On Krivine's Realizability Interpretation of Classical Second-Order Arithmetic

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Abstract

This article investigates Krivine's realizability interpretation of classical second-order arithmetic and its recent extension handling countable choice. We will start by presenting a two-step interpretation which first eliminates classical logic via a negative translation and then applies standard realizability interpretation. We then argue that a slight variant of Krivine's interpretation corresponds to this two-step interpretation. This variant can be viewed as the continuation passing style variant of Krivine's original interpretation, and as such uses standard λ-terms and avoids the use of new continuation constants in the interpretation of classical logic.

1 Introduction

This article investigates Krivine's realizability interpretation of classical second-order arithmetic [7], and its recent extension handling also countable choice [8]. Krivine's interpretation makes use of extensions of untyped λ-calculus with continuation constructs (for the realization of classical logic) and a new operation labelled *quote* (for the interpretation of countable choice) which associates indices to closed terms.

Our goal in this paper is to study the connection between Krivine's interpretation of countable choice and the recent modified realizability interpretation due to U. Berger and the first author [3] (based on [2]) which makes use of *bar recursion*. Such study has been suggested in [8]. It is clear, however, that a direct inter-definability result cannot hold, since Krivine's 'quote' works even when all functions on natural numbers in the model are computable whereas bar recursion requires the presence of noncomputable sequences of natural numbers because otherwise the Kleene tree[1] would be well founded.

Therefore, rather than comparing the functionals involved, we will focus on the corresponding realizability interpretations. The main problem in relating the

[1] See [11], chapter 4, section 7.6.

$$
\frac{\Gamma, A \vdash_{\mathsf{c2}} B}{\Gamma \vdash_{\mathsf{c2}} A \to B} \qquad \frac{\Gamma \vdash_{\mathsf{c2}} A}{\Gamma \vdash_{\mathsf{c2}} \forall x A} \qquad \frac{\Gamma \vdash_{\mathsf{c2}} A}{\Gamma \vdash_{\mathsf{c2}} \forall X} \qquad \Gamma \vdash_{\mathsf{c2}} A_i
$$

$$
\frac{\Gamma \vdash_{\mathsf{c2}} A \quad \Gamma \vdash_{\mathsf{c2}} A \to B}{\Gamma \vdash_{\mathsf{c2}} B} \qquad \frac{\Gamma \vdash_{\mathsf{c2}} \forall x A}{\Gamma \vdash_{\mathsf{c2}} A[s/x]} \qquad \frac{\Gamma \vdash_{\mathsf{c2}} \forall X A}{\Gamma \vdash_{\mathsf{c2}} A[B/X]} \qquad \frac{\Gamma \vdash_{\mathsf{c2}} (A \to B) \to A}{\Gamma \vdash_{\mathsf{c2}} A}
$$

Figure 1: Classical Second-order Arithmetic (System c2)

two realizability interpretations of countable choice is the use of different language paradigms, namely *functionals of arbitrary finite types* is used in [3] while Krivine [8] works in a language for *objects and sets*. What we propose in this paper is to view (a slight variant of) Krivine's classical realizability interpretation as a combination of a negative translation (in the style of Gödel's negative translation [5]) followed by a standard intuitionistic realizability interpretation. That is the line which has been followed in [3], in the setting of finite types.

The presentation of Krivine's classical realizability interpretation given here differs slightly form the one presented in [8]. The main change is that we will be using standard λ-terms, avoiding the use of new continuation constants in the interpretation of classical logic.

The focus of this paper is on the relation between the two *realizability interpretations*. The problem of relating the functionals used to witness the interpretations, namely Krivine's *quote* and *bar recursion*, remains open.

2 Classical Second-Order Arithmetic

As in Krivine [8], we will use the axiomatisation of *classical second-order arithmetic* described in Figure 1. The usual restrictions apply in the universal introduction rules, that x and X are not free in $\Gamma \equiv A_1, \ldots, A_n$. We assume that the language of c2 contains function symbols f_ϕ for each recursive function $\phi : \mathbb{N}^k \to \mathbb{N}$. In particular, we have constants 0 (zero) and 1+ (successor). Equality $x = y$ is defined as $\forall X(X(x) \to X(y))$. Besides the logical rules shown in Figure 1, the system c2 contains the usual universal axioms for zero and successor, and the defining equations for all recursive functions. Finally, the system c2 contains the *induction axiom* $\forall x\, \mathsf{Int}(x)$, where

$$
\mathsf{Int}(x) :\equiv \forall X(\forall y(X(y) \to X(1 + y)) \to (X(0) \to X(x))).
$$

When writing formulas of the classical system $\mathsf{c2}$, the following abbreviations will be used:

$$\bot \equiv \forall X.X \qquad \neg A \equiv A \to \bot \qquad \exists^c xA \equiv \neg\forall x\neg A$$

Provability in classical second-order arithmetic will be denoted by the subscript $\mathsf{c2}$ in the provability sign, i.e. $\Gamma \vdash_{\mathsf{c2}} A$. For conciseness, we will normally present derivations in the system $\mathsf{c2}$ in a natural deduction style.

