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Modified Bar Recursion

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This paper studies modified bar recursion, a higher type recursion scheme which has been used in (BBC98) and (BO05) for a realizability interpretation of classical analysis. A complete clarification of its relation to Spector's and Kohlenbach's bar recursion, the fan functional, Gandy's functional Γ and Kleene's notion of S1-S9 computability is given.

1. Introduction

Spector (Spe62) extended Gödel's Dialectica interpretation of classical arithmetic (Göd58) to analysis by introducing a new scheme of recursion, known as bar recursion. Berardi, Bezem and Coquand (BBC98) gave an alternative interpretation of classical analysis based on realizability and another form of recursion, which the authors, in (BO05), called *modified bar recursion.* In (BBC98), the interpretation is based on a special form of realizability by infinitary terms. As it is shown in (BO05), a simple combination of the A-translation and Kreisel's modified realizability can also be used. In this paper we determine the computational strength of modified bar recursion in relation to other forms of recursion. Our main results are:

- Section 4. MBR of the lowest type is equivalent to Gandy's functional Γ (GH77). Hence, one can view MBR as a higher-type generalisation of Γ.
- Section 5. The type structure $\mathcal{M}$ of strongly majorizable functionals (shown in (Bez85) to be a model of Spector's bar recursion) is a model of MBR. This will be used to determine the strength of the different forms of bar recursion.
- Section 6. Spector's bar recursion is definable from MBR.
- Section 7. MBR is not S1-S9 computable over the total continuous functionals, from which we can conclude that modified bar recursion is strictly stronger than Spector's.

2. Bar recursion in finite types

In this section we introduce various forms of bar recursion and review some of the known results about these schemes.

2.1. *Heyting arithmetic in finite types*

Our background system is the same as in (BO05), that is, an extension of Heyting Arithmetic to the language of finite types, HA^ω. The finite types are built from the basic type $\mathbb{N}$ via the following type constructions: function types $\rho \to \sigma$, product types $\rho \times \sigma$, and finite sequences ρ^*. We set $\rho^\omega :\equiv \mathbb{N} \to \rho$. The *level* of a type is defined by $\mathsf{level}(\mathbb{N}) = 0$, $\mathsf{level}(\rho \times \sigma) = \max(\mathsf{level}(\rho), \mathsf{level}(\sigma))$, $\mathsf{level}(\rho^*) = \mathsf{level}(\rho)$, $\mathsf{level}(\rho \to \sigma) = \max(\mathsf{level}(\rho) + 1, \mathsf{level}(\sigma))$. The terms of HA^ω are a version of the terms of Gödel's system $\mathbf{T}$ (Göd58) in λ-calculus notation with constructors and recursors for finite sequences added. We will often write $r(s_1, \ldots, s_n)$ for an iterated application $rs_1 \ldots s_n$. The atomic formulas of HA^ω are equations $r \overset{\tau}{=} s$, for each finite type τ, and a symbol $\perp$ for absurdity. Composite formulas are built from atomic ones by means of the logical connectives $\wedge$, $\vee$, $\to$ and the quantifiers $\forall x^\tau$ and $\exists x^\tau$ where τ is an arbitrary type. Negation is defined as $\neg A :\equiv A \to \perp$. The axioms and logical rules of HA^ω are those of (many-sorted) intuitionistic logic plus induction axioms for arbitrary formulas and the usual equality axioms including β-equality, $(\lambda x.r)s = r[s/x]$. Our system is 'neutral', in Troelstra's terminology (Tro73), that is, we neither assume decidability for $\overset{\tau}{=}$ if $\mathsf{level}(\tau) > 0$, nor do we assume that equality between functionals is extensional. If required we will add extensionality via the axioms

$$\mathsf{EXT} \quad : \quad \forall f^{\rho\to\sigma}, g^{\rho\to\sigma}\, (\forall x^\rho\, (fx \overset{\rho}{=} gx) \to f \overset{\sigma}{=} g)$$

for all types ρ, σ. We set $\mathsf{E\text{-}HA}^\omega :\equiv \mathsf{HA}^\omega + \mathsf{EXT}$.

We will use the variables $i, j, k, l, m, n \colon \mathbb{N}$; $s, t \colon \rho^*$; $\alpha, \beta \colon \rho^\omega$ unless the type of these variables is stated explicitly otherwise. Other letters will be used for different types in different contexts. Type information will be omitted when irrelevant or inferable from the context. By $<$ we will always mean the usual ordering on $\mathbb{N}$ (defined by a suitable closed term). We let k^ρ denote the canonical lifting of a number $k \in \mathbb{N}$ to type ρ, e.g. $k^{\rho\to\sigma} :\equiv \lambda x^\rho.k^\sigma$. We will also use the following notations:

