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Unifying Functional Interpretations

Paulo Oliva

Abstract This article presents a parametrised functional interpretation. Depending on the choice of two parameters one obtains well-known functional interpretations such as Gödel's Dialectica interpretation, Diller-Nahm's variant of the Dialectica interpretation, Kohlenbach's monotone interpretations, Kreisel's modified realizability and Stein's family of functional interpretations. A functional interpretation consists of a formula interpretation and a soundness proof. I show that all these interpretations only differ on two design choices: firstly, on the amount of the counter-examples for A which became witnesses for $\neg A$ when defining the formula interpretation, and, secondly, the inductive information about the witnesses of A which is considered in the proof of soundness. Sufficient conditions on the parameters are also given which ensure the soundness of the resulting functional interpretation. The relation between the parametrised interpretation and the recent bounded functional interpretation is also discussed.

1 Introduction

In [9] Gödel developed his *Dialectica interpretation* (also known as *functional interpretation*) with the goal of proving relative consistency of first-order arithmetic. The consistency of arithmetic was reduced to that of a quantifier-free calculus based on the language of finite types. He successfully showed that quantifier dependencies can be totally captured by functional dependencies, so that logic is eliminated in favour of higher-order objects. Around the same time, Kreisel observed that similar proof interpretations give in fact much more than just relative consistency results. The interpretations can also be used to make explicit computational information hidden in the logical structure of the proof. In [14] Kreisel then gives a clear account of Gödel's Dialectica interpretation and uses it to define the *constructive truth*

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of mathematical theorems. In the same paper Kreisel also sketches an "alternative interpretation", which was further developed in [15] and came to be called *modified realizability.*

It is normally held that one weakening of Gödel's Dialectica interpretation is that it assumes decidability of atomic formulas, known as the *contraction problem.* This happens because when interpreting the contraction axiom the Dialectica interpretation must, in loose terms, pick one counter-example out of two, which can be done by checking which one is indeed a counter-example. This checking relies on the decidability of atomic formulas. Moreover, it might as well be that both candidates are indeed counter-examples, which implies that the choice at this point is not unique, making the Dialectica interpretation non-canonical (cf. [11], section 2.3.1). A variant of the Dialectica interpretation in which this problem is circumvented was then suggested in [4], and is known as the *Diller-Nahm variant of Dialectica interpretation.* The trick suggested is simply to collect all such counter-examples, postponing the actual decision. In [18] Stein showed that this idea could be generalised, and he defines a family of interpretations parametrised by the type level from which counter-examples are collected.

In [12; 13] Kohlenbach observes that Howard's majorizability relation [10] can be used to define *monotone* versions of both Gödel's Dialectica interpretation and Kreisel's modified realizability, where majorants, rather than precise witnesses, are obtained from proof. This allows for new (even ineffective) principles to be interpreted. More recently [7], a new functional interpretation based on Bezem's strong majorizability relation [3] has been developed – labelled *bounded functional interpretation.* The main motivation for this new interpretation was to obtain effective versions of conservation results for weak König's lemma in the setting of feasible analysis (cf. [5; 8]). The interpretation also provides a new solution to the contraction problem.

The goal of this article is to show that all these functional interpretation can be viewed as special cases of a single parametrised interpretation via a careful instantiation of two parameters. The two parameters capture two degrees of freedom in the definition of a functional interpretation: (1) the interpretation of a negated formula $\neg A$ given the interpretation of A, (2) the witnessing information which is inductively carried from axioms to the conclusion.

It is important to stress that those are not the only degrees of freedom in the definition of a functional interpretation. What I intend to show, however, is that all the functional interpretations mentioned above coincide except at those two points. Sufficient conditions on the two parameters are presented and a single soundness theorem for the parametrised interpretation is proved.

Note that the approach presented here is purely *syntactic*, and intends to show that different functional interpretations appear very different simply due to a non-uniform use of notation. Therefore, getting the appropriate logical formal system and abstract definition of a functional interpretation have been the most labour intensive part of this work. Once those are in place, it is actually easy to see the striking similarities between the various functional interpretations.

Hopefully, this common syntactic framework can also help in the development of a common *semantical* understanding of functional interpretation, for instance, in the lines of [11].

1.1 Functional interpretations Let us abbreviate by $\boldsymbol{x}$ and $\boldsymbol{t}$ sequences of variables $x_0, \ldots, x_n$ and terms $t_0, \ldots, t_m$, respectively. In this article, a *functional interpretation* of a formal system T into a system S is taken to be a pair of effective mappings:

- a *formula interpretation* which maps formulas of T into formulas of S with two (possibly empty) disjoint sequences of free-variables $\boldsymbol{x}$ and $\boldsymbol{y}$

$$A \quad \mapsto \quad |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$$

such that A is equivalent (in some reasonable model) to $\exists \boldsymbol{x} \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$. Quantifier dependencies of the form $\forall\exists$ are interpreted via functional dependencies with the help of the schema of choice

$$\mathsf{AC} \quad : \quad \forall y \exists x A(x,y) \rightarrow \exists f \forall y A(fy, y)$$

using a multi-sorted language.

- a *soundness proof* which maps a T-proof of A into an S-proof of B

$$(\pi : A)_{\mathsf{T}} \quad \mapsto \quad (\tilde{\pi} : B)_{\mathsf{S}}$$

for some formula B such that $B \rightarrow \exists \boldsymbol{x} \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$. Above $(\pi : A)_{\mathsf{T}}$ denotes that π is a T-proof of A. The soundness proof in which B is the formula $\forall \boldsymbol{y} |A|^{\boldsymbol{t}}_{\boldsymbol{y}}$, for some sequence of terms $\boldsymbol{t}$, will be called *standard soundness*. It is normally the case that the proof $\tilde{\pi}$ is obtained modularly from the proof π, i.e. each lemma in π corresponds to a lemma in $\tilde{\pi}$.

The system T is referred to as the *interpreted system* while S is called the *verifying system*. In the formula $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ the sequence of variables $\boldsymbol{x}$ marks the computational information required by A, or the constructive content of A. Any sequence of terms $\boldsymbol{t}$ for which $\forall \boldsymbol{y} |A|^{\boldsymbol{t}}_{\boldsymbol{y}}$ holds is called a *witness* for A. The sequence of variables $\boldsymbol{y}$ marks the position of the possible *counter-examples* for concrete potential witnesses $\boldsymbol{t}$, i.e. in order to show that $\boldsymbol{t}$ is not a witness for A one must to produce a sequence of terms $\boldsymbol{s}$ such that $\neg |A|^{\boldsymbol{t}}_{\boldsymbol{s}}$. Therefore, the soundness proof component of the interpretation gives a way of translating a proof of A into a proof of some formula B which implies the existence of witnesses for A.

Due to the modularity of the soundness proof, a functional interpretation of T can be easily extended to an interpretation of extensions of T given some conditions on the new axioms and rules. For that reason I will mainly focus on the core of the interpretation, i.e. the interpretation of intuitionistic predicate logic. Extensions of the core interpretation will be discussed in Section 2.3.

1.2 The interpreted system: IL Tables 1 and 2 describe a deduction system, which I will refer to by IL, for intuitionistic predicate logic. IL is basically a natural deduction system in sequent style, with contexts Γ and Δ modelled as multisets. Notice, however, that the formulation of the elimination rules for disjunction, implication, and existential quantifier deviate from the standard presentations of natural deduction systems.

The logical rules of IL are more basic than those described in e.g. [20] (section 2.1.8) in the sense that $\vee$E, $\rightarrow$E, and $\exists$E together with the cut rule allows one to derive the corresponding rules of [20]. In the other direction, $\vee$E, $\rightarrow$E, and $\exists$E of IL can be directly derived from the corresponding rules of [20] simply with the help of the identity axiom $A \vdash A$. The new rule (cut) of IL is also derivable in the system of [20] via a detour of $\rightarrow$I followed by $\rightarrow$E. Another difference is that contraction of assumptions is done *explicitly* via the contraction rule (con), since contexts are being viewed as multisets, rather than sets. The treatment of the contexts as multisets implies that in the rule $\rightarrow$I a single copy of A is removed from the context, while in $\rightarrow$E a single copy of A is added to the context.

As usual, there are two side conditions on the quantifier rules. In the rule $\forall$I, the variable z must not appear free in Γ. The side condition for $\exists$E is that the variable z must not appear free in Γ nor in B.

Recall that $\neg A$ is defined as $A \to \bot$. For the rest of the article I write $\Gamma \vdash \Delta$ for provability in IL. When referring to provability in some extension T of IL the system will be explicitly attached to the provability symbol as $\Gamma \vdash_{\mathsf{T}} A$.

$$A \vdash A \quad (\text{id}) \qquad\qquad \bot \vdash A \quad (\text{efq})$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\,\wedge\mathrm{E}_l \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\,\wedge\mathrm{E}_r \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\,\wedge\mathrm{I}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\,\vee\mathrm{I}_l \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\,\vee\mathrm{I}_r \qquad \frac{\Gamma, A \vdash C \quad \Delta, B \vdash C}{\Gamma, \Delta, A \vee B \vdash C}\,\vee\mathrm{E}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\,{\to}\mathrm{I} \qquad \frac{\Gamma \vdash A \to B}{\Gamma, A \vdash B}\,{\to}\mathrm{E}$$

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}\,(\text{wkn}) \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,(\text{con}) \qquad \frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\,(\text{cut})$$

Table 1 Propositional fragment of IL

The main motivation for defining IL as above is purely to *localise* context manipulations such as those performed in the rules of contraction and cut. Avoiding rules which involve change of context points naturally to a natural deduction formulation. One should notice, however, that the natural deduction treatment of disjunction, implication and existential elimination involve a hidden application of the cut rule, which we intend to localise in IL. Therefore, we adopt a more primitive formulation of those rules. What results at the end is (an apparently awkward) hybrid system between a natural deduction and a Gentzen formulation of intuitionistic logic. As we will see, this formulation will prove very useful in pinpointing the few places where the various functional interpretation mentioned in the introduction differ.

$$\frac{\Gamma \vdash A(z)}{\Gamma \vdash \forall z A(z)}\,\forall\mathrm{I} \qquad \frac{\Gamma \vdash \forall z A(z)}{\Gamma \vdash A(s)}\,\forall\mathrm{E} \qquad \frac{\Gamma \vdash A(s)}{\Gamma \vdash \exists z A(z)}\,\exists\mathrm{I} \qquad \frac{\Gamma, A(z) \vdash B}{\Gamma, \exists z A(z) \vdash B}\,\exists\mathrm{E}$$

Table 2 Quantifier rules

1.3 The verifying system: T^{ω} Since we wish to define a *parametrised* interpretation of IL, the verifying system will not be fully specified until we have considered concrete instantiations of the parameters. Nevertheless, we can prove properties of the parametrised interpretation by making use of an *abstract* formal system T^{ω}, in the sense that we just describe the essential properties (which we refer to as *conditions*) that any verifying system T^{ω} must have for any given instantiation of the parametrised interpretation, in order to ensure soundness.

As mentioned above, quantifier dependencies will be replaced by functional dependencies. This means that T^{ω} will be an extension of IL over the language of finite types. The set of *finite types* $\mathcal{T}$ is inductively defined as follows: $o \in \mathcal{T}$ and if $\rho, \sigma \in \mathcal{T}$ then $\rho \to \sigma \in \mathcal{T}$. We often omit the parenthesis in e.g. $\rho \to (\tau \to \sigma)$ writting simply $\rho \to \tau \to \sigma$, assuming right associativity of the functional type construction. We also define the *type level* of each element of $\mathcal{T}$ inductively as follows: $\mathsf{level}(o) := 0$ and $\mathsf{level}(\rho \to \tau) := \max\{\mathsf{level}(\rho)+1, \mathsf{level}(\tau)\}$. The types in $\mathcal{T}$ of the form $o \to \ldots \to o$ are called the *pure types*. Since the pure types are in one-to-one correspondence with the natural number, we often write n instead of the pure type of type level n. We leave open how the treatment of higher-order equality is handled in the *verifying* system T^ω, as this is not essential for the interpretation. As we discuss in Section 2.3, however, the treatment of higher-order equality in the *interpreted* theory (once we extend IL to a language of finite types) is essential.

We use $f, g, h, u, v, w, x, y, z$ for variables and s, t, r, q for terms of arbitrary type. The variables i, j, k, m, n will be used to range over the basic type o. In Section 3.4, however, we will make use of i, j to quantify over an arbitrary but fixed pure type.

The first assumption we will make on the higher-order system T^ω is that it enjoys the property of combinatorial completeness. In particular, given any term t and variable x there is a functional term $\lambda x.t$ such that

$$(\beta) \ \vdash_{\mathsf{T}^\omega} A((\lambda x.t)\, s) \leftrightarrow A(t[s/x])$$

for arbitrary formulas A.

Our second assumption is that the following substitution rule is admissible in T^ω

$$(\boldsymbol{t}/\boldsymbol{x}) \ \text{ if } \Gamma(\boldsymbol{x}) \vdash_{\mathsf{T}^\omega} A(\boldsymbol{x}) \text{ then } \Gamma(\boldsymbol{t}) \vdash_{\mathsf{T}^\omega} A(\boldsymbol{t})$$

for any context Γ and formula A, where the sequence of terms $\boldsymbol{t}$ has the same type as the sequence of variables $\boldsymbol{x}$, and $\boldsymbol{t}$ is free for $\boldsymbol{x}$ in A and Γ.

The third assumption we make on T^ω is that its language has two constants tt and ff (of type o) such that $\vdash_{\mathsf{T}^\omega} \mathsf{tt} \neq \mathsf{ff}$, and a family of ternary constants $\mathsf{if}(n^o, x^\rho, y^\rho)$ (for each type $\rho \in \mathcal{T}$) such that

$$(\mathrm{C}_{\mathsf{tt}}) \ \ n = \mathsf{tt} \vdash_{\mathsf{T}^\omega} A(\mathsf{if}(n, x, y)) \leftrightarrow A(x)$$

$$(\mathrm{C}_{\mathsf{ff}}) \ \ n \neq \mathsf{tt} \vdash_{\mathsf{T}^\omega} A(\mathsf{if}(n, x, y)) \leftrightarrow A(y)$$

for arbitrary formulas $A(x)$. We will also assume that in T^ω the logical constructor for disjunction $A \vee B$ is replaced by a (more primitive) ternary constructor $A \vee_s B$ – so-called *flagged disjunction* – where A, B are formulas and s is a term of basic type. The logical rules for the flagged disjunction, which replace those for the standard disjunction, are given as

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee_{\mathsf{tt}} B}\,\vee\mathrm{I}_l \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee_{\mathsf{ff}} B}\,\vee\mathrm{I}_r \qquad \frac{\Gamma, s = \mathsf{tt}, A \vdash C \quad \Delta, s \neq \mathsf{tt}, B \vdash C}{\Gamma, \Delta, A \vee_s B \vdash C}\,\vee\mathrm{E}$$

The flagged disjunction $A \vee_n B$ is the logical counter-part of the if-then-else term construction, and can be viewed as an abbreviation for either $(n = \mathsf{tt} \wedge A) \vee (n \neq \mathsf{tt} \wedge B)$ or $(n = \mathsf{tt} \to A) \wedge (n \neq \mathsf{tt} \to B)$ – in which case the rules above should be viewed as derivable rules. In order to derive $\vee$E with the second abbreviation, however, one needs the extra assumption $n = \mathsf{tt} \vee n \neq \mathsf{tt}$. The standard disjunction $A \vee B$ can be defined via the flagged disjunction as $A \vee B :\equiv \exists n (A \vee_n B)$.

Remark 1.1 Note that if T^ω is an arithmetical theory the constants tt, ff can be simply taken to be numerals $0, 1$ and $\mathsf{if}(n, x, y)$ can be defined via recursion. In fact, since all functional interpretations considered here have been developed with a focus on arithmetical theories, we shall assume that $\mathsf{tt} \equiv 0, \mathsf{ff} \equiv 1$. When interpreting purely logical theories such as IL, we assume that $\mathsf{if}(n, x, y)$ is taken as a primitive with axiom schema as given above.

We will also assume that the language (predicates and constants) of IL are included in the language of T^ω, and that there is an injective mapping of variables of IL into the ground type variables of T^ω. Therefore, to each atomic formula of IL there corresponds a unique atomic formula of T^ω.

Finally, the parametrised interpretation of IL into a theory T^ω will contain an uninterpreted bounded universal quantifier $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$, where $A(\boldsymbol{x})$ is a formula with a distinguished sequence of free-variables $\boldsymbol{x}$, and $\boldsymbol{t}$ is a sequence of terms. This should be viewed as an *abbreviation* rather than a new formula construct, and the symbol $\sqsubset$ is merely part of the abbreviation. For instance, $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$ could be an abbreviation, for instance, for either $A(\boldsymbol{t})$ or $\forall \boldsymbol{x} A(\boldsymbol{x})$. For each fixed choice of the abbreviation $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$ we will consider of particular interest the following class of formulas of T^ω.

