp(m-1))]$. So, by Proposition 4.2 (a), we conclude that

$\min(\mathrm{R}^{-}sln, sn\mathrm{R}^{-}sln) \in [\min(\mathrm{R}^{-}sl(k-1), s(k-1)\mathrm{R}^{-}sl(k-1)); \max(\mathrm{R}^{+}tp(m-1), t(m-1)\mathrm{R}^{+}tp(m-1))]$.

The cases $\mathrm{R}^{-}sh(n+1)$ and $\mathrm{R}^{-}rh(n+1)$ are similar.

Finally, we just prove that $\mathrm{R}^{+}tp(n+1) := \max(\mathrm{R}^{+}tpn, tn\mathrm{R}^{+}tpn) \in [\min(\mathrm{R}^{-}sl(k-1), s(k-1)\mathrm{R}^{-}sl(k-1)); \max(\mathrm{R}^{+}tp(m-1), t(m-1)\mathrm{R}^{+}tp(m-1))]$, since the cases $\mathrm{R}^{+}th(n+1)$ and $\mathrm{R}^{+}rh(n+1)$ are similar.

By induction hypothesis we know that $\mathrm{R}^{+}tpn \in [\mathrm{R}^{-}sl(k-1); \mathrm{R}^{+}tp(m-1)]$. With an argument already used it is possible to see that $\mathrm{R}^{+}tpn \in [\min(\mathrm{R}^{-}sl(k-1), s(k-1)\mathrm{R}^{-}sl(k-1)); \max(\mathrm{R}^{+}tp(m-1), t(m-1)\mathrm{R}^{+}tp(m-1))]$. From $tn\mathrm{R}^{+}tpn \in [s(k-1)\mathrm{R}^{-}sl(k-1); t(m-1)\mathrm{R}^{+}tp(m-1)]$ we also can see that $tn\mathrm{R}^{+}tpn \in [\min(\mathrm{R}^{-}sl(k-1), s(k-1)\mathrm{R}^{-}sl(k-1)); \max(\mathrm{R}^{+}tp(m-1), t(m-1)\mathrm{R}^{+}tp(m-1))]$. Applying Proposition 4.1 (a), we conclude $\max(\mathrm{R}^{+}tpn, tn\mathrm{R}^{+}tpn) \in [\min(\mathrm{R}^{-}sl(k-1), s(k-1)\mathrm{R}^{-}sl(k-1)); \max(\mathrm{R}^{+}tp(m-1), t(m-1)\mathrm{R}^{+}tp(m-1))]$.

□

Since the other arithmetical constants, 0^0 and $\mathrm{S}^{0\to 0}$ are self-confined, i.e. $0 \in [0;0]$ and $\mathrm{S} \in [\mathrm{S};\mathrm{S}]$, the Heyting arithmetic in all finite types HA^{ω} is an example of a confined theory.

5 Applications and Conclusions

In this section we outline some of the advantages of working with intervals, rather than just upper bounds, and prove that Heyting arithmetic is a confined theory.

First, one should notice that the usual bounded quantifiers such as $\forall n \leq b\, A(n)$ can be viewed as a bounded quantification of the form $\forall n \in [0, b]\, A(n)$, so that nothing is lost by working with intervals. One of the gains, however, is the ability to more precisely capture theorems which also involve lower bounds. Consider, for instance, Bertrand's Postulate[2]

$$\forall n(n \geq 3 \to \exists p \in [n, 2n-2]\, \mathsf{Prime}(p)).$$

In this case the existential quantification over p would be considered truly bounded, since it is bounded both from above and from bellow, whereas the universal quantification is unbounded from above (but bounded from below). The confined realizability allows for the propagation of both upper and lower bounds. This seems to be particularly useful in number theory (cf. study on distribution of primes [5]) where results commonly rely on a number being big enough, which is precisely the information contained in lower bounds.

But rather than just providing more accurate "bounds", the confined interpretation can be used to deal with data types that do not have a least element, such as integers, rationals and real numbers. In these cases, a simple upper bound might not convey useful information, when the number in question is negative, for instance. A possible solution would be to work with upper bounds on the absolute value of the number, i.e. $|p^{\mathbb{Q}}|$. But even that information could be rather weak if working with large numbers.

The approach we suggest is to add integers ($\mathbb{Z}$) and rationals ($\mathbb{Q}$) as basic data types. The $\in^*$ relation can be naturally extended as

$$n^{\mathbb{Z}} \in^*_{\mathbb{Z}} [k^{\mathbb{Z}}; l^{\mathbb{Z}}] \;:=\; k \leq_{\mathbb{Z}} n \leq_{\mathbb{Z}} l$$

$$p^{\mathbb{Q}} \in^*_{\mathbb{Q}} [q^{\mathbb{Q}}; r^{\mathbb{Q}}] \;:=\; q \leq_{\mathbb{Q}} p \leq_{\mathbb{Q}} r.$$

And, given the representation of reals as Cauchy sequences of rationals (let us write $\mathbb{R}$ as an abbreviation for $\mathbb{N} \to \mathbb{Q}$), we obtain the following derivable relation

$$x^{\mathbb{R}} \in^*_{\mathbb{R}} [y^{\mathbb{R}}; z^{\mathbb{R}}] \;\equiv\; \forall [n, m] \forall i \in [n, m](xi, yi, zi \in [yn, zm])$$

[2] For every integer n bigger than 3 there is a prime number between n and $2n-2$.

using Definition 1.9. I.e. real numbers are bounded by above and below by a monotone pair of sequences of rational numbers (not necessarily Cauchy).

In this paper we have not touched the issue of characterising the confined interpretation. The main reason for this is that the characterisation principles are similar to those of the bounded realizability interpretation (mainly when working with natural numbers $\mathbb{N}$). As argued above, in the context of natural numbers the confined interpretation provides a refinement of the witnessing information already provided by the bounded realizability, but both interpretations have the same strength regarding the class of interpretable formulas.

It would be also interesting to investigate a confined variant of the Dialectica interpretation, similar to the bounded Dialectica [4]. We have chosen to start our study looking at the realizability interpretation, given its simplicity. We strongly suspect that interesting applications might only come out from a confined Dialectica interpretation, since that would allow one to cover proofs which involve a mild amount of classical logic (Markov principle).

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