$$
\begin{array}{rcl}
\langle x_0, \ldots, x_{n-1} \rangle & := & \text{the finite sequence with elements } x_0, \ldots, x_{n-1} \\
|s| & := & \text{the length of } s \text{, i.e. } |\langle x_0, \ldots, x_{n-1} \rangle| = n \\
s_k & := & \text{the } k\text{-th element of } s \text{ if } k < |s| \text{ and 0 otherwise,} \\
s * t & := & \text{the concatenation of } s \text{ and } t \\
s * x & := & s * \langle x \rangle \\
s * \alpha & := & \lambda k.(\mathsf{if}\ k{<}|s|\ \mathsf{then}\ s_k\ \mathsf{else}\ \alpha(k{-}|s|)) \quad \text{(appending } \alpha \text{ to } s\text{)} \\
s \mathbin{@} \alpha & := & \lambda k.(\mathsf{if}\ k < |s|\ \mathsf{then}\ s_k\ \mathsf{else}\ \alpha(k)) \quad \text{(overwriting } \alpha \text{ with } s\text{)} \\
(\overline{\alpha, n}) & := & \lambda k.(\mathsf{if}\ k < n\ \mathsf{then}\ \alpha(k)\ \mathsf{else}\ 0) \\
\hat{s} & := & \lambda k.(\mathsf{if}\ k < |s|\ \mathsf{then}\ s_k\ \mathsf{else}\ 0) \\
\check{s} & := & \lambda k.(\mathsf{if}\ k < |s|\ \mathsf{then}\ s_k\ \mathsf{else}\ 1) \\
\overline{\alpha}k & := & \langle \alpha(0), \ldots, \alpha(k-1) \rangle \\
\beta \in s & := & \overline{\beta}|s| \overset{\rho^*}{=} s.
\end{array}
$$

It will suffice to work with a naive notion of model for HA^ω. By a *type structure* we mean a family $\mathcal{S}$ of sets $\mathcal{S}_\rho$, for each finite type ρ, equipped with enough structure to interpret the terms of HA^ω in a reasonable way (there is no need to make this more precise). The type structures considered in this paper are the model $\mathcal{C}$ of *total continuous functionals* of Kleene (Kle59) and Kreisel (Kre59), the model $\widehat{\mathcal{C}}$ of *partial continuous functionals* of Scott (Sco70) and Ershov (Ers77) (see also (Nor99)), and the model $\mathcal{M}$ of (strongly) *majorizable functionals* introduced by Howard (How73) and Bezem (Bez85).

2.2. *Forms of bar recursion*

Definition 2.1. We define four versions of bar recursion, Spector's original bar recursion SBR (Spe62), Kohlenbach's version KBR (Koh90), modified bar recursion MBR (BBC98; BO05) and a weak version, wMBR, of MBR. The types of the variables in the formulas below can be derived from the information given.

$$\mathsf{SBR}_{\rho,\tau}(G,Y,H,s) \stackrel{\tau}{=} \begin{cases} G(s) & \text{if } Y(\hat{s}) < |s| \\ H(s, \lambda x^\rho.\mathsf{SBR}_{\rho,\tau}(G,Y,H,s*x)) & \text{otherwise} \end{cases} \tag{1}$$

$$\mathsf{KBR}_{\rho,\tau}(G,Y,H,s) \stackrel{\tau}{=} \begin{cases} G(s) & \text{if } Y(\hat{s}) \stackrel{\mathbb{N}}{=} Y(\check{s}) \\ H(s, \lambda x^\rho.\mathsf{KBR}_{\rho,\tau}(G,Y,H,s*x)) & \text{otherwise} \end{cases} \tag{2}$$

$$\mathsf{MBR}_\rho(Y,H,s) \stackrel{\mathbb{N}}{=} Y(s \mathbin{@} H(s, \lambda x^\rho.\mathsf{MBR}_\rho(Y,H,s*x))) \tag{3}$$

$$\mathsf{wMBR}_\rho(Y,H,s) \stackrel{\mathbb{N}}{=} Y(s \mathbin{@} \lambda k.H(s, \lambda x^\rho.\mathsf{wMBR}_\rho(Y,H,s*x))) \tag{4}$$

Note that in (3) the parameter H has result type ρ^ω while in (4) H has result type ρ. Hence wMBR is a special case of MBR, more precisely, if MBR satisfies (3), then $\lambda Y,H,s.\mathsf{MBR}(Y,\lambda t,f,k.H(t,f),s)$ satisfies (4). This weaker form of modified bar recursion is sufficient for interpreting both countable and dependent choice (see Theorem 2.5).

Formally, each of the equations (1-4) is to be understood as a formula with one free variable describing the functional being introduced. Other free variables are implicitly universally quantified. For example (3) is shorthand for

$$\mathsf{MBR}_\rho(\Phi) \quad : \quad \forall Y,H,s \quad (\Phi(Y,H,s) = Y(s \mathbin{@} H(s, \lambda x^\rho.\Phi(Y,H,s*x)))).$$

We call such a formula a *defining axiom of a functional*, or simply a *functional*. We will let the symbols F, G range over arbitrary functionals given in this way. Sometimes we also write $\mathsf{F}(\Phi)$, $\mathsf{G}(\Phi)$ if we want to mention the free variable Φ. We say that a type structure $\mathcal{S}$ *satisfies* F, or F *exists in* $\mathcal{S}$, if there is an interpretation of Φ in $\mathcal{S}$ such that $\mathsf{F}(\Phi)$ holds.