In the following we consider λ-terms with the usual β-reduction $(\lambda x.t)s \rightsquigarrow t[s/x]$. We write Λ for the set of closed λ-terms. Instead of Krivine's "stacks" of closed λ-terms we will use finite sequences based on the following coding of pairs and projections. We write $\langle t, s\rangle$ as an abbreviation for $\lambda u.uts$ and for $i = 0, 1$ we write $(t)_i$ for $t(\lambda x_0\lambda x_1.x_i)$ which coding validates the reduction $(\langle t_0, t_1\rangle)_i \rightsquigarrow t_i$. We write $S \to T$ for

$$\{t \in \Lambda \ : \ \forall s{\in}S\,(ts \in T)\}.$$

and $S \times T$ for

$$\{\langle t, s\rangle \ : \ s{\in}S \text{ and } t{\in}T\}.$$

whenever $S, T \subseteq \Lambda$. A subset T of Λ is called *saturated* if it is closed under β-expansion, i.e. if $t \in T$ and $s \rightsquigarrow t$ then $s \in T$. Let $\mathcal{T} \subseteq \mathcal{P}(\Lambda)$ denote the set of saturated subsets of Λ. It is clear that if T is saturated then $S \to T$ is also saturated; and if S, T are saturated then $S \times T$ is saturated.

3 Relativisation to the Integers

The induction axiom of $\mathsf{c2}$ states that every element of the model is an integer. As argued in [8], this is too strong to be interpreted directly. Therefore, before we proceed with the realizability interpretation of $\mathsf{c2}$, we will need an initial relativisation of the first-order quantifiers of $\mathsf{c2}$ to the integers. The relativisation simply replaces first-order universal quantifiers $\forall x \ldots$ by $\forall x(\mathsf{Int}(x) \to \ldots)$. Let us denote by $(A)^{\mathsf{Int}}$ the relativisation of the first-order quantifiers of A to the integers. Moreover, let $\mathsf{c2r}$ denote the system $\mathsf{c2}$ where the defining equations for the recursive function are relativised, and the induction axiom is replaced by the axiom schema

$$\Delta_f \quad : \quad \forall\vec{y}(\mathsf{Int}(\vec{y}) \to \mathsf{Int}(f\vec{y}))$$

for each function symbol f of the language. Notice that the integer relativisation of the induction axiom, $\forall x(\mathsf{Int}(x) \to (\mathsf{Int})^{\mathsf{Int}}(x))$, then becomes provable. I.e. assuming

$\mathsf{Int}(x)$ and $X(0)$ and $\mathsf{Int}(y), X(y) \vdash X(1+y)$ we can derive $X(x)$. Making use of conjunctions for conciseness, the proof is sketched below[2]

$$\dfrac{\mathsf{Int}(0)\wedge X(0) \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\mathsf{Int}(y)\wedge X(y)]_{(1)}}{\mathsf{Int}(y)}}{\mathsf{Int}(1+y)} \qquad \dfrac{[\mathsf{Int}(y)\wedge X(y)]_{(1)} \quad \mathsf{Int}(y)\wedge X(y)\vdash X(1+y)}{X(1+y)}}{\mathsf{Int}(1+y)\wedge X(1+y)}}{\mathsf{Int}(y)\wedge X(y)\vdash \mathsf{Int}(1+y)\wedge X(1+y)}\,(1)}{\dfrac{\mathsf{Int}(x)\wedge X(x)}{X(x)}}\,(\text{by } \mathsf{Int}(x))$$

We have also used that both $\mathsf{Int}(0)$ and $\mathsf{Int}(y) \to \mathsf{Int}(1+y)$ are easily derivable.

Lemma 3.1 *Let $\vec{x}$ be all the first-order free-variables of Γ, A. If $\Gamma \vdash_{\mathsf{c2}} A$ then* $(\Gamma)^{\mathsf{Int}}, \mathsf{Int}(\vec{x}) \vdash_{\mathsf{c2r}} (A)^{\mathsf{Int}}$.

Proof. The lemma follows from a simple induction on the derivation $\Gamma \vdash_{\mathsf{c2}} A$. The only non-trivial case is in the elimination of the first-order universal quantifier:

$$\dfrac{\Gamma \vdash_{\mathsf{c2}} \forall x A(x)}{\Gamma \vdash_{\mathsf{c2}} A(s[y])}$$

The Int translation of the premise only implies the translation of the conclusion if we have a derivation of $(\Gamma)^{\mathsf{Int}}, \mathsf{Int}(y) \vdash \mathsf{Int}(s[y])$, which follows from Δ. As argued above, the integer relativisation of the induction axiom is provable in $\mathsf{c2r}$. $\square$

4 A Negative Translation

Having dealt with the issue of the induction axiom, we now show how classical logic can be eliminated via a form of negative translation (cf. [4, 5]). This will lead us to an intuitionistic version of $\mathsf{c2r}$ which we will call $\mathsf{i2r}$. This is based on the standard intuitionistic system $\mathsf{i2}$ whose logical rules are described in Figure 2 plus axioms for the constants 0 and 1+, function symbols and equations for all recursive functions, and the axioms Δ_f of the previous section.

Definition 4.1 (P-translation) *Let P be a new (uninterpreted) predicate constant. Consider the following translation of formulas of* $\mathsf{c2}$ *into formulas of* $\mathsf{i2}$*:*

[2] Thanks to Thierry Coquand for bringing this proof to our attention.