Definition 1.2 ($\sqsubset$-bounded formulas) Let the abbreviation $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$ be fixed. The class of $\sqsubset$-*bounded formulas* of T^ω (we denote arbitrary formulas in this class by $A_{\rm b}$ and $B_{\rm b}$) are those built out of atomic formulas via conjunction ($A_{\rm b} \wedge B_{\rm b}$), flagged disjunction ($A_{\rm b} \vee_s B_{\rm b}$), implication ($A_{\rm b} \to B_{\rm b}$), and bounded universal quantifier ($\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A_{\rm b}(\boldsymbol{x})$). Formulas of the form $\forall \boldsymbol{x} A_{\rm b}$ will be called $\forall \sqsubset$-*bounded formulas.*

In order to guarantee that this abbreviation behaves as a *universal quantifier* we shall assume the following: for each $\sqsubset$-bounded formulas $A_{\rm b}, B_{\rm b}$, and context Γ

(A_1) if $\Gamma \vdash_{\mathsf{T}^\omega} A_{\rm b}$ then $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, \Gamma \vdash_{\mathsf{T}^\omega} \forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A_{\rm b}$

(A_2) $\vdash_{\mathsf{T}^\omega} \forall \boldsymbol{x}, \boldsymbol{y} \sqsubset \boldsymbol{r}, \boldsymbol{t}\, (A_{\rm b} \wedge B_{\rm b}) \leftrightarrow (\forall \boldsymbol{x} \sqsubset \boldsymbol{r}\, A_{\rm b} \wedge \forall \boldsymbol{y} \sqsubset \boldsymbol{t}\, B_{\rm b})$

(A_3) $\vdash_{\mathsf{T}^\omega} \forall \boldsymbol{x}, \boldsymbol{y} \sqsubset \boldsymbol{r}, \boldsymbol{t}\, (A_{\rm b} \vee_s B_{\rm b}) \leftrightarrow (\forall \boldsymbol{x} \sqsubset \boldsymbol{r}\, A_{\rm b} \vee_s \forall \boldsymbol{y} \sqsubset \boldsymbol{t}\, B_{\rm b})$

(A_4) $\vdash_{\mathsf{T}^\omega} \forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A_{\rm b} \leftrightarrow A_{\rm b}$, if $\boldsymbol{x} \not\in \mathsf{FV}(A_{\rm b})$

assuming, in the cases of (A_2) and (A_3), that $\boldsymbol{x} \not\in \mathsf{FV}(B_{\rm b})$ and $\boldsymbol{y} \not\in \mathsf{FV}(A_{\rm b})$. Notice that ($\mathrm{A}_3$) is only required to hold for the flagged disjunction, which is an intuitionistically reasonable assumption given decidability of equality for the basic type o. Moreover, from (A_1) and (A_4) one can conclude that $\Gamma \vdash_{\mathsf{T}^\omega} A_{\rm b}$ implies $\Gamma \vdash_{\mathsf{T}^\omega} \forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A_{\rm b}$, if $\boldsymbol{x} \not\in \mathsf{FV}(\Gamma)$.

Moreover, in order to ensure that the abbreviation behaves as a *bounded quantifier* we will make use of three further conditions. For all $\sqsubset$-bounded formulas $A_{\rm b}(\boldsymbol{y})$ and a fixed sequence of free-variables $\boldsymbol{y}$ there must exist sequences of terms $\mathbf{b_1}, \mathbf{b_2}$, and $\mathbf{b_3}$ (over the free-variables of $A_{\rm b}(\boldsymbol{y})$ other than $\boldsymbol{y}$) in the language of T^ω such that

(B_1) $\forall \boldsymbol{y} \sqsubset \mathbf{b_1} \boldsymbol{x}\, A_{\rm b}(\boldsymbol{y}) \vdash_{\mathsf{T}^\omega} A_{\rm b}(\boldsymbol{x})$

(B_2) $\forall \boldsymbol{y} \sqsubset \mathbf{b_2} \boldsymbol{y}_0 \boldsymbol{y}_1\, A_{\rm b}(\boldsymbol{y}) \vdash_{\mathsf{T}^\omega} \forall \boldsymbol{y} \sqsubset \boldsymbol{y}_i\, A_{\rm b}(\boldsymbol{y})$, for $i \in \{0, 1\}$

(B_3) $\forall \boldsymbol{y} \sqsubset \mathbf{b_3} \boldsymbol{h} \boldsymbol{b}\, A_{\rm b}(\boldsymbol{y}) \vdash_{\mathsf{T}^\omega} \forall \boldsymbol{z} \sqsubset \boldsymbol{b} \forall \boldsymbol{y} \sqsubset \boldsymbol{h} \boldsymbol{z}\, A_{\rm b}(\boldsymbol{y})$

where the application $\boldsymbol{h}\boldsymbol{z}$ of one sequence $\boldsymbol{h}$ to a sequence of arguments $\boldsymbol{z}$ is an abbreviation for $h_0 \boldsymbol{z}, \ldots, h_n \boldsymbol{z}$. Intuitively, the conditions above capture the requirement that: every (sequence of) element(s) $\boldsymbol{x}$ is effectively and uniformly bounded by $\mathbf{b_1} \boldsymbol{x}$; any element bounded by either $\boldsymbol{y}_0$ or $\boldsymbol{y}_1$ is also bounded by $\mathbf{b_2} \boldsymbol{y}_0 \boldsymbol{y}_1$; and if $\boldsymbol{z}$ is bounded by $\boldsymbol{b}$ then any element bounded by $\boldsymbol{h}\boldsymbol{z}$ is also bounded by $\mathbf{b_3} \boldsymbol{h} \boldsymbol{b}$.

In the following I will denote by c^ρ the lifting of an arbitrary o-type constant c^o of T^ω to the type ρ, which can always be done via λ-abstractions.

2 Parametrised functional interpretation of IL

The goal of this article is to show that *both* the formula interpretation and the soundness proof can be parametrised, so that instantiations of those parameters will give rise to most of the known functional interpretations. We will, however, start with a parametrisation of the formula interpretation together with a standard soundness, assuming the properties of the parameter abbreviation $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$ outlined in Section 1.3 above. In Section 4, we then introduce a second parameter abbreviation to be used in the soundness proof, and show that further conditions on the second parameter allow us to prove a parametrised soundness for the parametrised formula interpretation.

2.1 Parametrised formula interpretation Suppose that the interpretations for A and B have already been given, so that $\boldsymbol{x}$ and $\boldsymbol{v}$ are witnesses for A and B if $\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ and $\forall \boldsymbol{w} |B|^{\boldsymbol{v}}_{\boldsymbol{w}}$ respectively. According to modified realizability, a witness for the implication $A \to B$ is simply a sequence of functionals $\boldsymbol{f}$ producing witnesses for B given witnesses for A, i.e.

$$\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to \forall \boldsymbol{w} |B|^{\boldsymbol{f}\boldsymbol{x}}_{\boldsymbol{w}}$$

This is very much in the spirit of the BHK interpretation of intuitionistic logic, where such functionals $\boldsymbol{f}$ are associated with proofs of $A \to B$. A proof of the implication $A \to B$, however, provides another construction which is normally disregarded: for each $\boldsymbol{x}$ and $\boldsymbol{w}$, the conclusion $|B|^{\boldsymbol{f}\boldsymbol{x}}_{\boldsymbol{w}}$ follows from *finitely* many instantiations of the premise. The required instantiations can be read-off (uniformly on $\boldsymbol{x}, \boldsymbol{w}$) from the proof, giving rise to a second construction $\boldsymbol{g}$ satisfying the stronger statement

$$\forall \boldsymbol{y} \in \boldsymbol{g}\boldsymbol{x}\boldsymbol{w}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to |B|^{\boldsymbol{f}\boldsymbol{x}}_{\boldsymbol{w}}$$

where $\boldsymbol{g}\boldsymbol{x}\boldsymbol{w}$ is a finite set. Moreover, if the formula $|A|^{\boldsymbol{w}}_{\boldsymbol{y}}$ is decidable it is possible to obtain a functional $\boldsymbol{g}'$ which selects a single element from the finite set $\boldsymbol{g}\boldsymbol{x}\boldsymbol{w}$ so that

$$|A|^{\boldsymbol{x}}_{\boldsymbol{g}'\boldsymbol{x}\boldsymbol{w}} \to |B|^{\boldsymbol{f}\boldsymbol{x}}_{\boldsymbol{w}}$$

The element $\boldsymbol{g}'\boldsymbol{x}\boldsymbol{w}$ can, for instance, be taken to be any element $\boldsymbol{y} \in \boldsymbol{g}\boldsymbol{x}\boldsymbol{w}$ such that $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ does not hold, or an arbitrary element if $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ holds for all (finitely many) elements of $\boldsymbol{g}\boldsymbol{x}\boldsymbol{w}$.

The three choices of the interpretation of $A \to B$ described above correspond to modified realizability, the Diller-Nahm interpretation and Gödel's original functional interpretation. This papers considers a parametrisation of the amount of information produced by $\boldsymbol{g}$ by defining the interpretation of implication as

$$\forall \boldsymbol{y} \sqsubset \boldsymbol{g}\boldsymbol{x}\boldsymbol{w}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to |B|^{\boldsymbol{f}\boldsymbol{x}}_{\boldsymbol{w}}$$

leaving open what $\forall \boldsymbol{y} \sqsubset \boldsymbol{g}\boldsymbol{x}\boldsymbol{w}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ stands for. Under some basic conditions on the choice of the abbreviation $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$ (described in Section 1.3) a standard soundness theorem can be proved for such parametrised interpretation. This implies that any instantiations of the abbreviation satisfying these conditions will give rise to a different functional interpretation. As examples of other interpretations we will present Stein's family of functional interpretations and the recent bounded functional interpretation.

Definition 2.1 (Parametrised formula interpretation) To each formula A of IL we associate a $\sqsubset$-bounded formula $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ of T^ω as follows.

$$|P| \;:\equiv\; P$$

for atomic formulas P, where the variables of IL are mapped to variables of type o in T^ω. Notice that for atomic formulas the tuples of witnesses and counter-examples are both empty. Assume we have already defined $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ and $|B|^{\boldsymbol{v}}_{\boldsymbol{w}}$, we define

$$
\begin{array}{lll}
|A \wedge B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \wedge |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
|A \vee B|^{\boldsymbol{x},\boldsymbol{v},n}_{\boldsymbol{y},\boldsymbol{w}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \vee_n |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
|A \to B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} & :\equiv & \forall \boldsymbol{y} \sqsubset \boldsymbol{gxw}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to |B|^{\boldsymbol{fx}}_{\boldsymbol{w}} \\
|\forall z A(z)|^{\boldsymbol{f}}_{\boldsymbol{y},z} & :\equiv & |A(z)|^{\boldsymbol{f}z}_{\boldsymbol{y}} \\
|\exists z A(z)|^{\boldsymbol{x},z}_{\boldsymbol{y}} & :\equiv & |A(z)|^{\boldsymbol{x}}_{\boldsymbol{y}}
\end{array}
$$

In the clause for implication if $\boldsymbol{v}$ (respectively $\boldsymbol{y}$) is the empty sequence then $\boldsymbol{f}$ (respectively $\boldsymbol{g}$) is also taken to be the empty sequence. Similarly, if in the clause for universal quantification if $\boldsymbol{x}$ is the empty sequence then $\boldsymbol{f}$ is also taken to be the empty sequence.

Notice that, except for the bounded quantifier in the treatment of implication, the interpretation of an arbitrary formula A is a quantifier-free formula. It should be noted, however, that the types of variables in the resulting formula might be of arbitrary level depending on the logical complexity (i.e. the nesting of universal quantifiers and implications) in the formula A.

We present now a standard soundness proof for the parametrised formula interpretation. This should be viewed as a preparation for the next step: a parametrisation of the soundness proof. The following theorem will be shown to be a special case of the parametrised soundness proof.

Theorem 2.2 (Standard soundness) *Let T^ω and the abbreviation $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$ be chosen so that the conditions outlined in Section 1.3 hold. If $\Gamma \vdash A$ then there are sequences of terms $\boldsymbol{t}, \boldsymbol{r} \in \mathcal{L}(\mathsf{T}^\omega)$ such that*

$$\forall \boldsymbol{w} \sqsubset \boldsymbol{r}\, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}} \vdash_{\mathsf{T}^\omega} |A|^{\boldsymbol{t}}_{\boldsymbol{y}}$$

where if $\mathsf{FV}(A) \cup \mathsf{FV}(\Gamma) \equiv \{\boldsymbol{a}\}$ then $\mathsf{FV}(\boldsymbol{t}) \subseteq \{\boldsymbol{a}, \boldsymbol{v}\}$ and $\mathsf{FV}(\boldsymbol{r}) \subseteq \{\boldsymbol{a}, \boldsymbol{v}, \boldsymbol{y}\}$.

Proof By induction on the structure of the IL-proof of $\Gamma \vdash A$. The axioms of identity $A \vdash A$ are associated with the T^ω-derivation of $\forall \boldsymbol{y'} \sqsubset \mathbf{b_1} \boldsymbol{y}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y'}} \vdash |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$, which is guaranteed by condition (B_1). Moreover, given that falsity is an atomic formula we have $|\perp| \equiv \perp$, which means that we can associate the *ex-falso sequitur quodlibet* axioms $\perp \vdash A$ with new instances $\perp \vdash |A|^{\boldsymbol{c}}_{\boldsymbol{y}}$, where $\boldsymbol{c}$ is a sequence of arbitrary terms c^ρ of appropriate type.

For each logical rule we show how the terms for the interpretation of the conclusion can be obtained given terms for the interpretation of the premises. If a substitution is performed, we will explicitly show the relevant free-variables in the relevant terms. We will assume without loss of generality that the multisets Γ, Δ consist of a single formula, since manipulations of formulas in Γ, Δ are done pointwise.

Conjunction introduction ($\wedge$I)

$$\dfrac{\dfrac{\forall \boldsymbol{w}_0 \sqsubset \boldsymbol{r}_0\, |\Gamma|^{\boldsymbol{v}_0}_{\boldsymbol{w}_0} \vdash |A|^{\boldsymbol{t}_0}_{\boldsymbol{y}_0} \qquad \forall \boldsymbol{w}_1 \sqsubset \boldsymbol{r}_1\, |\Delta|^{\boldsymbol{v}_1}_{\boldsymbol{w}_1} \vdash |B|^{\boldsymbol{t}_1}_{\boldsymbol{y}_1}}{\forall \boldsymbol{w}_0 \sqsubset \boldsymbol{r}_0\, |\Gamma|^{\boldsymbol{v}_0}_{\boldsymbol{w}_0}, \forall \boldsymbol{w}_1 \sqsubset \boldsymbol{r}_1\, |\Delta|^{\boldsymbol{v}_1}_{\boldsymbol{w}_1} \vdash |A|^{\boldsymbol{t}_0}_{\boldsymbol{y}_0} \wedge |B|^{\boldsymbol{t}_1}_{\boldsymbol{y}_1}}\,\wedge\mathrm{I}}{\forall \boldsymbol{w}_0 \sqsubset \boldsymbol{r}_0\, |\Gamma|^{\boldsymbol{v}_0}_{\boldsymbol{w}_0}, \forall \boldsymbol{w}_1 \sqsubset \boldsymbol{r}_1\, |\Delta|^{\boldsymbol{v}_1}_{\boldsymbol{w}_1} \vdash |A \wedge B|^{\boldsymbol{t}_0,\boldsymbol{t}_1}_{\boldsymbol{y}_0,\boldsymbol{y}_1}}\,(\mathrm{D.2.1})$$

The reader can check that the free-variables condition on witnessing terms is maintained, i.e. if (for $i \in \{0, 1\}$) the free-variables of $\boldsymbol{t}_i$ are included in $\{\boldsymbol{v}_i\} \cup \mathsf{FV}(A_i, \Gamma_i)$, then the free-variables of $\boldsymbol{t}_0, \boldsymbol{t}_1$ are trivially included in $\{\boldsymbol{v}_0, \boldsymbol{v}_1\} \cup \mathsf{FV}(\Gamma_0, \Gamma_1, A_0, A_1)$. Similarly for the free-variables of $\boldsymbol{r}_0$ and $\boldsymbol{r}_1$. We will not focus on this point for the rest of this proof.