Theorem 2.2 (Ers77, BO05). The models $\mathcal{C}$ and $\widehat{\mathcal{C}}$ satisfy all four variants of bar recursion.

The existence of modified bar recursion in $\widehat{\mathcal{C}}$ and $\mathcal{C}$ follows from the fact that these models validate the *continuity axiom*

$\mathsf{Cont} \quad : \quad \forall F^{\rho^\omega \to \mathbb{N}} \forall \alpha \, \exists n \, \forall \beta \, (\overline{\alpha} n = \overline{\beta} n \to F(\alpha) = F(\beta)).$

Remark 2.3. The continuity axiom holds in fact for functionals F of any type $\rho^\omega \to \iota$ with ι of level 0 (the reason is that elements of type level 0 are finite). Hence in the definitions of MBR and wMBR (Definition 2.1) the result type $\mathbb{N}$ could be replaced by any type of level 0. The same applies to the definition of the fan functional (cf. Section 2.5). However, as shown in (BO05), the defining equation of wMBR becomes inconsistent with HA^ω if the type $\mathbb{N}$ is replaced by $\mathbb{N} \to \mathbb{N}$.

Theorem 2.4 (Bez85, Koh90). $\mathcal{M}$ satisfies SBR but not KBR.

We will show in Section 5 that $\mathcal{M}$ satisfies MBR (and hence also wMBR).

2.3. *Interpreting classical analysis*

In the following we review the main result of (BO05) on the extraction of witnesses from proofs of Σ^0_1-formulas (i.e. formulas of the form $\exists y^{\mathbb{N}} A(x^\rho, y)$, with $A(x^\rho, y^{\mathbb{N}})$ atomic) in classical analysis using MBR. By *classical analysis* we mean PA^ω (= HA^ω+ classical logic) extended by the axiom scheme of *countable dependent choice* (HK66)

$\mathsf{DC} \quad : \quad \forall n \, \forall x \, \exists y \, A(n, x, y) \to \forall x \exists f \, (f(0) = x \wedge \forall n \, A(n, f(n), f(n+1))).$

In order to prove the correctness of the extracted witnesses one needs the continuity axiom (Section 2.2) and *relativized quantifier free bar induction*

$\mathsf{rBI_{qf}} \quad : \quad (\mathrm{I}) \wedge (\mathrm{II}) \to P(\langle \, \rangle),$

where P is a quantifier free predicate of finite sequences and

$(\mathrm{I}) \equiv \forall \alpha \in S \, \exists n \, P(\overline{\alpha} n),$
$(\mathrm{II}) \equiv \forall s \in S \, (\forall x \, (S(x, |s|) \to P(s * x)) \to P(s)),$

using the abbreviations $\alpha \in S :\equiv \forall n S(\alpha(n), n)$ and $s \in S :\equiv \forall i < |s| S(s_i, i)$. We call any n satisfying $\forall \beta \, (\overline{\alpha} n = \overline{\beta} n \to F(\alpha) = F(\beta))$ a *point of continuity of F at α*.

Note that, classically, $\mathsf{rBI_{qf}}$ follows from DC, and the latter is valid in $\widehat{\mathcal{C}}$, $\mathcal{C}$ and $\mathcal{M}$ since in all three models ρ^ω is interpreted as the full set-theoretic function space.

Below, a *Horn formula* is a formula of the form $\forall x_1, \ldots, x_n (A_1 \wedge \ldots \wedge A_m \to B)$ where $A_1, \ldots, A_m, B$ are atomic.

Theorem 2.5 (BO05). Let $\mathcal{H}$ be a set of Horn formulas and let $A(x^\rho, y^{\mathbb{N}})$ be an atomic formula. From any proof

$\mathsf{PA}^\omega + \mathcal{H} + \mathsf{DC} \vdash \forall x \, \exists y \, A(x, y)$

one can extract a term r such that

$\mathsf{HA}^\omega + \mathcal{H} + \mathsf{rBI_{qf}} + \mathsf{Cont} + \mathsf{wMBR}(\Phi) \vdash \forall x \, A(x, r(\Phi, x)).$

Moreover, viewing Φ as a constant and $\mathsf{wMBR}(\Phi)$ as a family of rewrite rules, then for every closed term s^ρ the term $r(\Phi, s)$ reduces by leftmost-outermost reduction to a numeral $\underline{n}$ such that $A(s, \underline{n})$ holds in the model $\mathcal{C}$.