$\dfrac{\Gamma \vdash_{\mathsf{i2}} A_0 \wedge A_1}{\Gamma \vdash_{\mathsf{i2}} A_{i \in \{0,1\}}}$	$\dfrac{\Gamma \vdash_{\mathsf{i2}} A_0 \quad \Gamma \vdash_{\mathsf{i2}} A_1}{\Gamma \vdash_{\mathsf{i2}} A_0 \wedge A_1}$	$\dfrac{\Gamma, A \vdash_{\mathsf{i2}} B}{\Gamma \vdash_{\mathsf{i2}} A \to B}$	$\dfrac{\Gamma \vdash_{\mathsf{i2}} A \to B \quad \Gamma \vdash_{\mathsf{i2}} A}{\Gamma \vdash_{\mathsf{i2}} B}$
$\dfrac{\Gamma, A \vdash_{\mathsf{i2}} B}{\Gamma, \exists x A \vdash_{\mathsf{i2}} B}$	$\dfrac{\Gamma, \exists x A \vdash_{\mathsf{i2}} B}{\Gamma, A[s/x] \vdash_{\mathsf{i2}} B}$	$\dfrac{\Gamma, A \vdash_{\mathsf{i2}} B}{\Gamma, \exists X A \vdash_{\mathsf{i2}} B}$	$\dfrac{\Gamma, \exists X A \vdash_{\mathsf{i2}} B}{\Gamma, A[C/X] \vdash_{\mathsf{i2}} B}$
$\dfrac{\Gamma \vdash_{\mathsf{i2}} A}{\Gamma \vdash_{\mathsf{i2}} \forall x A}$	$\dfrac{\Gamma \vdash_{\mathsf{i2}} \forall x A}{\Gamma \vdash_{\mathsf{i2}} A[s/x]}$	$\dfrac{\Gamma \vdash_{\mathsf{i2}} A}{\Gamma \vdash_{\mathsf{i2}} \forall X A}$	$\dfrac{\Gamma \vdash_{\mathsf{i2}} \forall X A}{\Gamma \vdash_{\mathsf{i2}} A[C/X]}$

Figure 2: Intuitionistic Second-order Arithmetic (System i2)

$$
\begin{aligned}
(X)_P &:\equiv X \\
(A \to B)_P &:\equiv ((A)_P \to P) \wedge (B)_P \\
(\forall x A)_P &:\equiv \exists x (A)_P \\
(\forall X A)_P &:\equiv \exists X (A)_P
\end{aligned}
$$

where X is a second-order variable.

Intuitively, the translation is a generalised negative translation, in which a fixed predicate constant P takes the role of $\bot$. Note that, in the present context, we need not double negate prime formulas because there is no logical difference between quantifying over single negated predicates and double negated predicates (and $(A)_P \to P$ is intuitionistically equivalent to the formula obtained from A by replacing every occurrence of X by $X \to P$). As discussed in [1] and [10], such negative translation can also be used to analyse Shoenfield's Dialectica interpretation of classical logic.

Lemma 4.2 (Storage operator) *The following equivalence is derivable in* i2r

$$((\mathsf{Int}(y))_P \to P) \leftrightarrow ((\mathsf{Int}(y) \to P) \to P)$$

Proof. Consider for instance the forward implication. That is equivalent to

$$(\mathsf{Int}(y) \to P) \to (((\mathsf{Int}(y))_P \to P) \to P).$$

Assuming $\mathsf{Int}(y) \to P$ and $(\mathsf{Int}(y))_P \to P$ we must derive P. For conciseness let us write $\neg A$ for $A \to P$. Note that $(\mathsf{Int}(y))_P \to P$ is equivalent to

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[((A)_P \to P) \wedge (B)_P]_{(1)}}{(B)_P} \qquad \dfrac{\dfrac{[(A)_P]_{(2)} \qquad \dfrac{[((A)_P \to P) \wedge (B)_P]_{(1)}}{(A)_P \to P}}{P}}{(B)_P \to P}}{P}\;(1)}{(((A)_P \to P) \wedge (B)_P) \to P)}\;(\mathrm{def})}{(A \to B)_P \to P} \qquad [(+)]_{(3)}}{(A)_P \to P}
$$

$$
\dfrac{\dfrac{[(A)_P]_{(2)} \qquad (A)_P \to P}{P}}{(A)_P \to P}\;(2)
$$

Figure 3: Derivability of the negative translation of Peirce's law

$$\forall X(\forall k(\neg Xk \to \neg X(1+k)) \to (\neg X0 \to \neg Xy)))$$

which in turn implies

$$(+)\quad \forall k(\neg\neg\mathsf{Int}(k) \to \neg\neg\mathsf{Int}(1+k)) \to (\neg\neg\mathsf{Int}(0) \to \neg\neg\mathsf{Int}(y)))$$

instantiating X with $\neg\mathsf{Int}$. A sketch of whole derivation can be presented as:

$$
\dfrac{\neg\mathsf{Int}(y) \qquad \dfrac{\neg\neg\mathsf{Int}(0) \qquad \dfrac{\neg\neg\mathsf{Int}(k) \to \neg\neg\mathsf{Int}(1+k)}{\neg\neg\mathsf{Int}(0) \to \neg\neg\mathsf{Int}(y)}\;(+)}{\neg\neg\mathsf{Int}(y)}}{P}
$$

using that $\neg\neg\mathsf{Int}(0)$ and $\neg\neg\mathsf{Int}(k) \to \neg\neg\mathsf{Int}(1+k)$ are derivable. This is the derivation which gives rise to the so-called storage operator [6]. The other implication is even simpler. $\square$

For a sequence of formulas $\Gamma \equiv A_1, \ldots, A_n$, let $(\Gamma)_P \to P$ be an abbreviation for the sequence $(A_1)_P \to P, \ldots, (A_n)_P \to P$.

Theorem 4.3 *If* $\Gamma \vdash_{\mathsf{c2r}} A$ *then* $(\Gamma)_P \to P \vdash_{\mathsf{i2r}} (A)_P \to P$.