Conjunction elimination (∧E).

$$\dfrac{\dfrac{\dfrac{\forall \boldsymbol{w} \sqsubset \boldsymbol{r}[\boldsymbol{y}_0, \boldsymbol{y}_1] \, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}} \vdash |A \wedge B|^{\boldsymbol{t}_0, \boldsymbol{t}_1}_{\boldsymbol{y}_0, \boldsymbol{y}_1}}{\forall \boldsymbol{w} \sqsubset \boldsymbol{r}[\boldsymbol{y}_0, \boldsymbol{y}_1] \, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}} \vdash |A|^{\boldsymbol{t}_0}_{\boldsymbol{y}_0} \wedge |B|^{\boldsymbol{t}_1}_{\boldsymbol{y}_1}} \text{ (D.2.1)}}{\forall \boldsymbol{w} \sqsubset \boldsymbol{r}[\boldsymbol{y}_0, \boldsymbol{c}] \, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}} \vdash |A|^{\boldsymbol{t}_0}_{\boldsymbol{y}_0} \wedge |B|^{\boldsymbol{t}_1}_{\boldsymbol{c}}} \; (\boldsymbol{c}/\boldsymbol{y}_1)}{\forall \boldsymbol{w} \sqsubset \boldsymbol{r}[\boldsymbol{y}_0, \boldsymbol{c}] \, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}} \vdash |A|^{\boldsymbol{t}_0}_{\boldsymbol{y}_0}} \; \wedge\mathrm{E}_l$$

where $\boldsymbol{c}$ is a tuple of arbitrary terms of appropriate type. The case $\wedge\mathrm{E}_r$ is treated similarly.
Disjunction introduction (∨I). Let $\boldsymbol{c}$ be a tuple of arbitrary terms of appropriate type.

$$\dfrac{\dfrac{\forall \boldsymbol{y} \sqsubset \boldsymbol{r} \, |\Gamma|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash |A|^{\boldsymbol{t}_0}_{\boldsymbol{w}_0}}{\forall \boldsymbol{y} \sqsubset \boldsymbol{r} \, |\Gamma|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash |A|^{\boldsymbol{t}_0}_{\boldsymbol{w}_0} \vee_0 |B|^{\boldsymbol{c}}_{\boldsymbol{w}_1}} \; \vee\mathrm{I}_l}{\forall \boldsymbol{y} \sqsubset \boldsymbol{r} \, |\Gamma|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash |A \vee B|^{0, \boldsymbol{t}_0, \boldsymbol{c}}_{\boldsymbol{w}_0, \boldsymbol{w}_1}} \text{ (D.2.1)} \qquad \dfrac{\dfrac{\forall \boldsymbol{y} \sqsubset \boldsymbol{r} \, |\Gamma|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash |B|^{\boldsymbol{t}_1}_{\boldsymbol{w}_1}}{\forall \boldsymbol{y} \sqsubset \boldsymbol{r} \, |\Gamma|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash |A|^{\boldsymbol{c}}_{\boldsymbol{w}_0} \vee_1 |B|^{\boldsymbol{t}_1}_{\boldsymbol{w}_1}} \; \vee\mathrm{I}_r}{\forall \boldsymbol{y} \sqsubset \boldsymbol{r} \, |\Gamma|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash |A \vee B|^{1, \boldsymbol{c}, \boldsymbol{t}_1}_{\boldsymbol{w}_0, \boldsymbol{w}_1}} \text{ (D.2.1)}$$

Disjunction elimination (∨E). For the sake of simplicity we will omit the assumptions Γ and Δ in the case of disjunction elimination.

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall \boldsymbol{y}_0 \sqsubset \boldsymbol{r}_0 \, |A|^{\boldsymbol{x}_0}_{\boldsymbol{y}_0} \vdash |C|^{\boldsymbol{t}_0}_{\boldsymbol{w}}}{n = 0, \forall \boldsymbol{y}_0 \sqsubset \boldsymbol{r}_0 \, |A|^{\boldsymbol{x}_0}_{\boldsymbol{y}_0} \vdash |C|^{\mathsf{if}(n, \boldsymbol{t}_0, \boldsymbol{t}_1)}_{\boldsymbol{w}}} \; (\mathrm{C}_{\mathsf{tt}}) \qquad \dfrac{\forall \boldsymbol{y}_1 \sqsubset \boldsymbol{r}_1 \, |B|^{\boldsymbol{x}_1}_{\boldsymbol{y}_1} \vdash |C|^{\boldsymbol{t}_1}_{\boldsymbol{w}}}{n \neq 0, \forall \boldsymbol{y}_1 \sqsubset \boldsymbol{r}_1 \, |B|^{\boldsymbol{x}_1}_{\boldsymbol{y}_1} \vdash |C|^{\mathsf{if}(n, \boldsymbol{t}_0, \boldsymbol{t}_1)}_{\boldsymbol{w}}} \; (\mathrm{C}_{\mathsf{ff}})}{\forall \boldsymbol{y}_0 \sqsubset \boldsymbol{r}_0 \, |A|^{\boldsymbol{x}_0}_{\boldsymbol{y}_0} \vee_n \forall \boldsymbol{y}_1 \sqsubset \boldsymbol{r}_1 \, |B|^{\boldsymbol{x}_1}_{\boldsymbol{y}_1} \vdash |C|^{\mathsf{if}(n, \boldsymbol{t}_0, \boldsymbol{t}_1)}_{\boldsymbol{w}}} \; \vee\mathrm{E}}{\forall \boldsymbol{y}_0, \boldsymbol{y}_1 \sqsubset \boldsymbol{r}_0, \boldsymbol{r}_1 \, (|A|^{\boldsymbol{x}_0}_{\boldsymbol{y}_0} \vee_n |B|^{\boldsymbol{x}_1}_{\boldsymbol{y}_1}) \vdash |C|^{\mathsf{if}(n, \boldsymbol{t}_0, \boldsymbol{t}_1)}_{\boldsymbol{w}}} \; (\mathrm{A}_3)}{\forall \boldsymbol{y}_0, \boldsymbol{y}_1 \sqsubset \boldsymbol{r}_0, \boldsymbol{r}_1 \, |A \vee B|^{n, \boldsymbol{x}_0, \boldsymbol{x}_1}_{\boldsymbol{y}_0, \boldsymbol{y}_1} \vdash |C|^{\mathsf{if}(n, \boldsymbol{t}_0, \boldsymbol{t}_1)}_{\boldsymbol{w}}} \text{ (D.2.1)}$$

Implication introduction (→I)

$$\dfrac{\dfrac{\dfrac{\forall \boldsymbol{z} \sqsubset \boldsymbol{q} \, |\Gamma|^{\boldsymbol{u}}_{\boldsymbol{z}}, \forall \boldsymbol{y} \sqsubset \boldsymbol{r} \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash |B|^{\boldsymbol{t}}_{\boldsymbol{w}}}{\forall \boldsymbol{z} \sqsubset \boldsymbol{q} \, |\Gamma|^{\boldsymbol{u}}_{\boldsymbol{z}} \vdash \forall \boldsymbol{y} \sqsubset \boldsymbol{r} \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to |B|^{\boldsymbol{t}}_{\boldsymbol{w}}} \to \mathrm{I}}{\forall \boldsymbol{z} \sqsubset \boldsymbol{q} \, |\Gamma|^{\boldsymbol{u}}_{\boldsymbol{z}} \vdash \forall \boldsymbol{y} \sqsubset (\lambda \boldsymbol{x} \lambda \boldsymbol{w} . \boldsymbol{r}) \boldsymbol{x} \boldsymbol{w} \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to |B|^{(\lambda \boldsymbol{x} . \boldsymbol{t}) \boldsymbol{x}}_{\boldsymbol{w}}} \; (\beta)}{\forall \boldsymbol{z} \sqsubset \boldsymbol{q} \, |\Gamma|^{\boldsymbol{u}}_{\boldsymbol{z}} \vdash |A \to B|^{(\lambda \boldsymbol{x} \lambda \boldsymbol{w} . \boldsymbol{r}), \lambda \boldsymbol{x} . \boldsymbol{t}}_{\boldsymbol{x}, \boldsymbol{w}}} \text{ (D.2.1)}$$

where $\lambda \boldsymbol{x} . \boldsymbol{t}$ abbreviates $\lambda \boldsymbol{x} . t_0, \ldots, \lambda \boldsymbol{x} . t_n$ (similarly for $\lambda \boldsymbol{x} \lambda \boldsymbol{w} . \boldsymbol{r}$).
Implication elimination (→E)

$$\dfrac{\dfrac{\forall \boldsymbol{z} \sqsubset \boldsymbol{q} \, |\Gamma|^{\boldsymbol{u}}_{\boldsymbol{z}} \vdash |A \to B|^{\boldsymbol{r}, \boldsymbol{t}}_{\boldsymbol{x}, \boldsymbol{w}}}{\forall \boldsymbol{z} \sqsubset \boldsymbol{q} \, |\Gamma|^{\boldsymbol{u}}_{\boldsymbol{z}} \vdash \forall \boldsymbol{y} \sqsubset \boldsymbol{r} \boldsymbol{x} \boldsymbol{w} \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to |B|^{\boldsymbol{t} \boldsymbol{x}}_{\boldsymbol{w}}} \text{ (D.2.1)}}{\forall \boldsymbol{z} \sqsubset \boldsymbol{q} \, |\Gamma|^{\boldsymbol{u}}_{\boldsymbol{z}}, \forall \boldsymbol{y} \sqsubset \boldsymbol{r} \boldsymbol{x} \boldsymbol{w} \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash |B|^{\boldsymbol{t} \boldsymbol{x}}_{\boldsymbol{w}}} \to \mathrm{E}$$

Cut

$$\dfrac{\dfrac{\dfrac{\forall \boldsymbol{y}_0 \sqsubset \boldsymbol{r}_0[\boldsymbol{z}] \, |\Gamma|^{\boldsymbol{x}_0}_{\boldsymbol{y}_0} \vdash |A|^{\boldsymbol{s}}_{\boldsymbol{z}}}{\forall \boldsymbol{z} \sqsubset \boldsymbol{q}' \, \forall \boldsymbol{y}_0 \sqsubset \boldsymbol{r}_0[\boldsymbol{z}] \, |\Gamma|^{\boldsymbol{x}_0}_{\boldsymbol{y}_0} \vdash \forall \boldsymbol{z} \sqsubset \boldsymbol{q}' \, |A|^{\boldsymbol{s}}_{\boldsymbol{z}}} \; (\mathrm{A}_1) \quad \dfrac{\forall \boldsymbol{y}_1 \sqsubset \boldsymbol{r}'_1 \, |\Delta|^{\boldsymbol{x}_1}_{\boldsymbol{y}_1}, \forall \boldsymbol{z} \sqsubset \boldsymbol{q} \, |A|^{\boldsymbol{v}}_{\boldsymbol{z}} \vdash |B|^{\boldsymbol{t}}_{\boldsymbol{w}}}{\forall \boldsymbol{y}_1 \sqsubset \boldsymbol{r}'_1 \, |\Delta|^{\boldsymbol{x}_1}_{\boldsymbol{y}_1}, \forall \boldsymbol{z} \sqsubset \boldsymbol{q}' \, |A|^{\boldsymbol{s}}_{\boldsymbol{z}} \vdash |B|^{\boldsymbol{t}'}_{\boldsymbol{w}}} \; (\boldsymbol{s}/\boldsymbol{v})}{\forall \boldsymbol{z} \sqsubset \boldsymbol{q}' \, \forall \boldsymbol{y}_0 \sqsubset \boldsymbol{r}_0[\boldsymbol{z}] \, |\Gamma|^{\boldsymbol{x}_0}_{\boldsymbol{y}_0}, \forall \boldsymbol{y}_1 \sqsubset \boldsymbol{r}'_1 \, |\Delta|^{\boldsymbol{x}_1}_{\boldsymbol{y}_1} \vdash |B|^{\boldsymbol{t}'}_{\boldsymbol{w}}} \; (\mathrm{cut})}{\forall \boldsymbol{y}_0 \sqsubset \mathbf{b_3}(\lambda \boldsymbol{z} . \boldsymbol{r}_0[\boldsymbol{z}]) \boldsymbol{q}' \, |\Gamma|^{\boldsymbol{x}}_{\boldsymbol{y}_0}, \forall \boldsymbol{y}_1 \sqsubset \boldsymbol{r}'_1 \, |\Delta|^{\boldsymbol{x}_1}_{\boldsymbol{y}_1} \vdash |B|^{\boldsymbol{t}'}_{\boldsymbol{w}}} \; (\mathrm{B}_3)$$

where $\boldsymbol{t}', \boldsymbol{q}'$ and $\boldsymbol{r}_1'$ are obtained from $\boldsymbol{t}, \boldsymbol{q}$ and $\boldsymbol{r}_1$ via the substitution $\boldsymbol{s}/\boldsymbol{v}$.
Weakening

$$\frac{\forall \boldsymbol{u} \sqsubset \boldsymbol{r}\, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{u}} \vdash |B|^{\boldsymbol{t}}_{\boldsymbol{w}}}{\forall \boldsymbol{u} \sqsubset \boldsymbol{r}\, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{u}}, \forall \boldsymbol{y} \sqsubset \boldsymbol{c}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash |B|^{\boldsymbol{t}}_{\boldsymbol{w}}}\ (\mathrm{wkn})$$

Contraction. We only show the relevant variables $\boldsymbol{x}_1, \boldsymbol{x}_2$ in the terms $\boldsymbol{r}_0, \boldsymbol{r}_1$ and $\boldsymbol{t}$:

$$\cfrac{\cfrac{\cfrac{\forall \boldsymbol{y} \sqsubset \boldsymbol{r}_0[\boldsymbol{x}_0, \boldsymbol{x}_1]\, |A|^{\boldsymbol{x}_0}_{\boldsymbol{y}}, \forall \boldsymbol{y} \sqsubset \boldsymbol{r}_1[\boldsymbol{x}_0, \boldsymbol{x}_1]\, |A|^{\boldsymbol{x}_1}_{\boldsymbol{y}} \vdash |B|^{\boldsymbol{t}[\boldsymbol{x}_0, \boldsymbol{x}_1]}_{\boldsymbol{w}}}{\forall \boldsymbol{y} \sqsubset \boldsymbol{r}_0[\boldsymbol{x}, \boldsymbol{x}]\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}, \forall \boldsymbol{y} \sqsubset \boldsymbol{r}_1[\boldsymbol{x}, \boldsymbol{x}]\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash |B|^{\boldsymbol{t}[\boldsymbol{x}, \boldsymbol{x}]}_{\boldsymbol{w}}}\ (\boldsymbol{x}/\boldsymbol{x}_0, \boldsymbol{x}/\boldsymbol{x}_1)}{\forall \boldsymbol{y} \sqsubset \mathbf{b_2}(\boldsymbol{r}_0[\boldsymbol{x}, \boldsymbol{x}], \boldsymbol{r}_1[\boldsymbol{x}, \boldsymbol{x}])\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}, \forall \boldsymbol{y} \sqsubset \mathbf{b_2}(\boldsymbol{r}_0[\boldsymbol{x}, \boldsymbol{x}], \boldsymbol{r}_1[\boldsymbol{x}, \boldsymbol{x}])\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash |B|^{\boldsymbol{t}[\boldsymbol{x}, \boldsymbol{x}]}_{\boldsymbol{w}}}\ (\mathrm{B}_2)}{\forall \boldsymbol{y} \sqsubset \mathbf{b_2}(\boldsymbol{r}_0[\boldsymbol{x}, \boldsymbol{x}], \boldsymbol{r}_1[\boldsymbol{x}, \boldsymbol{x}])\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash |B|^{\boldsymbol{t}[\boldsymbol{x}, \boldsymbol{x}]}_{\boldsymbol{w}}}\ (\mathrm{con})$$

Universal quantifier ($\forall$I/E)

$$\cfrac{\cfrac{\forall \boldsymbol{w} \sqsubset \boldsymbol{r}\, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}} \vdash |A(z)|^{\boldsymbol{t}}_{\boldsymbol{y}}}{\forall \boldsymbol{w} \sqsubset \boldsymbol{r}\, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}} \vdash |A(z)|^{(\lambda z.\boldsymbol{t})z}_{\boldsymbol{y}}}\ (\beta)}{\forall \boldsymbol{w} \sqsubset \boldsymbol{r}\, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}} \vdash |\forall z A(z)|^{\lambda z.\boldsymbol{t}}_{\boldsymbol{y}, z}}\ (\mathrm{D.2.1}) \qquad \cfrac{\cfrac{\forall \boldsymbol{w} \sqsubset \boldsymbol{r}[z]\, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}} \vdash |\forall z A(z)|^{\boldsymbol{t}}_{\boldsymbol{y}, z}}{\forall \boldsymbol{w} \sqsubset \boldsymbol{r}[s]\, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}} \vdash |\forall z A(z)|^{\boldsymbol{t}}_{\boldsymbol{y}, s}}\ (s/z)}{\forall \boldsymbol{w} \sqsubset \boldsymbol{r}[s]\, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}} \vdash |A(s)|^{\boldsymbol{t}s}_{\boldsymbol{y}}}\ (\mathrm{D.2.1})$$

Existential quantifier ($\exists$I/E)

$$\frac{\forall \boldsymbol{w} \sqsubset \boldsymbol{r}\, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}} \vdash |A(s)|^{\boldsymbol{t}}_{\boldsymbol{y}}}{\forall \boldsymbol{w} \sqsubset \boldsymbol{r}\, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}} \vdash |\exists z A(z)|^{\boldsymbol{t}, s}_{\boldsymbol{y}}}\ (\mathrm{D.2.1}) \qquad \frac{\forall \boldsymbol{u} \sqsubset \boldsymbol{r}\, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{u}}, \forall \boldsymbol{y} \sqsubset \boldsymbol{q}\, |A(z)|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash |B|^{\boldsymbol{t}}_{\boldsymbol{w}}}{\forall \boldsymbol{u} \sqsubset \boldsymbol{r}\, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{u}}, \forall \boldsymbol{y} \sqsubset \boldsymbol{q}\, |\exists z A(z)|^{\boldsymbol{x}, z}_{\boldsymbol{y}} \vdash |B|^{\boldsymbol{t}}_{\boldsymbol{w}}}\ (\mathrm{D.2.1})$$

□

In contrast to *cut elimination*, in a functional interpretation the *cut rule is interpreted* again by another instance of the cut rule with the help of condition (B_3) which, due to the presence of the parameter relation, states an abstract form of functional application or composition. On the other hand, the *quantifier rules are eliminated* via the use of higher-order functionals. In this respect, one might view functional interpretations as quantifier-elimination procedures. Obviously, new quantifiers might be introduced in the interpretation of negation – via the abbreviation $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$ – and quantifier rules might be introduced again in order to deal with conditions (B_1), (B_2) and (B_3).