The term r in Theorem 2.5 is obtained by a modified realizability interpretation of the negative- and A-translated classical proof where the corresponding translation of DC is realized using MBR (the translations of Horn formulas have trivial realizers). The statement about the computation of the witnessing numeral follows from Plotkin's adequacy theorem (Plo77). In (Ber05) it is shown that a suitable formulation of wMBR even leads to a strongly normalizing system.

Compared with Spector's result (Spe62), Theorem 2.5 seems to be proof-theoretically weaker since the verification of the correctness of the extracted program r takes place in the basis system $\mathsf{HA}^\omega + \mathsf{Cont} + \mathsf{rBI}_{\mathsf{qf}}$ while Spector works with a quantifier-free verifying system and needs neither continuity nor bar induction. However, there are practical arguments in favour of Theorem 2.5. For example, unlike in (Spe62), Theorem 2.5 does not require decidability of atomic formulas and seems to work best with a (user friendly) natural deduction calculus.

2.4. *Definability*

While (BO05) was mainly concerned with proof-theoretic issues, we will in this paper focus on the relative computational strengths of the different forms of bar recursion, that is, the question whether one form can be defined from the other.

Definition 2.6. A functional F is *definable* from a functional G in a theory Δ if there is a closed term t such that

$$\mathsf{E\text{-}HA}^\omega + \Delta + \mathsf{G}(\Phi) \vdash \mathsf{F}(t(\Phi)),$$

where $\mathsf{G}(\Phi)$ is the statement that Φ satisfies the defining equation of G – see Definition 2.1. If Δ is empty, then we simply say that F is definable from G, or that F is *reducible* to G. Definability of a scheme from one or more schemes is defined similarly. Moreover, we say that F is *equivalent* to G if F is definable from G and vice-versa.

For example, wMBR is definable from MBR, by the remark after Definition 2.1.

2.5. *The fan functional*

As a first nontrivial example of definability we recall a result from (BO05) about the *fan functional*

$$\mathsf{FAN}(\Phi) \quad : \quad \forall Y \forall \alpha, \beta \leq \lambda x.1 \big(\overline{\alpha}(\Phi(Y)) = \overline{\beta}(\Phi(Y)) \to Y\alpha \overset{\mathbb{N}}{=} Y\beta\big),$$

where $\alpha \leq \lambda x.1$ means $\alpha(k) \leq 1$ for all $k \in \mathbb{N}$. A functional Φ satisfying $\mathsf{FAN}(\Phi)$ computes a modulus of uniform continuity for every functional $Y^{\mathbb{N}^\omega \to \mathbb{N}}$ restricted to the Cantor space.

Theorem 2.7 (BO05). The fan functional is definable from $\mathsf{MBR}+\mathsf{KBR}$ in $\mathsf{Cont}+\mathsf{rBI}_{\mathsf{qf}}$.

The proof uses a construction of (Ber90).

3. The equivalence of MBR and wMBR

As another example of definability we show that MBR_ρ can be reduced to weak modified bar recursion at type ρ^ω.

Theorem 3.1. MBR_ρ is definable from $\mathsf{wMBR}_{\rho^\omega}$.

Proof. Note that, by Definition 2.6, we may assume that equality is extensional. For an element x of type ρ and each natural number i we define the function $[x]_i : \rho^\omega$

$$[x]_i(k) := \begin{cases} x & \text{if } k = i \\ 0^\rho & \text{otherwise.} \end{cases}$$

For a sequence $s = \langle s_0, \ldots, s_n \rangle$ of type ρ^* we define

$$\mathsf{up}(s) := \langle [s_0]_0, \ldots, [s_n]_n \rangle,$$

(note that $\mathsf{up}(s)$ has type $(\rho^\omega)^*$) and for a sequence $s = \langle \alpha_0, \ldots, \alpha_n \rangle$ of type $(\rho^\omega)^*$ we define

$$\mathsf{down}(s) := \langle \alpha_0(0), \ldots, \alpha_n(n) \rangle,$$

which has type ρ^*. Observe that

(i) $\mathsf{down}(\mathsf{up}(s)) = s$,
(ii) $\mathsf{up}(s) * [x]_{|s|} = \mathsf{up}(s * x)$,
(iii) $\lambda k.(\mathsf{up}(s) \mathbin{@} \lambda k.\alpha)(k)(k) = s \mathbin{@} \alpha$.

Given functionals $Y : \rho^\omega \to \mathbb{N}$ and $H : \rho^* \times (\rho \to \mathbb{N}) \to \rho^\omega$ we define $\tilde{Y}$ of type $(\rho^\omega)^\omega \to \mathbb{N}$ and $\tilde{H}$ of type $(\rho^\omega)^* \times (\rho^\omega \to \mathbb{N}) \to \rho^\omega$ as follows

(iv) $\tilde{Y}(\alpha) := Y(\lambda k.\alpha(k)(k))$, (i.e. Y gets the diagonal of $\alpha : (\rho^\omega)^\omega$)
(v) $\tilde{H}(s, F) := H(\mathsf{down}(s), \lambda x^\rho.F([x]_{|s|}))$.