Proof. The proof proceeds by induction on the classical derivation $\Gamma \vdash_{\mathsf{c2r}} A$. Every instance of an axiom $\Gamma \vdash A_i$ translates into the simple derivation of $(\Gamma)_P \to$

$P, (A_i)_P \vdash_{\mathsf{i2r}} P$. For the rest of the proof we omit the subscript i2r in the provability sign.

Implication. In this case, for simplicity, let us ignore the context Γ. The implication elimination is treated as

$$\dfrac{\dfrac{\vdash (A \to B)_P \to P}{\vdash (((A)_P \to P) \wedge (B)_P) \to P}\,(\text{def}) \quad \dfrac{((A)_P \to P) \quad (B)_P \vdash (B)_P}{(B)_P \vdash ((A)_P \to P) \wedge (B)_P}}{\dfrac{(B)_P \vdash P}{\vdash (B)_P \to P}\,(1)}$$

and for implication introduction

$$\dfrac{\dfrac{\dfrac{\dfrac{(A)_P \to P \vdash (B)_P \to P \quad (B)_P \vdash (B)_P}{(A)_P \to P, (B)_P \vdash P}}{((A)_P \to P) \wedge (B)_P \vdash P}}{(A \to B)_P \vdash P}\,(\text{def})}{\vdash (A \to B)_P \to P}$$

Peirce's law. The the derivation of the translation of Peirce's law is shown in Figure 3, where (+) is $((A \to B)_P \to P) \wedge (A)_P \to P$.

Universal quantifiers. The rules for the second-order universal quantifier are translated into the rules for existential quantifiers:

$$\dfrac{\dfrac{\dfrac{(\Gamma)_P \to P \vdash (A(X))_P \to P}{(\Gamma)_P \to P, (A(X))_P \vdash P}}{(\Gamma)_P \to P, \exists X(A(X))_P \vdash P}}{(\Gamma)_P \to P \vdash (\forall X A(X))_P \to P} \qquad \dfrac{\dfrac{\dfrac{\dfrac{(\Gamma)_P \to P \vdash (\forall X A(X))_P \to P}{(\Gamma)_P \to P, \exists X(A(X))_P \vdash P}}{(\Gamma)_P \to P, (A(X))_P[(B)_P/X] \vdash P}}{(\Gamma)_P \to P, (A(B))_P \vdash P}}{(\Gamma)_P \to P \vdash (A(B))_P \to P}$$

using that $(A(B))_P \equiv (A(X))_P[(B)_P/X]$. Similarly one proceeds for first-order quantifiers.

Axioms Δ_f. We must show that Δ_f implies its P-translation. This follows from the fact that Δ_f implies

$$((\mathsf{Int}(y) \to P) \to P) \to (\mathsf{Int}(fy) \to P) \to P$$

and this implies $(\Delta_f)_P \to P$, by Lemma 4.2. $\square$

Note the fundamental use of conjunction introduction and elimination in the treatment of implication. As we will see, Krivine's classical realizability interpretation combines negative translation with a standard realizability interpretation into one package, which makes it more difficult to produce realizers, since at each λ-abstraction or term application one has to take into consideration the pair formulations and projections shown in the proof above. This is dealt with in a neat way by Krivine's abstract machine, via the use of stacks, but the price to pay is that realizers loose some of their semantic meaning, and become more "machine oriented".

5 Intuitionistic Realizability

We present now an intuitionistic realizability interpretation of i2r (which is also used by Krivine in [6]) and show that when combined with the P-translation of Section 4 and the integer relativisation of Section 3 we obtain a CPS translation of Krivine's classical realizability interpretation of second-order logic.

Let $\mathcal{M}$ be a term structure for second-order logic. The interpretation associates each formula A with a saturated set (i.e. an element of $\mathcal{T}$). In order to give an interpretation for a formula A with free-variables, we will make use of an *environment* ρ associating each first-order free-variable with an element of $\mathcal{M}$ and second-order free-variables with mappings from M^k to $\mathcal{T}$.

Let $\vec{x}$ consist of all free-variables (both first and second-order variables) of A, and let ρ be an assignment for the free-variables as described above. The set of realizers of $A(\vec{x})$ with respect to ρ, written $\{A(\vec{x})\}_\rho$, is defined inductively as follows: For the propositional part, the interpretation is given as:

$$\begin{array}{lcl} \{X(t)\}_\rho & :\equiv & \rho(X)(\rho(t)) \\ \{A \wedge B\}_\rho & :\equiv & \{A\}_\rho \times \{B\}_\rho \\ \{A \to B\}_\rho & :\equiv & \{A\}_\rho \to \{B\}_\rho. \end{array}$$

The interpretation of the first-order quantifiers is done as

$$\begin{array}{lcl} \{\exists x A\}_\rho & :\equiv & \bigcup_{a \in \mathcal{M}} \{A\}_{\rho[a/x]} \\ \{\forall x A\}_\rho & :\equiv & \bigcap_{a \in \mathcal{M}} \{A\}_{\rho[a/x]} \end{array}$$

while, for the second-order quantifier we set

$$\begin{array}{lcl} \{\exists X A\}_\rho & :\equiv & \bigcup_{S \in M^k \to \mathcal{T}} \{A\}_{\rho[S/X]} \\ \{\forall X A\}_\rho & :\equiv & \bigcap_{S \in M^k \to \mathcal{T}} \{A\}_{\rho[S/X]}. \end{array}$$

Theorem 5.1 (Soundness) *If* $\Gamma \vdash_{\mathsf{i2r}} A$ *then there exists a term* $t[\vec{x}]$ *such that, for all* ρ *and* $\vec{s} \in \{\Gamma\}_\rho$ *we have* $t[\vec{s}] \in \{A\}_\rho$.