2.2 Completeness We have seen how a proof of A in IL can be transformed into a proof (in the system T^ω) that some sequence of terms $\boldsymbol{t}$ is a witness for A. This section describes three (parametrised) schemata which, over T^ω, allows one to conclude A from the fact that A has witnesses. Those are: the schema of choice (described in Section 1.1), the schema of independence of premise for $\forall \sqsubset$-formulas

$$\mathsf{IP}_\sqsubset \quad : \quad (\forall \boldsymbol{x} A_{\mathrm{b}}(\boldsymbol{x}) \to \exists \boldsymbol{y} B(\boldsymbol{y})) \to \exists \boldsymbol{y}(\forall \boldsymbol{x} A_{\mathrm{b}}(\boldsymbol{x}) \to B(\boldsymbol{y})),$$

and the Markov principle for $\sqsubset$-bounded formulas

$$\mathsf{MP}_\sqsubset \quad : \quad (\forall \boldsymbol{x} A_{\mathrm{b}}(\boldsymbol{x}) \to B_{\mathrm{b}}) \to \exists \boldsymbol{b}(\forall \boldsymbol{x} \sqsubset \boldsymbol{b}\, A_{\mathrm{b}}(\boldsymbol{x}) \to B_{\mathrm{b}}).$$

Let $\mathsf{T}^\#$ denote the extension of T^ω with these three axiom schemata.

Theorem 2.3 *For arbitrary formulas A in the language of* IL, $\mathsf{T}^{\#}$ *proves* $A \leftrightarrow \exists \boldsymbol{x} \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$.

Proof By induction on the logical structure of A. The only non-trivial case being when A has the form $A \to B$

$$\begin{array}{rcl} A \to B & \overset{\mathrm{IH}}{\Longleftrightarrow} & \exists \boldsymbol{x} \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to \exists \boldsymbol{v} \forall \boldsymbol{w} |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\ & \overset{\mathsf{T}^{\omega}}{\Longleftrightarrow} & \forall \boldsymbol{x} (\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to \exists \boldsymbol{v} \forall \boldsymbol{w} |B|^{\boldsymbol{v}}_{\boldsymbol{w}}) \\ & \overset{\mathsf{IP}_{\sqsubset}}{\Longleftrightarrow} & \forall \boldsymbol{x} \exists \boldsymbol{v} (\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to \forall \boldsymbol{w} |B|^{\boldsymbol{v}}_{\boldsymbol{w}}) \\ & \overset{\mathsf{T}^{\omega}}{\Longleftrightarrow} & \forall \boldsymbol{x} \exists \boldsymbol{v} \forall \boldsymbol{w} (\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to |B|^{\boldsymbol{v}}_{\boldsymbol{w}}) \\ & \overset{\mathsf{MP}_{\sqsubset}}{\Longleftrightarrow} & \forall \boldsymbol{x} \exists \boldsymbol{v} \forall \boldsymbol{w} \exists \boldsymbol{b} (\forall \boldsymbol{y} \sqsubset \boldsymbol{b} \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to |B|^{\boldsymbol{v}}_{\boldsymbol{w}}) \\ & \overset{\mathsf{AC}}{\Longleftrightarrow} & \exists \boldsymbol{f}, \boldsymbol{g} \forall \boldsymbol{x}, \boldsymbol{w} (\forall \boldsymbol{y} \sqsubset \boldsymbol{g}\boldsymbol{x}\boldsymbol{w} \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to |B|^{\boldsymbol{f}\boldsymbol{x}}_{\boldsymbol{w}}) \\ & \overset{(\mathrm{D}.2.1)}{\Longleftrightarrow} & \exists \boldsymbol{f}, \boldsymbol{g} \forall \boldsymbol{x}, \boldsymbol{w} |A \to B|^{\boldsymbol{f}, \boldsymbol{g}}_{\boldsymbol{x}, \boldsymbol{w}} \end{array}$$

The case $\forall x A(x)$ also uses the principle of choice. □

Since we have left open what the abbreviation $\forall \boldsymbol{x} \sqsubset \boldsymbol{t} \, A(\boldsymbol{x})$ might stand for, we are not in a position to claim that the interpretation of the principles $\mathsf{MP}_{\sqsubset}$, $\mathsf{IP}_{\sqsubset}$ and AC will be trivialised by the interpretation, as is the case for each of the instantiations considered in Section 3.

2.3 Extensions of the parametrised interpretation In this section we indicate how Theorem 2.2 can be extended in several ways already on the level of the parametrised interpretation, i.e. prior to any concrete instantions of the parameter abbreviation $\forall \boldsymbol{x} \sqsubset \boldsymbol{t} \, A(\boldsymbol{x})$ are considered. For the first extension we will make use of the following definition.

Definition 2.4 (Purely universal formulas) We denote by $\mathcal{U}$ the class of formulas of the form $\forall \boldsymbol{y} A(\boldsymbol{y})$ where $A(\boldsymbol{y})$ is built out of atomic formulas via conjunction and implication.

It is easy to see that for formulas $B \in \mathcal{U}$ (say $B \equiv \forall \boldsymbol{y} A(\boldsymbol{y})$) we have $|B|_{\boldsymbol{y}} \equiv A(\boldsymbol{y})$. This implies that formulas B in $\mathcal{U}$ do not ask for witnesses (the tuple of witnessing variables is empty) and moreover B implies its parametrised interpretation, no matter what the choice of the abbreviation $\forall \boldsymbol{x} \sqsubset \boldsymbol{t} \, A(\boldsymbol{x})$ is. Therefore, in Theorem 2.2 axioms in $\mathcal{U}$ can be added to the interpreted theory IL given that those are also added to the verifying theory T^{ω}.

Another trivial generalisation of Theorem 2.2 is to extend the language of the interpreted theory IL to the language of all finite types. This gives rise to a system which we will refer to as IL^{ω}. One must simply notice that the interpretation of quantifiers given in Definition 2.1 can be easily generalised as

$$|\forall z^{\rho} A(z)|^{\boldsymbol{f}}_{\boldsymbol{y}, z} :\equiv |A(z^{\rho})|^{\boldsymbol{f}z}_{\boldsymbol{y}} \qquad\qquad |\exists z^{\rho} A(z)|^{\boldsymbol{x}, z}_{\boldsymbol{y}} :\equiv |A(z^{\rho})|^{\boldsymbol{x}}_{\boldsymbol{y}}$$

if the language of the interpreted theory already contains quantification over higher-order variables. In order to have an interpretation of IL^{ω} on the parametrised level (before instantiations) one needs to be careful about the treatment of higher-order equality. We would like to have a purely universal axiomatisation, i.e. by formulas in $\mathcal{U}$, so that the interpretations of the axioms are implied by the axioms themselves. For instance, we can adopt a *minimal* treatment of extensionality (see section 3.3 of [19]), in the sense that only equality between terms of the basic type o is taken as primitive in the language and the axiom schemata characterising the behaviour of the logical constants (combinators) Π and Σ are in the class $\mathcal{U}$. Obviously, as concrete instations of the parametrised interpretation are considered the amount of extensionality allowed will also vary.

The basic parametrised interpretation of intuitionistic logic can also be extended to deal with arithmetic. Let Heyting arithmetic in all finite types HA^{ω} be an extension of IL^{ω} with constants for zero and the successor function, together with the appropriate quantifier-free axioms, and the induction rule:

$$\dfrac{\vdash A(0) \qquad A(n) \vdash A(n+1)}{\vdash A(n)}\ \mathsf{IND}$$

Without loss of generality we can assume that the subproofs $\vdash A(0)$ and $A(n) \vdash A(n+1)$ do not contain extra assumptions, as this is enough for deriving all instances of the induction schema. The language of HA^ω also contains the recursor R for Gödel's primitive recursion in all finite types, with quantifier-free axioms (sub$_\mathsf{R}$)

$$\vdash P(\mathsf{R}xy0) \leftrightarrow P(x) \qquad\qquad \vdash P(\mathsf{R}xy(n+1)) \leftrightarrow P(yn(\mathsf{R}xyn))$$

where $P(\cdot)$ is a an atomic formula with a distinguished variable of appropriate type. It is easy to show that (sub$_\mathsf{R}$) can be extended to arbitrary formulas A. The system HA^ω has a parametrised functional interpretation since witnessing terms for $\vdash A(n)$ can be produced out of witnesses for $\vdash A(0)$ and $A(n) \vdash A(n+1)$ as

$$\dfrac{\dfrac{\dfrac{\vdash |A(0)|^{\boldsymbol{s}}_{\boldsymbol{y}}}{\vdash \forall \boldsymbol{y} |A(0)|^{\boldsymbol{s}}_{\boldsymbol{y}}}\ \forall\mathrm{I}}{\vdash \forall \boldsymbol{y} |A(0)|^{\mathsf{R}(\boldsymbol{s}, \lambda n.\boldsymbol{t}, 0)}_{\boldsymbol{y}}} \qquad \dfrac{\dfrac{\dfrac{\forall \boldsymbol{y}' \sqsubset \boldsymbol{q}[\boldsymbol{x}, \boldsymbol{y}]\, |A(n)|^{\boldsymbol{x}}_{\boldsymbol{y}'} \vdash |A(n+1)|^{\boldsymbol{tx}}_{\boldsymbol{y}}}{\forall \boldsymbol{y} |A(n)|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash \forall \boldsymbol{y} |A(n+1)|^{\boldsymbol{tx}}_{\boldsymbol{y}}}\ (\mathrm{A}_1)}{\forall \boldsymbol{y} |A(n)|^{\mathsf{R}(\boldsymbol{s}, \lambda n.\boldsymbol{t}, n)}_{\boldsymbol{y}} \vdash \forall \boldsymbol{y} |A(n+1)|^{\boldsymbol{t}\,\mathsf{R}(\boldsymbol{s}, \lambda n.\boldsymbol{t}, n)}_{\boldsymbol{y}}}\ (\mathsf{R}(\boldsymbol{s}, \lambda n.\boldsymbol{t}, n)/\boldsymbol{x})}{\forall \boldsymbol{y} |A(n)|^{\mathsf{R}(\boldsymbol{s}, \lambda n.\boldsymbol{t}, n)}_{\boldsymbol{y}} \vdash \forall \boldsymbol{y} |A(n+1)|^{\mathsf{R}(\boldsymbol{s}, \lambda n.\boldsymbol{t}, n+1)}_{\boldsymbol{y}}}\ (\mathrm{sub}_\mathsf{R})}{\vdash \forall \boldsymbol{y} |A(n)|^{\mathsf{R}(\boldsymbol{s}, \lambda n.\boldsymbol{t}, n)}_{\boldsymbol{y}}}\ \mathsf{IND}$$

We do not consider here extensions of the interpretation in order to deal with classical logic or comprehension principles (cf. [17; 2]), since those are normally dealt with via complementary translations such as the negative (double-negation) translation and Friedman's A-translation. The combinations of the interpretations discussed here with those translations, e.g. Kreisel's non-counterexample interpretation and Shoenfield's interpretation [16], are out of the scope of this article.

3 Instantiations of $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$

We will see next that by simply instantiating the abbreviation $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$ in the parametrised functional interpretation we obtain well-known functional interpretations, both the formula interpretation and the corresponding standard soundness theorem – see, however, the discussion about the more subtle bounded functional interpretation on Section 3.5. In each case we fix the verifying theory T^ω and explicitly give the families of sequences of terms $\mathbf{b_1}, \mathbf{b_2}$ and $\mathbf{b_3}$ (of T^ω) and show that conditions (B_1), (B_2) and (B_3) hold for such choices. Notice also that for each instantiation of $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$ the $\sqsubset$-bounded formulas constitute a concrete class of formulas, characterising in each case the (formula) range of each concrete interpretation.

3.1 Kreisel's modified realizability The first instantiation we consider is Kreisel's modified realizability, first discussed in [14] and further elaborated in [15]. Modified realizability is nowadays normally viewed as a higher-order variant of *Kleene's realizability* where realizers are functionals rather than numeric codes of partial recursive functions. Therefore, it is surprising that it was discovered via a detailed analysis of Gödel's Dialectica interpretation (see [14]). By showing that both modified realizability and the Dialectica interpretation are two straightforward instances of the parametrised interpretation we confirm Kreisel's impression that the two interpretation are rather similar.

The verifying system T^ω for modified realizability can be taken to be IL^ω, with an extensional treatment of equality. In most expositions modified realizability is defined as follows:

Definition 3.1 (Modified realizability [14; 15]) For each formula A of IL we associate a new formula $\boldsymbol{x}$ mr A of IL^ω ($\boldsymbol{x}$ is a sequence of fresh variable) inductively as follows:

$$\varepsilon \quad \mathsf{mr} \quad P \quad :\equiv \quad P$$

where ε is the empty sequence of variables. Assume we have already defined $\boldsymbol{x}$ mr A and $\boldsymbol{v}$ mr B, we define

$$\begin{array}{llll} \boldsymbol{x}, \boldsymbol{y} & \mathsf{mr} & A \wedge B & :\equiv & (\boldsymbol{x} \;\mathsf{mr}\; A) \wedge (\boldsymbol{v} \;\mathsf{mr}\; B) \\ \boldsymbol{x}, \boldsymbol{v}, n & \mathsf{mr} & A \vee B & :\equiv & (\boldsymbol{x} \;\mathsf{mr}\; A) \vee_n (\boldsymbol{v} \;\mathsf{mr}\; B) \\ \boldsymbol{f} & \mathsf{mr} & A \to B & :\equiv & \forall \boldsymbol{x}((\boldsymbol{x} \;\mathsf{mr}\; A) \to (\boldsymbol{f}\boldsymbol{x} \;\mathsf{mr}\; B)) \\ \boldsymbol{f} & \mathsf{mr} & \forall z A(z) & :\equiv & \forall z(\boldsymbol{f} z \;\mathsf{mr}\; A(z)) \\ \boldsymbol{x}, z & \mathsf{mr} & \exists z A(z) & :\equiv & \boldsymbol{x} \;\mathsf{mr}\; A(z). \end{array}$$

The free-variables of the formula $\boldsymbol{x}$ mr A are $\boldsymbol{x}$ and those already free in A.

In order to obtain modified realizability from the parametrised functional interpretation we take $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$ to be an abbreviation for $\forall \boldsymbol{x} A(\boldsymbol{x})$. In such case, the definition of implication given in Definition 2.1 when instantiated becomes

$$|A \to B|^{\boldsymbol{f}}_{\boldsymbol{x},\boldsymbol{w}} :\equiv \forall \boldsymbol{y}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to |B|^{\boldsymbol{f}\boldsymbol{x}}_{\boldsymbol{w}}$$

where the sequence of variables $\boldsymbol{g}$ can be assumed to be empty. Notice that conditions (B_1), (B_2) and (B_3) clearly hold for such choice, with the sequences of terms $\mathbf{b_1}, \mathbf{b_2}$ taken to be the empty sequence.

Lemma 3.2 *In Definition 2.1, let $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$ be an abbreviation for $\forall x A(x)$. Then for all formulas A in the language of* IL

$$\vdash_{\mathsf{T}^\omega} (\boldsymbol{x} \;\mathsf{mr}\; A) \leftrightarrow \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.$$

Proof By induction on the logical structure of A. The case in which A is atomic is trivial. For the composite cases, assume $(\boldsymbol{x} \;\mathsf{mr}\; A) \leftrightarrow \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ and $(\boldsymbol{v} \;\mathsf{mr}\; B) \leftrightarrow \forall \boldsymbol{w} |A|^{\boldsymbol{v}}_{\boldsymbol{w}}$. We then have:

$$\begin{array}{lcl} (\boldsymbol{f} \;\mathsf{mr}\; A \to B) & \stackrel{(\mathrm{D}.3.1)}{\Longleftrightarrow} & \forall \boldsymbol{x}((\boldsymbol{x} \;\mathsf{mr}\; A) \to (\boldsymbol{f}\boldsymbol{x} \;\mathsf{mr}\; B)) \\ & \stackrel{\mathrm{IH}}{\Longleftrightarrow} & \forall \boldsymbol{x}(\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to \forall \boldsymbol{w} |B|^{\boldsymbol{f}\boldsymbol{x}}_{\boldsymbol{w}}) \\ & \stackrel{\mathsf{T}^\omega}{\Longleftrightarrow} & \forall \boldsymbol{x}, \boldsymbol{w}(\forall \boldsymbol{y}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to |B|^{\boldsymbol{f}\boldsymbol{x}}_{\boldsymbol{w}}) \\ & \stackrel{(\mathrm{D}.2.1)}{\Longleftrightarrow} & \forall \boldsymbol{x}, \boldsymbol{w} |A \to B|^{\boldsymbol{f}}_{\boldsymbol{x},\boldsymbol{w}} \\ (\boldsymbol{x}, \boldsymbol{v} \;\mathsf{mr}\; A \wedge B) & \stackrel{(\mathrm{D}.3.1)}{\Longleftrightarrow} & (\boldsymbol{x} \;\mathsf{mr}\; A) \wedge (\boldsymbol{v} \;\mathsf{mr}\; B) \\ & \stackrel{\mathrm{IH}}{\Longleftrightarrow} & \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \wedge \forall \boldsymbol{w} |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\ & \stackrel{\mathsf{T}^\omega}{\Longleftrightarrow} & \forall \boldsymbol{y}, \boldsymbol{w}(|A|^{\boldsymbol{x}}_{\boldsymbol{y}} \wedge |B|^{\boldsymbol{v}}_{\boldsymbol{w}}) \\ & \stackrel{(\mathrm{D}.2.1)}{\Longleftrightarrow} & \forall \boldsymbol{y}, \boldsymbol{w} |A \wedge B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} \end{array}$$

$$
\begin{array}{rcl}
(\boldsymbol{x}, \boldsymbol{v}, n \;\mathsf{mr}\; A \vee B) & \overset{(D.3.1)}{\Longleftrightarrow} & (\boldsymbol{x} \;\mathsf{mr}\; A) \vee_n (\boldsymbol{v} \;\mathsf{mr}\; B) \\
& \overset{\mathrm{IH}}{\Longleftrightarrow} & \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \vee_n \forall \boldsymbol{w} |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
& \overset{\mathsf{T}^\omega}{\Longleftrightarrow} & \forall \boldsymbol{y}, \boldsymbol{w} (|A|^{\boldsymbol{x}}_{\boldsymbol{y}} \vee_n |B|^{\boldsymbol{v}}_{\boldsymbol{w}}) \\
& \overset{(D.2.1)}{\Longleftrightarrow} & \forall \boldsymbol{y}, \boldsymbol{w} |A \vee B|^{\boldsymbol{x}, \boldsymbol{v}, n}_{\boldsymbol{y}, \boldsymbol{w}}
\end{array}
$$

$$
\begin{array}{rcl}
(\boldsymbol{f} \;\mathsf{mr}\; \forall z A(z)) & \overset{(D.3.1)}{\Longleftrightarrow} & \forall z (\boldsymbol{f} z \;\mathsf{mr}\; A(z)) \\
& \overset{\mathrm{IH}}{\Longleftrightarrow} & \forall z (\forall \boldsymbol{y} |A(z)|^{\boldsymbol{f} z}_{\boldsymbol{y}}) \\
& \overset{(D.2.1)}{\Longleftrightarrow} & \forall z, \boldsymbol{y} |\forall z A(z)|^{\boldsymbol{f}}_{z, \boldsymbol{y}}
\end{array}
\qquad
\begin{array}{rcl}
(\boldsymbol{x}, z \;\mathsf{mr}\; \exists z A(z)) & \overset{(D.3.1)}{\Longleftrightarrow} & \boldsymbol{x} \;\mathsf{mr}\; A(z) \\
& \overset{\mathrm{IH}}{\Longleftrightarrow} & \forall \boldsymbol{y} |A(z)|^{\boldsymbol{x}}_{\boldsymbol{y}} \\
& \overset{(D.2.1)}{\Longleftrightarrow} & \forall \boldsymbol{y} |\exists z A(z)|^{\boldsymbol{x}, z}_{\boldsymbol{y}}
\end{array}
$$