Now $\mathsf{MBR}_\rho(Y, H, s)$ can be defined as $\mathsf{wMBR}_{\rho^\omega}(\tilde{Y}, \tilde{H}, \mathsf{up}(s))$ since

$$\begin{aligned}
&\mathsf{wMBR}_{\rho^\omega}(\tilde{Y}, \tilde{H}, \mathsf{up}(s)) = \\
&\tilde{Y}(\mathsf{up}(s) \mathbin{@} \lambda k.\tilde{H}(\mathsf{up}(s), \lambda x.\mathsf{wMBR}_{\rho^\omega}(\tilde{Y}, \tilde{H}, \mathsf{up}(s) * x))) \overset{(v),(i)}{=} \\
&\tilde{Y}(\mathsf{up}(s) \mathbin{@} \lambda k.H(s, \lambda x.\mathsf{wMBR}_{\rho^\omega}(\tilde{Y}, \tilde{H}, \mathsf{up}(s) * [x]_{|s|}))) \overset{(ii)}{=} \\
&\tilde{Y}(\mathsf{up}(s) \mathbin{@} \lambda k.H(s, \lambda x.\mathsf{wMBR}_{\rho^\omega}(\tilde{Y}, \tilde{H}, \mathsf{up}(s * x)))) \overset{(iv),(iii)}{=} \\
&Y(s \mathbin{@} H(s, \lambda x.\mathsf{wMBR}_{\rho^\omega}(\tilde{Y}, \tilde{H}, \mathsf{up}(s * x))))
\end{aligned}$$

□

Note that if $\mathsf{level}(\rho) > 0$, then ρ and ρ^ω are primitive recursively isomorphic and hence wMBR_ρ defines MBR_ρ. It is an open question whether this holds for types ρ of level 0 as well, i.e. whether $\mathsf{wMBR}_\mathbb{N}$ already defines $\mathsf{MBR}_\mathbb{N}$. Since, trivially, MBR defines wMBR we have:

Theorem 3.2. The schemes MBR and wMBR are equivalent.

4. The functional Γ

The functional Γ (introduced in (GH77)) is defined as

$$\Gamma(Y, s) \overset{\mathbb{N}}{=} Y(s * 0 * \lambda n^\mathbb{N}.\Gamma(Y, s * (n + 1))). \tag{5}$$

Using a continuity argument it is easy to see that in both models $\widehat{\mathcal{C}}$ and $\mathcal{C}$ this equation specifies a unique functional. Gandy and Hyland's purpose for defining the functional Γ

was to show that there exists a functional having a recursive associate but not being S1-S9 computable in the total continuous functionals, even with the fan functional as an oracle. In the following we show that modified bar recursion of the lowest type is equivalent to the functional Γ. Hence, one can view MBR as an extension of the functional Γ to higher types.

Theorem 4.1. The functional Γ is equivalent to $\mathsf{MBR}_{\mathbb{N}}$.

Proof. It is easy to see that Γ is definable from $\mathsf{MBR}_{\mathbb{N}}$. For the other direction the intuition is as follows. Uniformly in Y and H we can use Γ to compute the values of $\mathsf{MBR}_{\mathbb{N}}(Y, H, s) + 1$, where s varies. The advantage of doing this is that, if the sequence s contains only positive numbers, then the functional Y will be called at an infinite sequence α containing only one zero, namely the one introduced by the functional Γ. Say α has the form $s * 0 * \beta$. Therefore it is easy to transform α into the sequence $s * H(s, \beta)$. Moreover, since β is exactly $\lambda x.(\mathsf{MBR}_{\mathbb{N}}(Y, H, s * x) + 1)$ we are done. Now we give the formal proof. Define

$$s^+ := \langle s_0 + 1, \ldots, s_{|s|-1} + 1\rangle, \quad s^- := \langle s_0 - 1, \ldots, s_{|s|-1} - 1\rangle$$

$$\alpha_H(k) := \begin{cases} \alpha(k) - 1 & \text{if } \forall m \leq k\, (\alpha(m) > 0) \\ H((\overline{\alpha}m)^-, \lambda n.(\alpha(m+n+1) - 1))(k - m) & \text{otherwise,} \\ \quad \text{where } m \leq k \text{ is smallest such that } \alpha(m) = 0. & \end{cases}$$

$$Y_H(\alpha) := Y(\alpha_H) + 1.$$

Notice that by the choice of m in definition of α_H we are sure that $(\cdot)^-$ is only used for finite sequences of strictly positive numbers. Clearly, $(s^+ * 0 * \beta)_H = s * H(s, \lambda n.(\beta(n) - 1))$. Now we can define

$$\mathsf{MBR}(Y, H, s) := \Gamma(Y_H, s^+) - 1$$

since

$$\begin{aligned} \mathsf{MBR}(Y, H, s) &\equiv \Gamma(Y_H, s^+) - 1 \\ &\overset{(5)}{=} Y_H(s^+ * 0 * \lambda n.\Gamma(Y_H, s^+ * (n+1))) - 1 \\ &= Y_H(s^+ * 0 * \lambda n.\Gamma(Y_H, (s * n)^+)) - 1 \\ &= Y(s * H(s, \lambda n.(\Gamma(Y_H, (s * n)^+) - 1))) \\ &\equiv Y(s * H(s, \lambda n.\mathsf{MBR}(Y, H, s * n))). \end{aligned}$$