Proof. For conciseness we omit the universal quantifications over ρ and $\vec{s}$. We can transform the intuitionistic system into a "typing system" for λ-terms.

Axioms. For the axioms we have

$$\vec{s} \in \{\Gamma\} \;\Rightarrow\; s_i \in \{A_i\}$$

Arrow rules. In the case of the implication introduction and elimination:

$$\frac{\vec{s} \in \{\Gamma\}, r \in \{A\} \;\Rightarrow\; t[\vec{s}, r] \in \{B\}}{\vec{s} \in \{\Gamma\} \;\Rightarrow\; \lambda x.t[\vec{s}, x] \in \{A \to B\}} \qquad \frac{\vec{s} \in \{\Gamma\} \;\Rightarrow\; r[\vec{s}] \in \{A\} \quad \vec{s} \in \{\Gamma\} \;\Rightarrow\; t[\vec{s}] \in \{A \to B\}}{\vec{s} \in \{\Gamma\} \;\Rightarrow\; t[\vec{s}]r[\vec{s}] \in \{B\}}$$

First-order existential quantifier. Given the definition $\{\exists x A\}_\rho :\equiv \bigcup_{a \in \mathcal{M}} \{A\}_{\rho[a/x]}$ we get

$$\frac{\vec{s} \in \{\Gamma\}, r \in \{A(y)\} \;\Rightarrow\; t[\vec{s}, r] \in \{B\}}{\vec{s} \in \{\Gamma\}, r \in \{\exists y A(y)\} \;\Rightarrow\; t[\vec{s}, r] \in \{B\}} \qquad \frac{\vec{s} \in \{\Gamma\}, r \in \{\exists y A(y)\} \;\Rightarrow\; t[\vec{s}, r] \in \{B\}}{\vec{s} \in \{\Gamma\}, r \in \{A(s)\} \;\Rightarrow\; t[\vec{s}, r] \in \{B\}}$$

Second-order existential quantifier. Given the definition $\{\exists X A\}_\rho :\equiv \bigcup_{S \in M^k \to \mathcal{T}} \{A\}_{\rho[S/X]}$ we get

$$\frac{\vec{s} \in \{\Gamma\}, r \in \{A(X(s))\} \;\Rightarrow\; t[\vec{s}, r] \in \{B\}}{\vec{s} \in \{\Gamma\}, r \in \{\exists X A(X(s))\} \;\Rightarrow\; t[\vec{s}, r] \in \{B\}} \qquad \frac{\vec{s} \in \{\Gamma\}, r \in \{\exists X A(X(s)))\} \;\Rightarrow\; t[\vec{s}, r] \in \{B\}}{\vec{s} \in \{\Gamma\}, r \in \{A(C(s))\} \;\Rightarrow\; t[\vec{s}, r] \in \{B\}}$$

Axioms Δ_f. Let ϕ be the λ-term representing the recursive function f. It is easy to check that ϕ realizes Δ_f. $\Box$

6 Krivine's Classical Realizability Interpretation

In this section we will present a variant of Krivine's classical realizability interpretation which corresponds to the combination of the P-translation of Section 4 and the intuitionistic realizability of Section 5. We borrow most of the notation from [8]. Recall that a structure $\mathcal{M} = (M, f, \ldots, \perp\!\!\!\perp)$ for a second-order theory consists of a set M of individuals, an interpretation $f_{\mathcal{M}} : M^k \to M$ for each k-ary function symbol f, and a fixed saturated subset $\perp\!\!\!\perp$ of Λ. In the case of $\mathsf{c2r}$, classical second-order arithmetic, we always take $M = \mathbb{N}$ and the interpretation of each function symbol f_ϕ is the corresponding recursive function ϕ.

We will also use ρ as a mapping from the first-order variables of $\mathsf{c2r}$ to elements of M, and second-order variables of arity k to mappings from M^k to the set $\mathcal{T}$. Any such structure $\mathcal{M}$ gives rise to a model of $\mathsf{c2r}$ where to formulas saturated elements of $\mathcal{T}$ are assigned as follows.

For each formula A we will simultaneously define the sets $||A||_\rho$ and $|A|_\rho$ in $\mathcal{T}$. The set $|A|$ is supposed to contain *witnesses* for A relative to $\perp\!\!\!\perp$, while the set $||A||$ contains the set of *challenges* used in arguing that a term is not a witness for A. Recall that Λ denotes the set of all closed λ-terms. The saturated set $|A|$ is defined as

$$|A|_\rho :\equiv ||A||_\rho \to \perp\!\!\!\perp \qquad \text{i.e. } \{t \in \Lambda \,:\, \forall s \in ||A||_\rho (ts \in \perp\!\!\!\perp)\}$$

where the saturated set $||A||_\rho$ is defined inductively as:

$$\begin{array}{lcl} ||X(t)||_\rho & :\equiv & \rho(X)(\rho(t)) \\ ||A \to B||_\rho & :\equiv & |A|_\rho \times ||B||_\rho \\ ||\forall x A||_\rho & :\equiv & \bigcup_{a \in M} ||A||_{\rho[a/x]} \\ ||\forall X A||_\rho & :\equiv & \bigcup_{S \in M^k \to \mathcal{T}} ||A||_{\rho[S/X]}. \end{array}$$

Moreover, by the definition of $|A|_\rho$ given above we have:

$$\begin{array}{lcl} |A \to B|_\rho & \equiv & ||A \to B||_\rho \to \perp\!\!\!\perp \\ & \equiv & |A|_\rho \times ||B||_\rho \to \perp\!\!\!\perp \\ |\forall x A|_\rho & \equiv & \bigcap_{a \in M} |A|_{\rho[a/x]} \\ |\forall X A|_\rho & \equiv & \bigcap_{S \in M^k \to \mathcal{T}} |A|_{\rho[S/X]}. \end{array}$$