The equivalence between $\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \vee_n \forall \boldsymbol{w} |B|^{\boldsymbol{v}}_{\boldsymbol{w}}$ and $\forall \boldsymbol{y}, \boldsymbol{w} (|A|^{\boldsymbol{x}}_{\boldsymbol{y}} \vee_n |B|^{\boldsymbol{v}}_{\boldsymbol{w}})$ makes use of the decidability of the atomic formula $n = 0$. □

The soundness of the modified realizability interpretation follows directly from Theorem 2.2 and Lemma 3.2.

3.2 Gödel's Dialectica interpretation We show that, similarly to modified realizability, the Dialectica interpretation can be obtained as a straightforward instantiation of the parametrised interpretation. The Dialectica interpretation is normally defined as follows:

Definition 3.3 (Dialectica interpretation [1; 9]) For each formula A of IL we associate a new formulas A^D and A_D such that $A^D \equiv \exists \boldsymbol{x} \forall \boldsymbol{y} A_D(\boldsymbol{x}, \boldsymbol{y})$ (where A_D is quantifier-free) inductively as follows:

$$(P)^D \quad :\equiv \quad P, \quad \text{when } P \text{ is an atomic formula.}$$

Assume $A^D \equiv \exists \boldsymbol{x} \forall \boldsymbol{y} A_D(\boldsymbol{x}, \boldsymbol{y})$ and $B^D \equiv \exists \boldsymbol{v} \forall \boldsymbol{w} B_D(\boldsymbol{v}, \boldsymbol{w})$. We then define

$$
\begin{array}{lcl}
(A \wedge B)^D & :\equiv & \exists \boldsymbol{x}, \boldsymbol{v} \forall \boldsymbol{y}, \boldsymbol{w} (A_D(\boldsymbol{x}, \boldsymbol{y}) \wedge B_D(\boldsymbol{v}, \boldsymbol{w})) \\
(A \vee B)^D & :\equiv & \exists \boldsymbol{x}, \boldsymbol{v}, n \forall \boldsymbol{y}, \boldsymbol{w} (A_D(\boldsymbol{x}, \boldsymbol{y}) \vee_n B_D(\boldsymbol{v}, \boldsymbol{w})) \\
(A \to B)^D & :\equiv & \exists \boldsymbol{f}, \boldsymbol{g} \forall \boldsymbol{x}, \boldsymbol{w} (A_D(\boldsymbol{x}, \boldsymbol{g} \boldsymbol{x} \boldsymbol{w}) \to B_D(\boldsymbol{f} \boldsymbol{x}, \boldsymbol{w})) \\
(\forall z A(z))^D & :\equiv & \exists \boldsymbol{f} \forall z, \boldsymbol{w} A_D(\boldsymbol{f} z, \boldsymbol{x}, z) \\
(\exists z A(z))^D & :\equiv & \exists z, \boldsymbol{x} \forall \boldsymbol{y} A_D(\boldsymbol{x}, \boldsymbol{y}, z)
\end{array}
$$

where in each case $(\cdot)_D$ is defined as the maximal quantifier-free subformula of $(\cdot)^D$. For instance, $(A \vee B)_D \equiv A_D(\boldsymbol{x}, \boldsymbol{y}) \vee_n B_D(\boldsymbol{v}, \boldsymbol{w})$ and $(A \to B)_D \equiv A_D(\boldsymbol{x}, \boldsymbol{g} \boldsymbol{x} \boldsymbol{w}) \to B_D(\boldsymbol{f} \boldsymbol{x}, \boldsymbol{w})$.

In order to obtain Gödel's original functional interpretation from the parametrised functional interpretation we take $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$ to be an abbreviation for $A(\boldsymbol{t})$. In such case, the definition of implication can again on the meta-level be simplified to

$$|A \to B|^{\boldsymbol{f}, \boldsymbol{g}}_{\boldsymbol{x}, \boldsymbol{w}} :\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{g} \boldsymbol{x} \boldsymbol{w}} \to |B|^{\boldsymbol{f} \boldsymbol{x}}_{\boldsymbol{w}}$$

and that is what we use in the following. Notice also that the $\sqsubset$-bounded formulas are simply the quantifier-free formulas. Therefore, we can take the verifying system T^ω to be the quantifier-free fragment of IL^ω.

We must show that conditions (B_1), (B_2) and (B_3) hold for such choice of the abbreviation. Condition (B_1) holds by taking $\boxed{\mathbf{b_1} \boldsymbol{y} := \boldsymbol{y}}$. The fact that condition (B_2) holds now is not as trivial as in the case of modified realizability. For any quantifier-free formula $A_{\mathsf{qf}}(\boldsymbol{y})$ we must produce a sequence of terms $\mathbf{b_2}$ satisfying

$$A_{\mathsf{qf}}(\mathbf{b_2} \boldsymbol{y}_0 \boldsymbol{y}_1) \vdash_{\mathsf{T}^\omega} A_{\mathsf{qf}}(\boldsymbol{y}_0) \wedge A_{\mathsf{qf}}(\boldsymbol{y}_1).$$

This can be achieved e.g. if for each quantifier-free formula A_{qf} we can produce a term $t_{A_{\mathsf{qf}}}$ satisfying

$$\vdash_{\mathsf{T}^\omega} A_{\mathsf{qf}}(\boldsymbol{y}) \leftrightarrow t_{A_{\mathsf{qf}}}\boldsymbol{y} = 0.$$

If this is the case we can define $\boxed{\mathbf{b_2}\boldsymbol{y}_0\boldsymbol{y}_1 := \mathsf{if}(t_{A_{\mathsf{qf}}}\boldsymbol{y}_0, \boldsymbol{y}_1, \boldsymbol{y}_0)}$. Given the choice for the abbreviation above condition (B$_3$) reduces to $A_{\mathsf{qf}}(\mathbf{b_3}\boldsymbol{hb}) \vdash_{\mathsf{T}^\omega} A_{\mathsf{qf}}(\boldsymbol{hb})$, and we can simply take $\boxed{\mathbf{b_3}\boldsymbol{hb} := \boldsymbol{hb}}$.

Lemma 3.4 *In Definition 2.1, let $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$ be an abbreviation for $A(\boldsymbol{t})$. Then for all formulas A in the language of* IL

$$\vdash_{\mathsf{T}^\omega} A_D(\boldsymbol{x}, \boldsymbol{y}) \leftrightarrow |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.$$

Proof With the simplification outlined above for the case of implication, one can immediately see that the definition of $A_D(\boldsymbol{x}, \boldsymbol{y})$ coincides precisely with the definition of $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ (Definition 2.1) taking $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$ as an abbreviation for $A(\boldsymbol{t})$. □

The soundness of Gödel's Dialectica interpretation then follows directly from Theorem 2.2 and Lemma 3.4.

3.3 Diller-Nahm functional interpretation The Diller-Nahm functional interpretation is normally viewed as a variant of Gödel's Dialectica interpretation where decidability of atomic formulas is no longer necessary. That is achieved by *collecting* all candidate witnesses rather than *deciding* which candidate is indeed a witness (as in the Dialectica interpretation). The gain of not having to assume that atomic formulas are decidable comes with a cost: instead of producing witnessing terms given a proof of A the Diller-Nahm interpretation only produces a finite collection of candidate witnesses, with the assurance that one of those is indeed a witness.

For the Diller-Nahm interpretation we will assume that the verifying theory T^ω is defined over a language where the finite types $\mathcal{T}$ are extended with *finite sequence* constructions, i.e. if $\rho \in \mathcal{T}$ then $\rho^* \in \mathcal{T}$. Naturally, we also assume that T^ω contains contants (with appropriate defining axioms) for handling finite sequences, such as a length functor $\mathsf{len}(\cdot) : \rho^* \to o$, sufficient arithmetic for indexing the finite sequences, and a binary less-than relation on the basic type.

Definition 3.5 (Diller-Nahm interpretation [4]) For each formula A of IL we associate new formulas $A^\wedge$ and $A_\wedge$ of T^ω such that $A^\wedge \equiv \exists \boldsymbol{x} \forall \boldsymbol{y} A_\wedge(\boldsymbol{x}, \boldsymbol{y})$ (with $A_\wedge(\boldsymbol{x}, \boldsymbol{y})$ a 0-bounded – i.e. contains only bounded numerical quantifiers) inductively as follows:

$$(P)^\wedge \;:\equiv\; P$$

Assume $A^\wedge \equiv \exists \boldsymbol{x} \forall \boldsymbol{y} A_\wedge(\boldsymbol{x}, \boldsymbol{y})$ and $B^\wedge \equiv \exists \boldsymbol{v} \forall \boldsymbol{w} B_\wedge(\boldsymbol{v}, \boldsymbol{w})$. We then define

$$\begin{array}{lcl}
(A \wedge B)^\wedge & :\equiv & \exists \boldsymbol{x}, \boldsymbol{v} \forall \boldsymbol{y}, \boldsymbol{w} (A_\wedge(\boldsymbol{x}, \boldsymbol{y}) \wedge B_\wedge(\boldsymbol{v}, \boldsymbol{w})) \\
(A \vee B)^\wedge & :\equiv & \exists \boldsymbol{x}, \boldsymbol{v}, n \forall \boldsymbol{y}, \boldsymbol{w} (A_\wedge(\boldsymbol{x}, \boldsymbol{y}) \vee_n B_\wedge(\boldsymbol{v}, \boldsymbol{w})) \\
(A \to B)^\wedge & :\equiv & \exists \boldsymbol{f}, \boldsymbol{g} \forall \boldsymbol{x}, \boldsymbol{w} (\forall \boldsymbol{i} < \mathsf{len}(\boldsymbol{g}) A_\wedge(\boldsymbol{x}, \boldsymbol{g}_i \boldsymbol{x}\boldsymbol{w}) \to B_\wedge(\boldsymbol{f}\boldsymbol{x}, \boldsymbol{w})) \\
(\forall z A(z))^\wedge & :\equiv & \exists \boldsymbol{f} \forall z, \boldsymbol{w} A_\wedge(\boldsymbol{f} z, \boldsymbol{x}, z) \\
(\exists z A(z))^\wedge & :\equiv & \exists z, \boldsymbol{x} \forall \boldsymbol{y} A_\wedge(\boldsymbol{x}, \boldsymbol{y}, z).
\end{array}$$

In each case the formulas $(\cdot)_\wedge$ are defined in the spirit of Gödel's Dialectica interpretation, e.g. $(A \to B)_\wedge \equiv \forall \boldsymbol{i} < \mathsf{len}(\boldsymbol{g}) A_\wedge(\boldsymbol{x}, \boldsymbol{g}_i \boldsymbol{x}\boldsymbol{w}) \to B_\wedge(\boldsymbol{f}\boldsymbol{x}, \boldsymbol{w})$.

The only difference to Gödel's original functional interpretation is in the treatment of implication. Rather than asking for a tuple of functionals $\boldsymbol{g}^{\boldsymbol{\tau}}$ producing the concrete witnesses, in the Diller-Nahm interpretation $\boldsymbol{g}$ has type $\boldsymbol{\tau}^*$. It is also possible to define the Diller-Nahm interpretation without extending the finite type structure, by simply coding finite sequence via a pair of an infinite sequence $o \to \tau$ together with the length of the finite sequence. For simplicity we abbreviate $\forall \boldsymbol{i} < \mathsf{len}(\boldsymbol{g}) A(\boldsymbol{g}_i)$ by $\forall \boldsymbol{x} \in \boldsymbol{g}\, A(\boldsymbol{x})$, since finite sequences can also be viewed as finite multisets. Using this shorthand the treatment of implication can be rewritten as

$$(A \to B)^\wedge :\equiv \exists \boldsymbol{f}, \boldsymbol{g} \forall \boldsymbol{x}, \boldsymbol{w} (\forall \boldsymbol{y} \in \boldsymbol{g}\boldsymbol{x}\boldsymbol{w}\, A_\wedge(\boldsymbol{x}, \boldsymbol{y}) \to B_\wedge(\boldsymbol{f}\boldsymbol{x}, \boldsymbol{w})).$$

Therefore, the Diller-Nahm interpretation can be viewed as an instantiation of the parametrised functional interpretation where $\forall \boldsymbol{x}^\tau \sqsubset \boldsymbol{t}^{\tau^*} A(\boldsymbol{x})$ is an abbreviation for $\forall \boldsymbol{x} \in \boldsymbol{t}\, A(\boldsymbol{x})$. In order to see that this is a valid abbreviation, we must produce families of sequences of terms $\mathbf{b_1}, \mathbf{b_2}$ and $\mathbf{b_3}$ satisfying the conditions (B_1), (B_2) and (B_3). Those can be given independently of the formula A_{b} as

$$\boxed{\mathbf{b_1}\boldsymbol{y} := \langle y_0 \rangle, \ldots, \langle y_n \rangle} \qquad \boxed{\mathbf{b_2}\boldsymbol{y}_0\boldsymbol{y}_1 := \boldsymbol{y}_0 \cup \boldsymbol{y}_1} \qquad \boxed{\mathbf{b_3}\boldsymbol{h}\boldsymbol{b} := \textstyle\bigcup_{\boldsymbol{z} \in \boldsymbol{b}} \boldsymbol{h}\boldsymbol{z}}$$

where the union symbol above denotes pointwise union of finite multisets, which I assume to be definable in T^ω.

Lemma 3.6 *In Definition 2.1, let $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$ be an abbreviation for $\forall \boldsymbol{x} \in \boldsymbol{t}\, A(\boldsymbol{x})$. Then for all formulas A in the language of* IL

$$\vdash_{\mathsf{T}^\omega} A_\wedge(\boldsymbol{x}, \boldsymbol{y}) \leftrightarrow |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.$$

The soundness of the Diller-Nahm interpretation follows directly from Theorem 2.2 and Lemma 3.6.

3.4 Stein's family of interpretations In [18], a family of interpretation between Diller-Nahm and modified realizability is defined, parametrised by a number $n > 0$. The parameter n basically dictates the types of the universal quantifiers up to which the interpretation leaves "untouched", as done in the definition of modified realizability. Universal quantifiers in the premise of an implication of type level greater than n will be pulled out as a set of witnesses, similarly to the Diller-Nahm interpretation. The interpretation we define below is a slight reformulation of Stein's definition, in the sense that for each formula A, our definition of A^n is intuitionistically (but not syntactically) equivalent to his definition.

For the rest of this section we use the following notation: given a tuple of variable $\boldsymbol{x}$, we will denote by $\underline{\boldsymbol{x}}$ the sub-tuple containing the variables in $\boldsymbol{x}$ which have type level $\geq n$, whereas $\overline{\boldsymbol{x}}$ denotes the sub-tuple of the variables in $\boldsymbol{x}$ which have type level $< n$. The actual value of n will be clear from the context.