□

5. The model $\mathcal{M}$ of strongly majorizable functionals

The type structure $\mathcal{M}$ $(= \bigcup \mathcal{M}_\rho)$ of strongly majorizable functionals was introduced by Bezem (Bez85) as a variation of Howard's majorizable functionals (How73). The structure is a model of Gödel's system **T** extended with Spector's bar recursion SBR. The sets $\mathcal{M}_\rho$ are defined simultaneously with a strong majorizability relation $\text{s-maj}_\rho \subseteq \mathcal{M}_\rho \times \mathcal{M}_\rho$ by induction on types. We abbreviate s-maj_ρ by maj_ρ and by "majorizable" we always mean "strongly majorizable". We will often omit the type in the relation maj_ρ. For the basic type $\mathbb{N}$ we have $n \operatorname{maj}_{\mathbb{N}} m :\equiv n \geq m$ and $\mathcal{M}_{\mathbb{N}} :\equiv \mathbb{N}$. For $F^*, F \in \mathcal{M}_\rho \to \mathcal{M}_\tau$ we define

$$F^* \operatorname{maj}_{\rho\to\tau} F :\equiv \forall G^*, G \in \mathcal{M}_\rho \, (G^* \operatorname{maj}_\rho G \to F^*G^* \operatorname{maj}_\tau F^*G, FG),$$

and the set $\mathcal{M}_{\rho\to\tau}$ is defined to contain precisely the set-theoretic functionals $F \in \mathcal{M}_\rho \to \mathcal{M}_\tau$ which have a majorant. Majorizability for product types and finite sequences is defined pointwise. More precisely, for $\langle x^*, y^*\rangle, \langle x, y\rangle \in \mathcal{M}_\rho \times \mathcal{M}_\sigma$ we define

$$\langle x^*, y^*\rangle \operatorname{maj}_{\rho\times\sigma} \langle x, y\rangle :\equiv x^* \operatorname{maj}_\rho x \wedge y^* \operatorname{maj}_\sigma y$$

and $\mathcal{M}_{\rho\times\sigma} :\equiv \mathcal{M}_\rho \times \mathcal{M}_\sigma$; and for $s^*, s \in \mathcal{M}_\rho^*$ we set

$$s^* \operatorname{maj}_{\rho^*} s :\equiv |s| \leq |s^*| \wedge \forall i \leq |s^*| (s^*_i \operatorname{maj}_\rho s^*_i \wedge (i \leq |s| \to s^*_i \operatorname{maj}_{\rho^*} s_i)),$$

and $\mathcal{M}_{\rho^*} :\equiv \mathcal{M}^*_\rho$. It is easy to see that the standard properties of majorizability are preserved by this extension to our language containing product types and finite sequences. In particular, every element of $\mathcal{M}$ has a majorant.

In (Koh90) it is shown that KBR is provably not definable from SBR, since SBR yields a well defined functional in the model of (strongly) majorizable functionals $\mathcal{M}$ (cf. (Bez85)) and KBR does not. SBR, however, can be defined from KBR (cf. (Koh90)). In this section we show that a functional satisfying MBR exists in $\mathcal{M}$. We first show that there exists a set-theoretic functional

$$\Phi : \mathcal{M}_{\rho^\omega \to \mathbb{N}} \times \mathcal{M}_{\rho^* \times (\rho\to\mathbb{N}) \to \rho^\omega} \to \mathcal{M}_{\rho^* \to \mathbb{N}}$$

satisfying MBR, then we show that any such Φ has a majorant and therefore belongs to $\mathcal{M}$. For the rest of this section all variables (unless stated otherwise) are assumed to range over the type structure $\mathcal{M}$.