Note that $|A \to B|$ and $|A| \to |B|$ are not the same set. Nevertheless, if we use $t \bullet s$ as an abbreviation for $\lambda x.t\langle s, x\rangle$ then $t \in |A \to B|$ and $s \in |A|$ imply $t \bullet s \in |B|$ as follows: Suppose $t \in |A \to B|$ and $s \in |A|$ and $r \in ||B||$. Then $(t \bullet s)r \rightsquigarrow_\beta t\langle s, r\rangle \in \perp\!\!\!\perp$ and thus $(t \bullet s)r \in \perp\!\!\!\perp$ since $\perp\!\!\!\perp$ is saturated.

Similarly, even if a term $t[x]$ is such that $t[s]$ belongs to $|B|$ whenever $s \in |A|$, this does not imply $\lambda x.t[x] \in |A \to B|$. For the more general form of abstraction $\tilde{\lambda} x.t[x] \equiv \lambda y.t[y_0]y_1$, however, we do have that $\tilde{\lambda} x.t[x] \in |A \to B|$ as follows: Suppose $t[s] \in |B|$ whenever $s \in |A|$. Let $s \in |A|$ and $r \in ||B||$. We have $(\tilde{\lambda} x.t[x])\langle s, r\rangle \rightsquigarrow_\beta t[(\langle s, r\rangle)_0](\langle s, r\rangle)_1 \rightsquigarrow_\beta t[s]r \in \perp\!\!\!\perp$ and thus $(\tilde{\lambda} x.t[x])\langle s, r\rangle \in \perp\!\!\!\perp$ since $\perp\!\!\!\perp$ is saturated. Finally, note also that these two abbreviations satisfy $(\tilde{\lambda} x.t[x]) \bullet s \rightsquigarrow_{\beta\eta} t[s]$.

The reason why the correspondence between $|A \to B|$ and $|A| \to |B|$ is only up to equivalence comes from the fact that the realizability interpretation presented

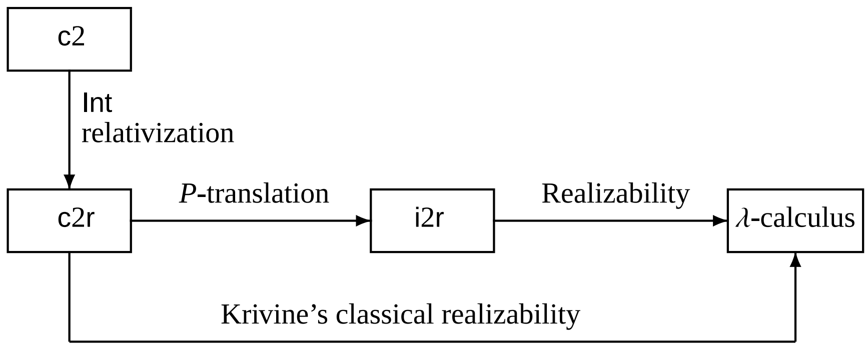

Figure 4: Decomposing Krivine's interpretation

above incorporates the negative translation discussed in Section 4. This negative translation, particularly the treatment of implication, relies on conjunction introduction and elimination (cf. proof of Theorem 4.3) which appears here as pair formation and projections.

Example 6.1 *Consider the interpretation of the closed formula* $\mathsf{Int}(n)$

$$|\mathsf{Int}(n)| \equiv \bigcap_{S \in M^k \to \mathcal{T}} \left(\bigcap_{a \in M} (|S(a) \to S(1+a)|) \times (|S(0)| \times \|S(n)\|) \right) \to \perp\!\!\!\perp$$

where n is the numeral $1{+}1{+}\ldots{+}0$. *Let* $S \in M^k \to \mathcal{T}$, $s \in \bigcap_{a \in M}(|S(a) \to S(1{+}a)|)$ *and* $r \in |S(0)|$ *be fixed. We have that a witness for* $\mathsf{Int}(n)$ *must be a closed term* t *such that* $(t \bullet s) \bullet r \in |S(n)|$. *For instance, this could be* $\tilde{\lambda} f.\tilde{\lambda} x. f^n(x)$, *where* $f^0 x := x$ *and* $f^{n+1} x := f \bullet f^n(x)$.

Definition 6.2 (Krivine realizability) *For every fixed choice of* $\perp\!\!\!\perp$, *a closed term* t *is said to* $\perp\!\!\!\perp$-realize A *if* $t \in |A|$, *i.e. if* $ts \in \perp\!\!\!\perp$ *whenever* $s \in \|A\|$.

We often omit $\perp\!\!\!\perp$ and just write "t realizes A". If $\perp\!\!\!\perp :\equiv \emptyset$ then $|A|$ is either the set of all closed terms or the empty set, and we get the usual notion of a formula being either true ($|A|$ being the whole set) or false ($|A|$ being empty). We are interested in closed λ-terms that $\perp\!\!\!\perp$-realize A (i.e. belong to $|A|$) no matter the choice of $\perp\!\!\!\perp$. *Those programs capture the computational content of the formula A.*

Theorem 6.3 ([8]) *Let the base saturated set* $\perp\!\!\!\perp$ *be fixed. Each proof of* $\Gamma \vdash_{\mathsf{c2r}} A$ *can be associated with a* λ*-term* $t[\vec{x}]$ *such that, for all* ρ *and* $\vec{u} \in |\Gamma|_\rho$ *we have* $t[\vec{u}] \in |A|_\rho$.