Definition 3.7 (Stein's family of interpretations [18]) For each positive natural number n, the interpretation of a formula A of IL^ω is a new formula A^n of the form $\exists \boldsymbol{x} \forall \underline{\boldsymbol{y}} \forall \overline{\boldsymbol{y}} A_n$, where A_n contains only universal quantifiers of type level $< n$, and no existential quantifier. The assignment is done inductively as follows:

$$(P)^n \quad :\equiv \quad P$$

Assume we have $A^n \equiv \exists \boldsymbol{x} \forall \underline{\boldsymbol{y}} \forall \overline{\boldsymbol{y}} A_n(\boldsymbol{x}, \boldsymbol{y})$ and $B^n \equiv \exists \boldsymbol{v} \forall \underline{\boldsymbol{w}} \forall \overline{\boldsymbol{w}} B_n(\boldsymbol{v}, \boldsymbol{w})$, we define

$$
\begin{array}{lll}
(A \wedge B)^n & :\equiv & \exists \boldsymbol{x}, \boldsymbol{v} \forall \underline{\boldsymbol{y}}, \underline{\boldsymbol{w}} \forall \overline{\boldsymbol{y}}, \overline{\boldsymbol{w}} (A_n(\boldsymbol{x}, \boldsymbol{y}) \wedge B_n(\boldsymbol{v}, \boldsymbol{w})) \\
(A \vee B)^n & :\equiv & \exists \boldsymbol{x}, \boldsymbol{v}, m \forall \underline{\boldsymbol{y}}, \underline{\boldsymbol{w}} \forall \overline{\boldsymbol{y}}, \overline{\boldsymbol{w}} (A_n(\boldsymbol{x}, \boldsymbol{y}) \vee_m B_n(\boldsymbol{v}, \boldsymbol{w})) \\
(A \to B)^n & :\equiv & \exists \boldsymbol{f}, \boldsymbol{g} \forall \underline{\boldsymbol{x}}, \underline{\boldsymbol{w}} \forall \overline{\boldsymbol{x}}, \overline{\boldsymbol{w}} (\forall \boldsymbol{i}^{n-1} \forall \overline{\boldsymbol{y}} A_n(\boldsymbol{x}, \boldsymbol{g}\boldsymbol{x}\boldsymbol{w}\boldsymbol{i}, \overline{\boldsymbol{y}}) \to B_n(\boldsymbol{f}\boldsymbol{x}, \boldsymbol{w})) \\
(\forall z^\rho A(z))^n & :\equiv & \exists \boldsymbol{f} \forall \underline{\boldsymbol{y}} \forall z \forall \overline{\boldsymbol{y}} A_n(\boldsymbol{f} z, \boldsymbol{y}, z) \\
(\exists z^\rho A(z))^n & :\equiv & \exists z, \boldsymbol{x} \forall \underline{\boldsymbol{y}} \forall \overline{\boldsymbol{y}} A_n(\boldsymbol{x}, \boldsymbol{y}, z)
\end{array}
$$

where $(\forall z^\rho A(x))_n :\equiv \forall z \forall \overline{\boldsymbol{y}} A_n(\boldsymbol{f} z, \boldsymbol{y}, z)$ if $\mathsf{level}(\rho) < n$, and $\forall \overline{\boldsymbol{y}} A_n(\boldsymbol{f} z, \boldsymbol{y}, z)$ otherwise. Above we are also abbreviating by $\boldsymbol{i}^{n-1}$ a sequence of variables all of pure type $n-1$.

Whereas in the Diller-Nahm interpretation one collects potential witnesses into finite multisets, in the case of Stein's family of interpretations, one collects the potential witnesses into an infinite set indexed by elements of the pure type $(n-1)$. Therefore, in the treatment of implication, the type of $\boldsymbol{g}\boldsymbol{x}\boldsymbol{w}$ is actually a finite sequence of functionals of type $(n-1) \to \boldsymbol{\rho}$, rather than a finite sequence of objects of type $\boldsymbol{\rho}$. For the sake of simplicity and intuition we write quantifications of the form $\forall \boldsymbol{i}^{n-1} A(\boldsymbol{t}\boldsymbol{i})$ as $\forall \boldsymbol{y} \in \mathsf{rng}(\boldsymbol{t})\, A(\boldsymbol{y})$. We can then more clearly write the treatment of implication as

$$(A \to B)^n :\equiv \exists \boldsymbol{f}, \boldsymbol{g} \forall \underline{\boldsymbol{x}}, \underline{\boldsymbol{w}} \forall \overline{\boldsymbol{x}}, \overline{\boldsymbol{w}} (\forall \underline{\boldsymbol{y}} \in \mathsf{rng}(\boldsymbol{g}\boldsymbol{x}\boldsymbol{w}) \forall \overline{\boldsymbol{y}} A_n(\boldsymbol{x}, \boldsymbol{y}) \to B_n(\boldsymbol{f}\boldsymbol{x}, \boldsymbol{w})).$$

We show now that also Stein's family of interpretations can be obtained from the parametrised functional interpretation via the following instantiation

$$(*) \quad \forall \boldsymbol{x}^{\boldsymbol{\tau}} \sqsubset^n \boldsymbol{t}^{(n-1) \to \underline{\boldsymbol{\tau}}} A(\boldsymbol{x}) :\equiv \forall \underline{\boldsymbol{x}} \in \mathsf{rng}(\boldsymbol{t}) \forall \overline{\boldsymbol{x}} A(\boldsymbol{x})$$

where $\underline{\boldsymbol{\tau}}$ is the type of the sequence $\underline{\boldsymbol{x}}$.

It is again easy to see that this choice complies with conditions (B_1, B_2, B_3). In the case of (B_1) we can take $\boxed{\mathbf{b_1} \boldsymbol{y} := \lambda i. \underline{\boldsymbol{y}}}$. As for ($B_2$), all we need is a functional t of type $(n-1) \to (o \times (n-1))$ whose range contains the set $(\{0,1\} \times (n-1))$. We can then take $\boxed{\mathbf{b_2} \boldsymbol{y}_0 \boldsymbol{y}_1 := \lambda i. \mathsf{if}((ti)_0, \underline{\boldsymbol{y}_0}(ti)_1, \underline{\boldsymbol{y}_1}(ti)_1)}$, where $(ti)_0$ and $(ti)_1$ represent the first and second projections of the pair of ti, respectively. In the case of condition (B_3), for simplicity, we consider singleton tuples only. We must produce a term b_3 satisfying

$$\forall y^\sigma \sqsubset \mathsf{b}_3 h b\, A_{\mathrm{b}}(y) \vdash_{\mathsf{T}^\omega} \forall x^\tau \sqsubset b \forall y^\sigma \sqsubset h x\, A_{\mathrm{b}}(y).$$

The only non-trivial situation is when $\mathsf{level}(\sigma) \geq n$ and $\mathsf{level}(\tau) < n$, i.e.

$$\forall y^\sigma \in \mathsf{b}_3 h A_{\mathrm{b}}(y) \vdash_{\mathsf{T}^\omega} \forall x^\tau \forall y^\sigma \in h x A_{\mathrm{b}}(y),$$

which stands for (b is not used)

$$\forall i^{n-1} A_{\mathrm{b}}(\mathsf{b}_3 h i) \vdash_{\mathsf{T}^\omega} \forall x^\tau \forall j^{n-1} A_{\mathrm{b}}(h x j).$$

Since $\mathsf{level}(\tau) < n$, we can then take $\boxed{\mathsf{b}_3 h i := h(ti)_0(ti)_1}$ using any surjective functional $t : (n-1) \to (\tau \times (n-1))$.

Lemma 3.8 *In Definition 2.1, let $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$ be an abbreviation for $\forall \boldsymbol{x} \sqsubset^n \boldsymbol{t}\, A(\boldsymbol{x})$ as defined above. Then for all formulas A (let $A^n \equiv \exists \boldsymbol{x} \forall \underline{\boldsymbol{y}} \forall \overline{\boldsymbol{y}} A_n(\boldsymbol{x}, \boldsymbol{y})$) in the language of* IL^ω

$$\vdash_{\mathsf{T}^\omega} A_n(\boldsymbol{x}, \boldsymbol{y}) \leftrightarrow |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.$$

Proof We only present here the case of implication. Assume $A^n \equiv \exists \boldsymbol{x} \forall \underline{\boldsymbol{y}} \forall \overline{y} A_n(\boldsymbol{x}, \boldsymbol{y})$ and $B^n \equiv \exists \boldsymbol{v} \forall \underline{\boldsymbol{w}} \forall \overline{\boldsymbol{w}} B_n(\boldsymbol{v}, \boldsymbol{w})$. Then

$$\begin{array}{rcl}(A \to B)_n(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{x}, \boldsymbol{w}) & \overset{\text{(D.3.7)}}{\Longleftrightarrow} & \forall \boldsymbol{i}^{n-1} \forall \overline{\boldsymbol{y}} A_n(\boldsymbol{x}, \boldsymbol{gxwi}, \overline{\boldsymbol{y}}) \to B_n(\boldsymbol{fx}, \boldsymbol{w}) \\ & \overset{\text{(abb)}}{\Longleftrightarrow} & \forall \underline{\boldsymbol{y}} \in \mathsf{rng}(\boldsymbol{gxw}) \forall \overline{\boldsymbol{y}} A_n(\boldsymbol{x}, \boldsymbol{y}) \to B_n(\boldsymbol{fx}, \boldsymbol{w}) \\ & \overset{(*)}{\Longleftrightarrow} & \forall \boldsymbol{y} \sqsubset^n \boldsymbol{gxw}\, A_n(\boldsymbol{x}, \boldsymbol{y}) \to B_n(\boldsymbol{fx}, \boldsymbol{w}) \\ & \overset{\text{IH}}{\Longleftrightarrow} & \forall \boldsymbol{y} \sqsubset^n \boldsymbol{gxw}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to |B|^{\boldsymbol{fx}}_{\boldsymbol{w}} \\ & \overset{\text{(D.2.1)}}{\Longleftrightarrow} & |A \to B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}}.\end{array}$$

□

The soundness of Stein's family of interpretations follows directly from Theorem 2.2 and Lemma 3.8.

3.5 Bounded functional interpretation In this section we show how the formula interpretation component of the recent *bounded functional interpretation*[1] [7] (b.f.i. for short) relates to our parametrised formula interpretation set out in Section 2.1. We will first need to extend Definition 2.1 to deal with bounded quantifiers, which will then allow for a simplification of the interpretation of disjunction. Although, we do not obtain the soundness of b.f.i. as a direct instantiation of Theorem 2.2, we indicate how conditions (B_1), (B_2) and (B_3) can be used to justify the design choices of the bounded functional interpretation.

Let IL^ω be an extension of IL to the language of finite types with a minimal treatment of extensionality, as discussed in Section 2.3. Moreover, let $\{\leq^*_\rho\}_{\rho \in \mathcal{T}}$ represent Bezem's family of strong majorizability relations [3], and $\mathsf{IL}^\omega_{\leq^*}$ denote the extension of IL^ω axiomatising[2] the family of relations $\{\leq^*_\rho\}_{\rho \in \mathcal{T}}$ as described in [7]. The type of the relation $\leq^*$ will always be clear from the context and we will omit the typing subscript henceforth. If $x \leq^* b$ we say that b majorizes x. Self-majorizing functionals $b^{\rho \to \tau}$ are called *monotone*, since for such functionals if $x \leq^* a$ then $bx \leq^* ba$. Contrary to [7], we will assume that monotone quantifiers (denoted in [7] by $\tilde{\forall}$ and $\tilde{\exists}$) are part of the language, with defining axiom schemata

$$\vdash \tilde{\forall} b A(b) \leftrightarrow \forall b (b \leq^* b \to A(b)) \qquad\qquad \vdash \tilde{\exists} b A(b) \leftrightarrow \exists b (b \leq^* b \wedge A(b))$$

We will let variables a, b, c, d, e range over monotone objects.

Definition 3.9 For each formula A of IL^ω let $[A] \in \mathcal{L}(\mathsf{IL}^\omega_{\leq^*})$ be obtained inductively as

$$\begin{array}{lcl}[P] & :\equiv & [P], \quad \text{for atomic formulas} \\ [A \star B] & :\equiv & [A] \star [B], \quad \text{for } \star \in \{\wedge, \vee, \to\} \\ [\forall x A(x)] & :\equiv & \tilde{\forall} b \forall x \leq^* b\, [A(x)] \\ [\exists x A(x)] & :\equiv & \tilde{\exists} b \exists x \leq^* b\, [A(x)]\end{array}$$

The formula $[A]$ can be viewed as a relativization of the quantifiers in A to Bezem's model of strongly majorizable functionals, since $\tilde{\forall} b \forall x \leq^* b\, A(x)$ and $\tilde{\exists} b \exists x \leq^* b\, A(x)$ are respectively equivalent to $\forall x (\tilde{\exists} b (x \leq^* b) \to A(x))$ and $\exists x (\tilde{\exists} b (x \leq^* b) \wedge A(x))$. Moreover, the equivalence between A and $[A]$ can be proved using the majorizability axioms

$$\mathsf{MAJ}^\rho \quad : \quad \forall x^\rho \tilde{\exists} b^\rho (x \leq^* b).$$

Note that the formulas $[A]$ only contain *monotone quantifiers* ($\tilde{\forall} b A(b)$ and $\tilde{\exists} b A(b)$) and *bounded quantifiers* ($\forall x \leq^* a\, A(x)$ and $\exists x \leq^* a\, A(x)$). Let us denote by $\mathcal{B}$ the class of formulas containing only those two kinds of quantifiers. We change slightly the definition of the b.f.i. given in [7] to focus on the interpretation of formulas in $\mathcal{B}$.

Definition 3.10 (Bounded functional interpretation [7]) For each formula $A \in \mathcal{B}$ we associate formulas $(A)^{\mathrm{B}}$ and A_{B} of $\mathsf{IL}^{\omega}_{\leq^*}$ such that $(A)^{\mathrm{B}} \equiv \tilde{\exists}\boldsymbol{b}\tilde{\forall}\boldsymbol{c}A_{\mathrm{B}}(\boldsymbol{b},\boldsymbol{c})$, with $A_{\mathrm{B}}(\boldsymbol{b},\boldsymbol{c})$ a bounded formula, as

$$(P)^{\mathrm{B}} \quad :\equiv \quad P$$

Assume $(A)^{\mathrm{B}} \equiv \tilde{\exists}\boldsymbol{b}\tilde{\forall}\boldsymbol{c}A_{\mathrm{B}}(\boldsymbol{b},\boldsymbol{c})$ and $(B)^{\mathrm{B}} \equiv \tilde{\exists}\boldsymbol{d}\tilde{\forall}\boldsymbol{e}B_{\mathrm{B}}(\boldsymbol{d},\boldsymbol{e})$. We then define

$$
\begin{array}{lll}
(A \wedge B)^{\mathrm{B}} & :\equiv & \tilde{\exists}\boldsymbol{b},\boldsymbol{d}\tilde{\forall}\boldsymbol{c},\boldsymbol{e}(A_{\mathrm{B}}(\boldsymbol{b},\boldsymbol{c}) \wedge B_{\mathrm{B}}(\boldsymbol{d},\boldsymbol{e}))\\
(A \vee B)^{\mathrm{B}} & :\equiv & \tilde{\exists}\boldsymbol{b},\boldsymbol{d}\tilde{\forall}\boldsymbol{c},\boldsymbol{e}(\tilde{\forall}\boldsymbol{c}' \leq^* \boldsymbol{c}\, A_{\mathrm{B}}(\boldsymbol{b},\boldsymbol{c}') \vee \tilde{\forall}\boldsymbol{e}' \leq^* \boldsymbol{e}\, B_{\mathrm{B}}(\boldsymbol{d},\boldsymbol{e}'))\\
(A \to B)^{\mathrm{B}} & :\equiv & \tilde{\exists}\boldsymbol{f},\boldsymbol{g}\tilde{\forall}\boldsymbol{b},\boldsymbol{e}(\tilde{\forall}\boldsymbol{c} \leq^* \boldsymbol{gbe}\, A_{\mathrm{B}}(\boldsymbol{b},\boldsymbol{c}) \to B_{\mathrm{B}}(\boldsymbol{fb},\boldsymbol{e}))\\
(\tilde{\forall}aA(a))^{\mathrm{B}} & :\equiv & \tilde{\exists}\boldsymbol{f}\tilde{\forall}a,\boldsymbol{c}A_{\mathrm{B}}(a,\boldsymbol{f}a,\boldsymbol{c})\\
(\tilde{\exists}aA(a))^{\mathrm{B}} & :\equiv & \tilde{\exists}a,\boldsymbol{b}\tilde{\forall}\boldsymbol{c}A_{\mathrm{B}}(a,\boldsymbol{b},\boldsymbol{c})\\
(\forall x \leq^* t\, A(x))^{\mathrm{B}} & :\equiv & \tilde{\exists}\boldsymbol{b}\tilde{\forall}\boldsymbol{c}\forall x \leq^* t\, A_{\mathrm{B}}(x,\boldsymbol{b},\boldsymbol{c})\\
(\exists x \leq^* t\, A(x))^{\mathrm{B}} & :\equiv & \tilde{\exists}\boldsymbol{b}\tilde{\forall}\boldsymbol{c}\exists x \leq^* t\,\tilde{\forall}\boldsymbol{c}' \leq^* \boldsymbol{c}\, A_{\mathrm{B}}(x,\boldsymbol{b},\boldsymbol{c}')
\end{array}
$$

where in each case $(\cdot)_{\mathrm{B}}$ is the maximal bounded subformula of $(\cdot)^{\mathrm{B}}$, and $\tilde{\forall}\boldsymbol{c} \leq^* \boldsymbol{t}\, A(\boldsymbol{c})$ abbreviates $\forall \boldsymbol{c} \leq^* \boldsymbol{t}\,(\boldsymbol{c} \leq^* \boldsymbol{c} \to A(\boldsymbol{c}))$.