Lemma 5.1 (Bez85, 1.4, 1.5). Let $\max^\rho$ be inductively defined as,

$$\begin{aligned}
&\max_{i\leq n}{}^{\mathbb{N}} m_i := \max\{m_0, \ldots, m_n\},\\
&\max_{i\leq n}{}^{\tau\to\rho} F_i := \lambda x^\tau . \max_{i\leq n}{}^{\rho} F_i \, x,\\
&\max_{i\leq n}{}^{\rho\times\sigma} \langle x_i, y_i\rangle := \langle \max_{i\leq n}{}^{\rho} x_i, \max_{i\leq n}{}^{\sigma} y_i\rangle,\\
&\max_{i\leq n}{}^{\rho^*} s_i := \langle \max_{i\leq n}(s_i)_0, \ldots, \max_{i\leq n}(s_i)_m\rangle,
\end{aligned}$$

where $m = \max\{|s_i|\}_{i\leq n}$ (we have slightly confused notation, as s_i above denotes the i-th finite sequence of a family, and $(s_i)_j$ denotes the j-th component of the finite sequence s_i). Moreover, for α^{ρ^ω} let us define $\alpha^+(n) := \max_{i\leq n}{}^{\rho} \alpha(i)$. Then,

$$\forall n(\alpha(n) \operatorname{maj} \beta(n)) \to \alpha^+ \operatorname{maj} \beta^+, \beta.$$

It is also easy to verify by induction on types that for any finite set $\{x_i\}_{i\leq n}$ of objects of type τ, $\max_{i\leq n}{}^{\tau} x_i \operatorname{maj} x_k$, for all $k \leq n$. Moreover, using the construction $(\cdot)^+$ one can show that $\mathcal{M}_{\rho^\omega}$ consists of all (set-theoretic) functions from $\mathbb{N}$ to $\mathcal{M}_\rho$.

Most of our results in this section rely on the next lemma which can be viewed as a *weak continuity property* of functionals Y of type $\rho^\omega \to \mathbb{N}$ in $\mathcal{M}$. It says that a *bound* on the value of $Y(\alpha)$ can be determined from an initial segment of α.

Lemma 5.2 (Weak continuity). $\forall Y^{\rho^\omega \to \mathbb{N}} \forall \alpha \, \exists n^{\mathbb{N}} \, \forall \beta \in \overline{\alpha}n \, (Y(\beta) \leq n)$.

Proof. Let Y and α be fixed, α^* maj α and Y^* maj Y. From the assumption

$(*)\ \forall n\ \exists \beta \in \overline{\alpha} n (Y(\beta) > n)$

we derive a contradiction. For any n, let β_n be the function whose existence we are assuming in $(*)$. Let

$$\beta^*_n(i) := \begin{cases} 0^\rho & i < n \\ [\beta_n(i)]^* & i \geq n, \end{cases}$$

where $[\beta_n(i)]^*$ denotes some majorant of $\beta_n(i)$. Define $\tilde{\alpha}(i) := \max\{\alpha^*(i), \max_{n \leq i} \beta^*_n(i)\}$. Let m be the value of $Y^*((\tilde{\alpha})^+)$. By Lemma 5.1 we have $(\tilde{\alpha})^+$ maj β_n, for all $n \in \mathbb{N}$, but from $(*)$ we should have $m = Y^*((\tilde{\alpha})^+) \geq Y(\beta_m) > m$, a contradiction. □

Lemma 5.3. $\forall \alpha^*, \alpha(\alpha^* \text{ maj } \alpha \to \forall n \forall \beta \in \overline{\alpha} n \exists \beta^* \in \overline{\alpha^*} n\ (\beta^* \text{ maj } \beta))$.

Proof. Let α^*, α be such that α^* maj α. Let n and $\beta \in \overline{\alpha} n$ be fixed. Define β^* as,

$$\beta^*(i) := \begin{cases} \alpha^*(i) & \text{if } i < n \\ \max^\rho\{\max_{j<i}{}^\rho \beta^*(j), [\beta(i)]^*\} & \text{otherwise,} \end{cases}$$

where $[\beta(i)]^*$ is some majorant of $\beta(i)$. First note that, for all i, $\beta^*(i)$ maj $\beta(i)$. We show that β^* maj β. Let $k \geq i$.

If $k < n$ then $\beta^*(k) = \alpha^*(k)$ maj $\alpha^*(i)$ maj $\alpha(i) = \beta(i)$.

If $k \geq n$ then $\beta^*(k) = \max^\rho\{\max_{i<k}{}^\rho \beta^*(i), [\beta(k)]^*\}$. This means that $\beta^*(k)$ majorizes both $[\beta(k)]^*$ (and hence $\beta(k)$) and $\beta^*(i)$ (and hence $\beta(i)$), for any $i < k$. □

Lemma 5.4. Define $\Upsilon : \mathcal{M}_{\rho^\omega \to \mathbb{N}} \to \mathcal{M}_{\rho^\omega} \to \mathcal{M}_{\mathbb{N}}$ as,

$$\Upsilon(Y)(\alpha) := \min n\ (\forall \beta \in \overline{\alpha} n (Y(\beta) \leq n)).$$

Then,

i) Υ maj Υ (hence $\Upsilon \in \mathcal{M}$)
ii) If Y^* maj Y then $\Upsilon(Y^*)$ maj Y
iii) $\Upsilon(Y)$ is continuous and $\Upsilon(Y)(\alpha)$ is a point of continuity for $\Upsilon(Y)$ on α.