Proof. Observe that for all ρ such that $\rho(P) = \perp\!\!\!\perp$ we have

(i) $t \in |A|_\rho$ if and only if $t \in \{(A)_P \to P\}_\rho$,

(ii) $t \in ||A||_\rho$ if and only if $t \in \{(A)_P\}_\rho$.

The result then follows from Theorems 4.3 and 5.1. □

The realizability interpretation above describes a procedure for associating computational information (in the form of a closed λ-term) to each proof in classical second-order arithmetic (see Figure 4).

Our presentation of Krivine's realizability interpretation differs from Krivine's [8] with respect to two points. First, we only need *standard* closed λ-terms in order to define realizers, whereas Krivine [8] uses also so-called *processes* (a pair of a λ-term and a *stack*). Second, our version of the interpretation of Peirce's law does not require continuation terms cc and k_r, but can be witnessed by a standard λ-term (see proof of Theorem 6.3).

The reason we do not need the continuation terms cc and k_r for the realizability interpretation is because those can be defined as:

$$\mathsf{cc} := \lambda f^{|(A \to B) \to A|} \lambda x^{||A||}. f \langle \lambda y^{|A| \times ||B||}.(y)_0 x, x \rangle$$

and for $r \in ||B||$

$$k_r := \lambda y^{|A| \times ||B||}.(y)_0 r.$$

It follows that the corresponding reductions for cc and k_r (used in [8]) are derivable

$$\mathsf{cc}\, tr \rightsquigarrow t\langle k_r, r\rangle \qquad\qquad k_r\langle t, s\rangle \rightsquigarrow tr$$

7 Krivine's Interpretation of Countable Choice

In this section we briefly revise Krivine's interpretation [8] of (the integer relativisation of) the *second-order countable choice*

$$\mathsf{cAC}_2 \quad : \quad \exists^c F_{(\cdot)} \forall x (A(x, F_x) \to \forall X A(x, X))$$

considering our reformulation of Krivine's realizability interpretation. The interpretation is split into two steps. The second step is discussed in Section 7.1, while the first is the interpretation of (the integer relativisation of) the *first-order countable choice*

$$\mathsf{cAC}_1 \quad : \quad \exists^c f \forall x (Z(x, f(x)) \to \forall y Z(x, y)).$$

The existence of the function f is in fact an abbreviation for the existence of a set F of pairs of numbers describing the graph of f, i.e.

$$\exists^c F(\mathsf{Func}(F) \wedge \forall x(\forall y(F(x,y) \to Z(x,y)) \to \forall y Z(x,y)))$$

where the formula $\mathsf{Func}(F)$ is $\forall n, m_0, m_1(\bigwedge_{i=0}^{1} F(n,m_i) \to m_0 = m_1)$. We argue that $(\mathsf{cAC}_1)^{\mathsf{Int}}$ is derivable in second-order arithmetic as follows. The function F can be produced by minimisation as

$$F(x,y) :\equiv \neg Z(x,y) \wedge \forall k^{\mathsf{Int}} < y\, Z(x,k).$$

It is easy to show that F as defined above satisfies Func. Moreover, assuming $\mathsf{Int}(x)$, the implication

$$\forall y^{\mathsf{Int}}(F(x,y) \to Z(x,y)) \vdash_{\mathsf{c2}} \forall y^{\mathsf{Int}} Z(x,y)$$

can be proven using induction (i.e. the assumption $\mathsf{Int}(x)$) and Peirce's law: for $y = 0$ the statement reduces to an instance of Peirce's law $((Z(x,0) \to \bot) \to Z(x,0)) \to Z(x,0)$. The induction hypothesis is used to eliminate $\forall k < y\, Z(x,k)$ in $F(x,y)$, so that we again get an instance of Peirce's law.

7.1 Full countable choice

The second step in Krivine's interpretation of countable choice is as follows. Assume a fixed surjection $n \mapsto t_n$ of the set of Church numerals into Λ. Let $u \mapsto n_u$ be some inverse of this mapping, so that $t_{n_u} = u$. We add to the λ-calculus a new constant χ with the respective conversion rule

$$\chi\langle u, r\rangle \rightsquigarrow u\langle n_u, r\rangle$$

i.e. χ simply applies the argument u to its corresponding Church numerals n_u (where n_u is reminiscent of LISP's `quote` u). The goal is to produce a term which realizes cAC_2. Krivine notices that cAC_2 is a consequence of the first-order countable choice cAC_1, and the following principle[3]

$$\mathsf{KA} \quad : \quad \exists^c S_{(\cdot)} \forall x(\forall n A(x, S_{x,n}) \to \forall X A(x,X))$$

which we will refer to as *Krivine's axiom*[4]. It basically says that for any property $A(X)$ there exists a family of sets $S_{x,n}$ such that quantifications over sets satisfying A can be reduced to quantification over the indexing set of the family of sets $S_{x,n}$. As shown above, the integer relativisation of cAC_1 is easily derivable in second-order logic and we are left with the integer relativisation of KA to interpret.

Instead of interpreting $(\mathsf{KA})^{\mathsf{Int}}$ one can actually produce a realizer for the stronger statement

[3] Take $Z(x,n) :\equiv A(x, S_{x,n}, Y)$ and $F_x :\equiv S_{x,f(x)}$.

[4] The interpretation of this principle has also been investigated in [9], in connection with a Curry-Howard correspondence for intuitionistic set theory.