What we intend to show is that the b.f.i. presented in [7] can either be viewed as a relativization of quantifiers (as described in Definition 3.9) followed by an interpretation of formulas in $\mathcal{B}$ (according to Definition 3.10); or as originally presented by combining these two steps into one, and giving a direct interpretation of the standard quantifiers as

$$(\forall x A(x))^{\mathrm{B}} :\equiv \tilde{\exists}\boldsymbol{f}\tilde{\forall}a,\boldsymbol{c}\forall x \leq^* a\, A_{\mathrm{B}}(x,\boldsymbol{f}a,\boldsymbol{c})$$

$$(\exists x A(x))^{\mathrm{B}} :\equiv \tilde{\exists}a,\boldsymbol{b}\tilde{\forall}\boldsymbol{c}\exists x \leq^* a\,\tilde{\forall}\boldsymbol{c}' \leq^* \boldsymbol{c}\, A_{\mathrm{B}}(x,\boldsymbol{b},\boldsymbol{c}')$$

In either case, what one obtains is an interpretation of formulas of the basic theory IL^{ω} (see Lemma 3.12). As observed by Ferreira, applying Definition 3.10 without the prior relativization would incur problems, since formulas $\tilde{\exists}aA(a)$ in general need not be monotone in a, which is necessary for the soundness proof of the b.f.i.

Therefore, the bounded functional interpretation of IL^{ω} can then be viewed as a quantifier relativization via the formula mapping $A \mapsto [A]$ followed by an instantiation of the parametrised interpretation (of formulas in $\mathcal{B}$) as

$$\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x}) :\equiv \tilde{\forall}\boldsymbol{x} \leq^* \boldsymbol{t}\, A(\boldsymbol{x})$$

where the interpretation of quantifiers given in Definition 2.1 is applied to the *monotone quantifiers*. Obviously, in order to compare Definitions 2.1 and 3.10 we must first extend the Definition 2.1 to deal with *bounded quantifiers*, which we do in a way similar to the b.f.i., i.e.

(*i*) $|\forall x \leq^* t\, A(x)|^{\boldsymbol{b}}_{\boldsymbol{c}} :\equiv \forall x \leq^* t\, |A(x)|^{\boldsymbol{b}}_{\boldsymbol{c}}$

(*ii*) $|\exists x \leq^* t\, A(x)|^{\boldsymbol{b}}_{\boldsymbol{c}} :\equiv \exists x \leq^* t\,\tilde{\forall}\boldsymbol{c}' \leq^* \boldsymbol{c}\, |A(x)|^{\boldsymbol{b}}_{\boldsymbol{c}'}$

Remark 3.11 The bounded quantification over $\boldsymbol{c}'$ in the interpretation of $\exists x \leq^* t\, A(x)$ is important since

$$\tilde{\forall}\boldsymbol{c}\exists x \leq^* t\, A_{\mathrm{b}}(\boldsymbol{c},x) \to \exists x \leq^* t\,\tilde{\forall}\boldsymbol{c}A_{\mathrm{b}}(\boldsymbol{c},x)$$

(which would be needed for the completeness of the interpretation) is generally false. On the other hand

$$\tilde{\forall}\boldsymbol{c}\exists x \leq^* t\,\tilde{\forall}\boldsymbol{c}' \leq^* \boldsymbol{c}\, A_{\mathrm{b}}(\boldsymbol{c}',x) \to \exists x \leq^* t\,\tilde{\forall}\boldsymbol{c}A_{\mathrm{b}}(\boldsymbol{c},x)$$

is a generalization of weak König's lemma, as shown in [7].

In a functional interpretation, disjunctions $A \vee B$ are interpreted by the existence of the flag n allowing the decidability of $A \vee_n B$. It is easy to see that given the presence of bounded quantifiers (and a limited amount of arithmetic) disjunctions can equivalently be interpreted via $\exists n \leq^* 1\,(A \vee_n B)$. Based on the interpretation of bounded existential quantifiers (*ii*) we can then simplify the interpretation of disjunction in Definition 2.1 directly as

(*iii*) $|A \vee B|^{\boldsymbol{b},\boldsymbol{d}}_{\boldsymbol{c},\boldsymbol{e}} :\equiv \forall \boldsymbol{c}' \leq^* \boldsymbol{c}\, |A|^{\boldsymbol{b}}_{\boldsymbol{c}'} \vee \forall \boldsymbol{e}' \leq^* \boldsymbol{e}\, |B|^{\boldsymbol{d}}_{\boldsymbol{e}'}$

Lemma 3.12 *Let Definition 2.1 be extended to deal with bounded quantifiers, and the interpretation of disjunction be simplified as in (i), (ii) and (iii) above. Moreover, let* $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$ *be an abbreviation for* $\forall \boldsymbol{x} \leq^* \boldsymbol{t}\, A(\boldsymbol{x})$*, and let Definition 3.10 be extended to deal with formulas in* IL^ω *directly, as originally presented in [7]. Then for all formulas* $A \in \mathcal{L}(\mathsf{IL}^\omega)$ *(let* $(A)^{\mathrm{B}} \equiv \tilde{\exists} \boldsymbol{b} \tilde{\forall} \boldsymbol{c} A_{\mathrm{B}}(\boldsymbol{b}; \boldsymbol{c})$*) we have*

$$\vdash_{\mathsf{IL}^\omega_{\leq^*}} A_{\mathrm{B}}(\boldsymbol{b}; \boldsymbol{c}) \leftrightarrow |[A]|^{\boldsymbol{b}}_{\boldsymbol{c}}$$

where we are separating via a semicolon the sequences of existentially and universally quantified variables in $(\cdot)_{\mathrm{B}}$*.*

Proof The proof is by induction on the logical structure of A. The only non-trivial cases are those of the quantifiers:

$$
\begin{array}{lcl}
(\forall x A(x))_{\mathrm{B}}(\boldsymbol{f}; \boldsymbol{c}, a) & \overset{(\mathrm{D}.3.10)}{\Longleftrightarrow} & \forall x \leq^* a\, (A(x))_{\mathrm{B}}(\boldsymbol{f} a; \boldsymbol{c}) \\
& \overset{\mathrm{IH}}{\Longleftrightarrow} & \forall x \leq^* a\, |[A(x)]|^{\boldsymbol{f} a}_{\boldsymbol{c}} \\
& \overset{(i)}{\Longleftrightarrow} & |\forall x \leq^* a\, [A(x)]|^{\boldsymbol{f} a}_{\boldsymbol{c}} \\
& \overset{(\mathrm{D}.2.1)}{\Longleftrightarrow} & |\tilde{\forall} a \forall x \leq^* a\, [A(x)]|^{\boldsymbol{f}}_{\boldsymbol{c}, a} \\
& \overset{(\mathrm{D}.3.9)}{\Longleftrightarrow} & |[\forall x A(x)]|^{\boldsymbol{f}}_{\boldsymbol{c}, a} \\
(\exists x A(x))_{\mathrm{B}}(\boldsymbol{b}, a; \boldsymbol{c}) & \overset{(\mathrm{D}.3.10)}{\Longleftrightarrow} & \exists x \leq^* a\, \tilde{\forall} \boldsymbol{c}' \leq^* \boldsymbol{c}\, (A(x))_{\mathrm{B}}(\boldsymbol{b}; \boldsymbol{c}') \\
& \overset{\mathrm{IH}}{\Longleftrightarrow} & \exists x \leq^* a\, \tilde{\forall} \boldsymbol{c}' \leq^* \boldsymbol{c}\, |[A(x)]|^{\boldsymbol{b}}_{\boldsymbol{c}'} \\
& \overset{(ii)}{\Longleftrightarrow} & |\exists x \leq^* a\, [A(x)]|^{\boldsymbol{b}}_{\boldsymbol{c}} \\
& \overset{(\mathrm{D}.2.1)}{\Longleftrightarrow} & |\tilde{\exists} a \exists x \leq^* a\, [A(x)]|^{\boldsymbol{b}, a}_{\boldsymbol{c}} \\
& \overset{(\mathrm{D}.3.9)}{\Longleftrightarrow} & |[\exists x A(x)]|^{\boldsymbol{b}, a}_{\boldsymbol{c}}
\end{array}
$$

□

Let $\max_\tau\{b^\tau, c^\tau\}$ be defined pointwise for all types $\tau \in \mathcal{T}$. Since only monotone functionals are considered by the bounded functional interpretation, we might assume that $\boldsymbol{b}$ and $\boldsymbol{c}$ in the interpreted formula $|A|^{\boldsymbol{b}}_{\boldsymbol{c}}$ are sequences of *monotone* variables. Therefore, in order to satisfy conditions (B_1, B_2, B_3) we can take

$$\boxed{\mathbf{b_1} \boldsymbol{b} := \boldsymbol{b}} \qquad \boxed{\mathbf{b_2} \boldsymbol{b}_0 \boldsymbol{b}_1 := \max\{\boldsymbol{b}_0, \boldsymbol{b}_1\}} \qquad \boxed{\mathbf{b_3} \boldsymbol{h} \boldsymbol{b} := \boldsymbol{h} \boldsymbol{b}}$$

since for monotone functionals b, b_0, b_1, h we have

$$
\left.
\begin{array}{rcl}
& & b \leq^* b \\
\bigwedge_{i=0}^{1} (y \leq^* b_i & \to & y \leq^* \max\{b_0, b_1\}) \\
\forall z \leq^* b (y \leq^* hz & \to & y \leq^* hb)
\end{array}
\right\}
$$

Notice that instanting $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$ with $\tilde{\forall} \boldsymbol{x} \leq^* \boldsymbol{t}\, A(\boldsymbol{x})$ without the prior relativization described in Definition 3.9 would render it impossible to satisfy already condition (B_1) since open terms in general do not have a majorant – unless its free-variables are assumed to be monotone.

The soundness theorem for the b.f.i. does not follow directly from Theorem 2.2 and Lemma 3.12, due to the initial relativization of quantifiers. A more elaborated soundness proof, however does go through as described in [7].

4 Parametrised soundness proof

This section describes how the standard soundness theorem (Theorem 2.2) for the parametrised formula interpretation can also be parametrised, via a second parameter abbreviation. This gives rise to a family of soundness theorems for a family of formula interpretations. The second parameter $\exists \boldsymbol{x} \prec \boldsymbol{t}\, A(\boldsymbol{x})$ should again be viewed as a formula abbreviation. For instance, we can choose $\exists \boldsymbol{x} \prec \boldsymbol{t}\, A(\boldsymbol{x})$ to mean $A(\boldsymbol{t})$. In fact, this instantiation will give us back Theorem 2.2 as a special case of our parametrised soundness theorem. We will show, however, that $\exists \boldsymbol{x} \prec \boldsymbol{t}\, A(x)$ can also be taken to be $\exists \boldsymbol{x} \leq^* \boldsymbol{t}\, A(\boldsymbol{x})$, where $\leq^*$ is Howard's majorizability relation (see [10]). This was first observed in [12], and gives rise to so-called *monotone* versions of the Dialectica interpretation and modified realizability. As we will see, according to the framework set up in Section 2, these monotone variants are a combination of the standard formula interpretations with a *monotone soundness proof.*

As conditions on the abbreviation $\exists \boldsymbol{x} \prec \boldsymbol{t}\, A(\boldsymbol{x})$ we will consider bounded versions of (B_1, B_2, B_3). For all $\sqsubset$-bounded formulas A_{b}, with free-variables $\boldsymbol{a}$, there are sequences of *closed* terms $\mathbf{b_1^*}, \mathbf{b_2^*}, \mathbf{b_3^*}$ such that

(B_1^*) $\vdash_{\mathsf{T}^\omega} \exists \boldsymbol{\nu} \prec \mathbf{b_1^*} \forall \boldsymbol{a}, \boldsymbol{x} (\forall \boldsymbol{y} \sqsubset \boldsymbol{\nu a x}\, A_{\mathrm{b}}(\boldsymbol{y}) \to A_{\mathrm{b}}(\boldsymbol{x}))$

(B_2^*) $\vdash_{\mathsf{T}^\omega} \exists \boldsymbol{\chi} \prec \mathbf{b_2^*} \forall \boldsymbol{a}, \boldsymbol{y}_0, \boldsymbol{y}_1 (\forall \boldsymbol{y} \sqsubset \boldsymbol{\chi a y}_0 \boldsymbol{y}_1\, A_{\mathrm{b}}(\boldsymbol{y}) \to \forall \boldsymbol{y} \sqsubset \boldsymbol{y}_i\, A_{\mathrm{b}}(\boldsymbol{y}))$, for $i \in \{0, 1\}$

(B_3^*) $\vdash_{\mathsf{T}^\omega} \exists \boldsymbol{\xi} \prec \mathbf{b_3^*} \forall \boldsymbol{a}, \boldsymbol{h}, \boldsymbol{b} (\forall \boldsymbol{y} \sqsubset \boldsymbol{\xi a h b}\, A_{\mathrm{b}}(\boldsymbol{y}) \to \forall \boldsymbol{z} \sqsubset \boldsymbol{b} \forall \boldsymbol{y} \sqsubset \boldsymbol{h z}\, A_{\mathrm{b}}(\boldsymbol{y}))$

i.e. we do not require the sequences of terms $\mathbf{b_1}, \mathbf{b_2}$ and $\mathbf{b_3}$ to be part of the language, but only *bounding* terms $\mathbf{b_1^*}, \mathbf{b_2^*}$ and $\mathbf{b_3^*}$ for those, according to the choice of the abbreviation $\exists \boldsymbol{x} \prec \boldsymbol{t}\, A(\boldsymbol{x})$.

Moreover, in order to ensure that $\exists \boldsymbol{x} \prec \boldsymbol{t}\, A(\boldsymbol{x})$ behaves as an existential quantifier we add the following two conditions. Firstly, for all $\sqsubset$-bounded formulas $A_{\mathrm{b}}(\boldsymbol{a}, \boldsymbol{x})$, contexts Γ (consisting also of $\sqsubset$-bounded formulas), and sequence of closed terms $\boldsymbol{s}$

(E_1) if $\forall \boldsymbol{a} \Gamma(\boldsymbol{a}, \boldsymbol{x}) \vdash_{\mathsf{T}^\omega} \forall \boldsymbol{a} A_{\mathrm{b}}(\boldsymbol{a}, \boldsymbol{x})$ then $\exists \boldsymbol{x} \prec \boldsymbol{s} \forall \boldsymbol{a} \Gamma(\boldsymbol{a}, \boldsymbol{x}) \vdash_{\mathsf{T}^\omega} \exists \boldsymbol{x} \prec \boldsymbol{s} \forall \boldsymbol{a} A_{\mathrm{b}}(\boldsymbol{a}, \boldsymbol{x})$.

Secondly, for each $\sqsubset$-bounded formula A_{b}, sequence of closed term $\boldsymbol{s}$ and sequence of terms $\boldsymbol{t}[\boldsymbol{x}]$ (all free-variables of $\boldsymbol{t}$ contained in $\boldsymbol{x}$),

(E_2) if $\vdash_{\mathsf{T}^\omega} \exists \boldsymbol{x} \prec \boldsymbol{s} \forall \boldsymbol{a} A_{\mathrm{b}}(\boldsymbol{t}[\boldsymbol{x}], \boldsymbol{a})$ then $\vdash_{\mathsf{T}^\omega} \exists \boldsymbol{y} \prec \boldsymbol{t}^* \forall \boldsymbol{a} A_{\mathrm{b}}(\boldsymbol{y}, \boldsymbol{a})$, for a sequence of closed term $\boldsymbol{t}^*$.

We call $\boldsymbol{t}^*$ $\prec$*-majorizing terms* for $\boldsymbol{t}$. In particular, when the tuple $\boldsymbol{x}$ is empty we have that $\vdash_{\mathsf{T}^\omega} \forall \boldsymbol{a} A_{\mathrm{b}}(\boldsymbol{t}, \boldsymbol{a})$ implies $\vdash_{\mathsf{T}^\omega} \exists \boldsymbol{y} \prec \boldsymbol{t}^* \forall \boldsymbol{a} A_{\mathrm{b}}(\boldsymbol{y}, \boldsymbol{a})$.

We show that condition (B_1^*, B_2^*, B_3^*) together with (E_1, E_2) are sufficient for proving the following parametrised version of the standard soundness theorem (Theorem 2.2).

Theorem 4.1 (Parametrised soundness) *Let the abbreviations* $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$ *and* $\exists \boldsymbol{x} \prec \boldsymbol{t}\, A(\boldsymbol{x})$ *be fixed, and* T^ω *be as in Section 1.3 with conditions* (B_1, B_2, B_3) *replaced by* (B_1^*, B_2^*, B_3^*). *Moreover, assume that conditions* (E_1, E_2) *hold. If* $\Gamma \vdash A$ *then there are sequences of closed terms* $\boldsymbol{t}, \boldsymbol{r} \in \mathcal{L}(\mathsf{T}^\omega)$ *such that*

$$\vdash_{\mathsf{T}^\omega} \exists \boldsymbol{f}, \boldsymbol{g} \prec \boldsymbol{t}, \boldsymbol{r} \forall \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{y}\, |\Gamma \to A|^{\boldsymbol{fa}, \boldsymbol{ga}}_{\boldsymbol{v}, \boldsymbol{y}}$$

where $\mathsf{FV}(\Gamma) \cup \mathsf{FV}(A) \equiv \{\boldsymbol{a}\}$.