(∗) $\exists S_{(\cdot)}\forall x(\forall n^{\mathsf{Int}} A(x, S_{x,n}) \to \forall X A(x, X))$

i.e. the variable x does not need to be relativised and the existential quantification over S can be treated intuitionistically.

Theorem 7.1 ([8]) $\chi \in |(*)|$.

Proof. Recall that when working with second-order arithmetic we always assume that the set of individuals in the model is $\mathbb{N}$. We will show that χ belongs to

$$\bigcup_{S_{(\cdot)} \in \mathbb{N}^k \to \mathcal{T}} \bigcap_{x \in \mathbb{N}} \left(\left(\bigcap_{n \in \mathbb{N}} |\mathsf{Int}(n) \to A(x, S_{x,n})| \times \|\forall X A(x, X)\| \right) \to \perp\!\!\!\perp \right).$$

First, note that for each index n of the enumeration (t_n), if

$$(r \in \|\forall X A(x, X)\|) \wedge (t_n \langle n, r \rangle \notin \perp\!\!\!\perp)$$

for some term r, then for some mapping $S \in \mathbb{N}^k \to \mathcal{T}$ we also have $r' \in \|A(x, S)\|$ and $(t_n \langle n, r' \rangle \notin \perp\!\!\!\perp)$, for some (potentially different) r'. By countable choice there is a sequence of mappings $S_{x,n}$ such that

(∗) if $\exists r((r \in \|\forall X A(x, X)\|) \wedge (t_n \langle n, r \rangle \notin \perp\!\!\!\perp))$ then
$\exists r((r \in \|A(x, S_{x,n})\|) \wedge (t_n \langle n, r \rangle \notin \perp\!\!\!\perp))$.

We show that χ belongs to the set

$$\bigcap_{x \in \mathbb{N}} \left(\left(\bigcap_{n \in \mathbb{N}} |\mathsf{Int}(n) \to A(x, S_{x,n})| \times \|\forall X A(x, X)\| \right) \to \perp\!\!\!\perp \right).$$

Fix x and let $u \in \bigcap_{n \in \mathbb{N}} |\mathsf{Int}(n) \to A(x, S_{x,n})|$ and $r \in \|\forall X A(x, X)\|$. By the assumption on u and the fact that $t_{n_u} = u$ (recall that $n \mapsto t_n$ is an enumeration of λ-terms and $u \mapsto n_u$ is inverse of that), we have $u\langle n_u, r\rangle = t_{n_u}\langle n_u, r\rangle$. We must show that $t_{n_u}\langle n_u, r\rangle \in \perp\!\!\!\perp$, which implies $\chi\langle u, r\rangle \in \perp\!\!\!\perp$. Assume $t_{n_u}\langle n_u, r\rangle \notin \perp\!\!\!\perp$. By (∗) we get that for some $r' \in \|A(x, S_{x,n_u})\|$ we have $t_{n_u}\langle n_u, r'\rangle \notin \perp\!\!\!\perp$. This is a contradiction to our assumption on u $(= t_{n_u})$. □

In the presence of classical logic and first-order countable choice, Krivine's axiom is in fact *equivalent* to second-order countable choice. One direction was discussed above. For the other direction, notice that KA follows by cAC_2 from

$$\forall x \exists^c S_{(\cdot)}(\forall n A(x, S_n) \to \forall X A(x, X))$$

which is classically equivalent to the trivial statement

$$\forall x(\forall S_{(\cdot)} \forall n A(x, S_n) \to \forall X A(x, X)).$$

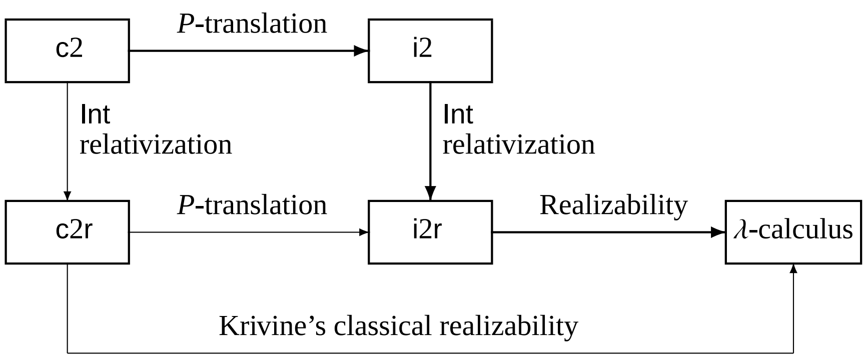

Figure 5: Alternative interpretation of c2

8 Conclusions

We have seen how a slight variant of Krivine's classical realizability interpretation can be viewed as a negative translation followed by a standard intuitionistic realizability interpretation. In particular, using this variant one does not need continuations constructs. Throughout the paper we have tried to make clear the important role of the integer relativisation of the first-order quantifiers.

One important advantage of splitting Krivine's interpretation into two steps is that we can, alternatively, perform the integer relativisation between the negative translation and the realizability interpretation (see highlighted path of Figure 5). This can be beneficial, given that the negative translation does not affect the induction axiom. One then has to produce realizers for Δ_f in the sense of the simpler intuitionistic realizability interpretation, and not in the sense of the classical realizability interpretation.

One might also wonder whether it is possible to translate Krivine's second-order interpretation of countable choice (via the use of the *quote*) into a language based on finite types. This does not seem to be helpful since the reduction from countable choice to *Krivine's axiom* (see Section 7.1) takes for granted the first-order countable choice. In the setting of finite types, however, already the first-order countable choice is strong enough to require the full power of bar recursion.

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