Proof Assuming conditions (E_1) and (E_2), the proof is a straightforwad generalisation of the proof of Theorem 2.2. We must simply be careful to show that in the treatment of the identity axiom, the rule of contraction, and the cut rule we only use the weaker conditions (B_1^*, B_2^*, B_3^*).
Consider a fixed instance $A \vdash A$ of the identity axiom. By (B_1^*) we have a T^ω-derivation of

$$\vdash \exists \boldsymbol{\nu} \prec \mathbf{b_1^*} \forall \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y} (\forall \boldsymbol{y}' \sqsubset \boldsymbol{\nu a x y} \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}'} \to |A|^{\boldsymbol{x}}_{\boldsymbol{y}})$$

which is equivalent to

$$\vdash \exists \boldsymbol{\nu} \prec \mathbf{b_1^*} \forall \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y} |A \to A|^{\boldsymbol{ta}, \boldsymbol{\nu a}}_{\boldsymbol{x}, \boldsymbol{y}}$$

for $\boldsymbol{t} := \lambda \boldsymbol{a} \lambda \boldsymbol{x} . \boldsymbol{x}$. By condition ($E_2$) we then get

$$\vdash \exists \boldsymbol{\nu}, \boldsymbol{f} \prec \mathbf{b_1^*}, \boldsymbol{t}^* \forall \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y} |A \to A|^{\boldsymbol{fa}, \boldsymbol{\nu a}}_{\boldsymbol{x}, \boldsymbol{y}}.$$

The contraction rule is treated as follows. Without loss of generality we can assume that the context Γ consists of only two copies of A. Assume also, by induction hypothesis, that we have closed terms $\boldsymbol{r}_0, \boldsymbol{r}_1, \boldsymbol{t}$ and a derivation of

$$\exists \boldsymbol{g}_0, \boldsymbol{g}_1, \boldsymbol{f} \prec \boldsymbol{r}_0, \boldsymbol{r}_1, \boldsymbol{t} \forall \boldsymbol{a}, \boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{w} |A \wedge A \to B|^{\boldsymbol{fa}, \boldsymbol{g}_0 \boldsymbol{a}, \boldsymbol{g}_1 \boldsymbol{a}}_{\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{w}}.$$

By condition (A_2), we have

$$\exists \boldsymbol{g}_0, \boldsymbol{g}_1, \boldsymbol{f} \prec \boldsymbol{r}_0, \boldsymbol{r}_1, \boldsymbol{t} \underbrace{\forall \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{w} (\forall \boldsymbol{y} \sqsubset \tilde{\boldsymbol{g}}_0 \boldsymbol{axw} \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \wedge \forall \boldsymbol{y} \sqsubset \tilde{\boldsymbol{g}}_0 \boldsymbol{axw} \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to |B|^{\tilde{\boldsymbol{f}} \boldsymbol{ax}}_{\boldsymbol{w}})}_{(i)}.$$

where $\tilde{\boldsymbol{g}}_0 \boldsymbol{axw} :\equiv \boldsymbol{g}_0 \boldsymbol{axxw}$ (similarly with $\tilde{\boldsymbol{g}}_1$ and $\tilde{\boldsymbol{f}}$). Consider also the following instance of (B_2^*)

$$\exists \boldsymbol{\chi} \prec \mathbf{b_2^*} \underbrace{\forall \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{w} (\forall \boldsymbol{y} \sqsubset \boldsymbol{\chi}(\boldsymbol{a}, \tilde{\boldsymbol{g}}_0 \boldsymbol{axw}, \tilde{\boldsymbol{g}}_1 \boldsymbol{axw}) \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to \forall \boldsymbol{y} \sqsubset \tilde{\boldsymbol{g}_j} \boldsymbol{axw} \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}).}_{(ii)}$$

for $j \in \{0, 1\}$. It is easy to check that (i) and (ii) imply

$$\forall \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{w} (\forall \boldsymbol{y} \sqsubset \boldsymbol{\chi}(\boldsymbol{a}, \tilde{\boldsymbol{g}}_0 \boldsymbol{axw}, \tilde{\boldsymbol{g}}_1 \boldsymbol{axw}) \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to |B|^{\tilde{\boldsymbol{f}} \boldsymbol{ax}}_{\boldsymbol{w}}) \qquad (\equiv \forall \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{w} |A \to B|^{\tilde{\boldsymbol{f}} \boldsymbol{a}, \tilde{\boldsymbol{\chi}} \boldsymbol{a}}_{\boldsymbol{x}, \boldsymbol{w}})$$

where $\tilde{\boldsymbol{\chi}} \boldsymbol{axw} := \boldsymbol{\chi}(\boldsymbol{a}, \tilde{\boldsymbol{g}}_0 \boldsymbol{axw}, \tilde{\boldsymbol{g}}_1 \boldsymbol{axw})$. Therefore, by condition (E_1) we have

$$\exists \boldsymbol{\chi}, \boldsymbol{g}_0, \boldsymbol{g}_1, \boldsymbol{f} \prec \mathbf{b_2^*}, \boldsymbol{r}_0, \boldsymbol{r}_1, \boldsymbol{t} \forall \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{w} |A \to B|^{\tilde{\boldsymbol{f}} \boldsymbol{a}, \tilde{\boldsymbol{\chi}} \boldsymbol{a}}_{\boldsymbol{x}, \boldsymbol{w}}.$$

Finally, by condition (E_2) (with $\boldsymbol{q}[\boldsymbol{\chi}, \boldsymbol{g}_0, \boldsymbol{g}_1] \equiv \lambda \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{w} . \boldsymbol{\chi}(\boldsymbol{a}, \boldsymbol{g}_0 \boldsymbol{axxw}, \boldsymbol{g}_1 \boldsymbol{axxw})$ and $\boldsymbol{s}[\boldsymbol{f}] \equiv \lambda \boldsymbol{a}, \boldsymbol{x} . \boldsymbol{faxx}$) this gives

$$\exists \boldsymbol{h}, \boldsymbol{f} \prec \boldsymbol{q}^*, \boldsymbol{s}^* \forall \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{w} |A \to B|^{\boldsymbol{fa}, \boldsymbol{ha}}_{\boldsymbol{x}, \boldsymbol{w}}.$$

For the cut rule assume we have derivations for (assume w.l.o.g. that Γ and Δ are single formulas)

$$\exists \boldsymbol{g}_0, \boldsymbol{h}_0 \prec \boldsymbol{q}_0, \boldsymbol{t}_0 \underbrace{\forall \boldsymbol{a}, \boldsymbol{v}_0, \boldsymbol{y} (\forall \boldsymbol{u}_0 \sqsubset \boldsymbol{g}_0 \boldsymbol{a} \boldsymbol{v}_0 \boldsymbol{y} \, |\Gamma|^{\boldsymbol{v}_0}_{\boldsymbol{u}_0} \to |A|^{\boldsymbol{h}_0 \boldsymbol{a} \boldsymbol{v}_0}_{\boldsymbol{y}})}_{(i)}$$

and (making use of condition (A_2))

$$\exists \boldsymbol{g}_1, \boldsymbol{h}_1, \boldsymbol{f} \prec \boldsymbol{q}_1, \boldsymbol{t}_1, \boldsymbol{s} \underbrace{\forall \boldsymbol{a}, \boldsymbol{v}_1, \boldsymbol{x}, \boldsymbol{w} (\forall \boldsymbol{u}_1 \sqsubset \boldsymbol{g}_1 \boldsymbol{a} \boldsymbol{v}_1 \boldsymbol{xw} \, |\Delta|^{\boldsymbol{v}_1}_{\boldsymbol{u}_1} \wedge \forall \boldsymbol{y} \sqsubset \boldsymbol{h}_1 \boldsymbol{a} \boldsymbol{v}_1 \boldsymbol{xw} \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to |B|^{\boldsymbol{fa} \boldsymbol{v}_1 \boldsymbol{x}}_{\boldsymbol{w}})}_{(ii)}$$

corresponding to the assumptions of the cut rule. Consider also the following instance of (B_3^*)

$$\exists \boldsymbol{\xi} \prec \mathbf{b_3^*} \underbrace{\forall \boldsymbol{b} (\forall \boldsymbol{u}_0 \sqsubset \boldsymbol{\xi}(\boldsymbol{g}_0 \boldsymbol{a} \boldsymbol{v}_0, \boldsymbol{h}_1 \boldsymbol{a} \boldsymbol{v}_1 \boldsymbol{xw}) \, |\Gamma|^{\boldsymbol{v}_0}_{\boldsymbol{u}_0} \to \forall \boldsymbol{y} \sqsubset \boldsymbol{h}_1 \boldsymbol{a} \boldsymbol{v}_1 \boldsymbol{xw} \, \forall \boldsymbol{u}_0 \sqsubset \boldsymbol{g}_0 \boldsymbol{a} \boldsymbol{v}_0 \boldsymbol{y} \, |\Gamma|^{\boldsymbol{v}_0}_{\boldsymbol{u}_0})}_{(iii)}$$

where $\boldsymbol{b} \equiv \boldsymbol{g}_0, \boldsymbol{h}_1, \boldsymbol{a}, \boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{x}, \boldsymbol{w}$. By condition ($A_1$) we can derive from $(i), (ii), (iii)$

$$\forall \boldsymbol{a}, \boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{w} | \Gamma \wedge \Delta \to B|^{\tilde{\boldsymbol{f}}\boldsymbol{a}, \tilde{\boldsymbol{\xi}}\boldsymbol{a}, \tilde{\boldsymbol{g}}_1\boldsymbol{a}}_{\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{w}}$$

where above we are using the abbreviations

$$\begin{array}{lll} \tilde{\boldsymbol{\xi}}\boldsymbol{a}\boldsymbol{v}_0\boldsymbol{v}_1\boldsymbol{w} & :\equiv & \boldsymbol{\xi}(\boldsymbol{g}_0\boldsymbol{a}\boldsymbol{v}_0, \boldsymbol{h}_1\boldsymbol{a}\boldsymbol{v}_1(\boldsymbol{h}_0\boldsymbol{a}\boldsymbol{v}_0)\boldsymbol{w}) \\ \tilde{\boldsymbol{g}}_1\boldsymbol{a}\boldsymbol{v}_0\boldsymbol{v}_1\boldsymbol{w} & :\equiv & \boldsymbol{g}_1\boldsymbol{a}\boldsymbol{v}_1(\boldsymbol{h}_0\boldsymbol{a}\boldsymbol{v}_0)\boldsymbol{w} \\ \tilde{\boldsymbol{f}}\boldsymbol{a}\boldsymbol{v}_0\boldsymbol{v}_1 & :\equiv & \boldsymbol{f}\boldsymbol{a}\boldsymbol{v}_1(\boldsymbol{h}_0\boldsymbol{a}\boldsymbol{v}_0). \end{array}$$

By (E_1) this gives

$$\exists \boldsymbol{\xi}, \boldsymbol{g}_0, \boldsymbol{g}_1, \boldsymbol{h}_0, \boldsymbol{h}_1, \boldsymbol{f} \prec \mathbf{b}^*_3, \boldsymbol{q}_0, \boldsymbol{q}_1, \boldsymbol{t}_0, \boldsymbol{t}_1, \boldsymbol{s}\, \forall \boldsymbol{a}, \boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{w} | \Gamma \wedge \Delta \to B|^{\tilde{\boldsymbol{f}}\boldsymbol{a}, \tilde{\boldsymbol{\xi}}\boldsymbol{a}, \tilde{\boldsymbol{g}}_1\boldsymbol{a}}_{\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{w}}.$$

Finally, by (E_2) we get

$$\exists \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} \prec \boldsymbol{s}^*, \boldsymbol{q}^*, \boldsymbol{t}^*\, \forall \boldsymbol{a}, \boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{w} | \Gamma \wedge \Delta \to B|^{\boldsymbol{f}\boldsymbol{a}, \boldsymbol{g}\boldsymbol{a}, \boldsymbol{h}\boldsymbol{a}}_{\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{w}}$$

for appropriate terms $\boldsymbol{s}, \boldsymbol{q}, \boldsymbol{t}$. The treatment of the logical rules, quantifiers, and other structural rules follow easily from the corresponding instances in the proof of Theorem 2.2 using (E_1) and (E_2). □

5 Final remarks

Table 3 summarises how different instantiations of $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$ and $\exists \boldsymbol{x} \prec \boldsymbol{t}\, A(\boldsymbol{x})$ give rise to different functional interpretations. The parametrised functional interpretation allows one to localise the differences between the various interpretations and to understand their properties. For instance, neither the Diller-Nahm interpretation nor the bounded functional interpretation require decidability of atomic formulas, but in interestingly different ways. In the case of the Diller-Nahm interpretation decidability is not necessary because potential witnesses are collected into finite sets, while in the bounded functional interpretation potential witnesses are collected into sets of functionals with a common majorant.

$\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$	$\exists \boldsymbol{x} \prec \boldsymbol{t}\, A(\boldsymbol{x})$	**Functional interpretations**
$A(\boldsymbol{t})$	$A(\boldsymbol{t})$	Dialectica interpretation (1958)
$\forall \boldsymbol{x} A(\boldsymbol{x})$	$A(\boldsymbol{t})$	Modified realizability (1962)
$\forall \boldsymbol{x} \in \boldsymbol{t}\, A(\boldsymbol{x})$	$A(\boldsymbol{t})$	Diller-Nahm interpretation (1962)
$\forall \underline{\boldsymbol{x}} \in \mathsf{rng}(\boldsymbol{t})\, \forall \overline{\boldsymbol{x}} A(\boldsymbol{x})$	$A(\boldsymbol{t})$	Stein's family of interpretations (1979)
$A(\boldsymbol{t})$	$\exists \boldsymbol{x} \leq^* \boldsymbol{t}\, A(\boldsymbol{x})$	Monotone Dialectica interpretation (1996)
$\forall \boldsymbol{x} A(\boldsymbol{x})$	$\exists \boldsymbol{x} \leq^* \boldsymbol{t}\, A(\boldsymbol{x})$	Monotone modified realizability (1998)
$\tilde{\forall} \boldsymbol{x} \leq^* \boldsymbol{t}\, A(\boldsymbol{x})$	$A(\boldsymbol{t})$	Bounded functional interpretation (2005)

Table 3 Instantiations of the parametrised functional interpretation

One should also notice that the parametrised functional interpretation can be applied directly to analyse proofs, leaving the instantiation to a later stage, after the (parametrised) witnessing term and (parametrised) verifying proof have been obtained. This can be achieved by adding to the language new bounded formulas $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$ and $\exists \boldsymbol{x} \prec \boldsymbol{t}\, A(\boldsymbol{x})$, families of constants $\mathbf{b}^*_1, \mathbf{b}^*_2, \mathbf{b}^*_3$ and axiom schemata corresponding to the relevant conditions. Starting with a proof of A what one then obtains is a (partially specified) proof of $\exists \boldsymbol{f} \prec \boldsymbol{t} \forall \boldsymbol{a}, \boldsymbol{y} | A|^{\boldsymbol{f}\boldsymbol{a}}_{\boldsymbol{y}}$, for some sequence of (partially specified) closed terms $\boldsymbol{t}$. These extracted terms $\boldsymbol{t}$ will potentially contain the new constants added, and the verifying proof will potentially make use of the new bounded formulas and their associated conditions. Extracting the *abstract* witnessing term $\boldsymbol{t}$ allows for a comparison between the terms extracted via different functional interpretation, namely, we know that terms extracted via different interpretations will have the same structure, and will only differ on the choices of $\mathbf{b}^*_1, \mathbf{b}^*_2$ and $\mathbf{b}^*_3$. This will be clear cut when analysing proofs of theorems whose interpretation do not contain the abbreviation

$\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$, e.g. implication-free theorems or theorems in prenex normal form. For such formulas A the interpretation $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ will be syntactically the same, no matter what the choice of the abbreviation $\forall \boldsymbol{x} \sqsubset \boldsymbol{t}\, A(\boldsymbol{x})$, although the extracted term $\boldsymbol{t}$ and the proof of $\exists \boldsymbol{f} \prec \boldsymbol{t} \forall \boldsymbol{a}, \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ will be possibly different.

The common framework presented above can also be used in the study and development of new functional interpretations. For instance, one might consider instantiating the second parameter abbreviation with $\exists \boldsymbol{x} A(\boldsymbol{x})$, which simply says that only the *existence* of witnesses is looked for. Such interpretation will obviously give less information when proofs are analysed. However, it will also require much less from the axioms and principles, as those need no longer have a witnessing term, but only the existence of those needs to be assumed. In particular, arbitrary *purely existential axioms* can be added to the interpreted theory, as long as those are also present in the verifying theory.

It is worth noting that the clear connection between conditions (B_1) and (B_3) and the categorical conditions of identity and composition, as investigated in [11]. The focus of this paper, however, has been on a purely syntactic comparison of the different functional interpretation. I believe that such common syntactic framework can pave the way to a better semantical understanding of functional interpretations, along the lines of [11].

Notes

1. A bounded version of modified realizability is currently under investigation [6].

2. Notice that, since the defining axioms for the relations $\{\leq^*_\rho\}_{\rho \in \mathcal{T}}$ are not purely universal, and its interpretation cannot be witnessed by a term in the language of IL^ω, in [7], the relation is axiomatised via a rule replacing the usual axiom $\forall v \forall u \leq^* v(su \leq^* tv \wedge tu \leq^* tv) \rightarrow s \leq^* t$. This entails the failure of the deduction theorem.

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Department of Computer Science
Queen Mary, University of London
Mile End Road, London E1 4NS
United Kingdom
pbo@dcs.qmul.ac.uk
http://www.dcs.qmul.ac.uk/˜pbo