Proof. First of all, we note that, by Lemma 5.2, the functional Υ is well defined.

i) Assume Y^* maj Y and α^* maj α. We show that $\Upsilon(Y^*)(\alpha^*) \geq \Upsilon(Y)(\alpha)$. Let $n = \Upsilon(Y^*)(\alpha^*)$ and suppose $m = \Upsilon(Y)(\alpha)$. Assume $n < m$. By the minimality condition in the definition of $\Upsilon(Y)$, there exists a $\beta \in \overline{\alpha}(m-1)$ such that $Y(\beta) \geq m$. But, since $n < m$, by Lemma 5.3, there exists a $\beta^* \in \overline{\alpha^*} n$ such that β^* maj β. Hence, $Y^*(\beta^*) \geq Y(\beta)$, a contradiction. For showing $\Upsilon(Y^*)(\alpha^*) \geq \Upsilon(Y^*)(\alpha)$ simply take $Y = Y^*$.

ii) Assume that Y^* maj Y and α^* maj α. By the definition of Υ we have $\Upsilon(Y^*)(\alpha^*) \geq Y^*(\alpha^*) \geq Y(\alpha)$. Moreover, $\Upsilon(Y^*)(\alpha^*) \geq \Upsilon(Y^*)(\alpha)$ follows from point (i).

iii) Let α be fixed and take $n = \Upsilon(Y)(\alpha)$. Suppose there exists a $\beta \in \overline{\alpha} n$ such that $\Upsilon(Y)(\beta) \neq n$. If $\Upsilon(Y)(\beta) < n$ we get, since $\alpha \in \overline{\beta} n$, that $\Upsilon(Y)(\alpha) < n$, a contradiction. Suppose $\Upsilon(Y)(\beta) > n$. Since $\beta \in \overline{\alpha} n$ we have, $\forall \gamma \in \overline{\beta} n (Y(\gamma) \leq n)$, also a contradiction. □

5.1. *Finding* $\Phi \in \mathcal{M}_{\rho^\omega \to \mathbb{N}} \times \mathcal{M}_{\rho^* \times (\rho \to \mathbb{N}) \to \rho^\omega} \to \mathcal{M}_{\rho^* \to \mathbb{N}}$ *satisfying* MBR

Throughout this section 5.1 we fix $Y \in \mathcal{M}_{\rho^\omega \to \mathbb{N}}$ and $H \in \mathcal{M}_{\rho^* \times (\rho \to \mathbb{N}) \to \rho^\omega}$. Our goal is to find $\Phi \in \mathcal{M}_{\rho^* \to \mathbb{N}}$ satisfying MBR, i.e. for all $s \in \mathcal{M}_{\rho^*}$

$$\Phi(s) = Y(s \mathbin{@} H(s, \lambda x.\Phi(s * x))) \tag{6}$$

It will be useful to view the elements s of $\mathcal{M}_{\rho^*}$ (finite sequences of elements in ρ) as the nodes of an infinite tree which we call T. The infinite paths of T are the elements of $\mathcal{M}_{\rho^\omega}$ (which is just $\mathcal{M}_\rho^\omega$ as shown in (Bez85)). The functional Φ we are looking for should assign values to the nodes of T according to equation (6). For each node s let us denote the set of nodes s' extending s by B_s.

First, we show that at each infinite path α (let $n := \Upsilon(Y)(\alpha)$) a functional $\phi_\alpha : B_{\overline{\alpha}n} \to \mathcal{M}_{\mathbb{N}}$ can be defined satisfying (6) for all s in its domain. Then we use Spector's bar recursion (which exists in $\mathcal{M}$) to extend ϕ_α to a functional $\Phi \in \mathcal{M}_{\rho^* \to \mathbb{N}}$ satisfying (6) for *all* $s \in \mathcal{M}_{\rho^*}$.

Let $\alpha \in \mathcal{M}_\rho^\omega$ be fixed, $n := \Upsilon(Y)(\alpha)$ and $K :\equiv \{0, 1, \ldots, n\}$. We show how to define a functional $\phi_\alpha(s)$ such that, for $s \in B_{\overline{\alpha}n}$, equation

$$\phi_\alpha(s) = Y(s \mathbin{@} H(s, \lambda x.\phi_\alpha(s * x))$$

holds. Here we note that, for $s \in B_{\overline{\alpha}n}$, by Lemma 5.2, the value $\phi_\alpha(s)$ must belong to K. Therefore, for those $s \in B_{\overline{\alpha}n}$, what we have is an instance of the more general equation,

$$\Psi(s) = G(s, \lambda x.\Psi(s * x)), \tag{7}$$

where $\mathsf{img}(G) \subseteq K$ ($\mathsf{img}(G)$ denotes the image set of the functional G). To see that modified bar recursion becomes an instance of (7), let

$$G(s, F) := Y(\overline{\alpha}n * s \mathbin{@} H(\overline{\alpha}n * s, F)).$$

Clearly, $\mathsf{img}(G) = \mathsf{img}(\lambda s, F.Y(\overline{\alpha}n{*}s \mathbin{@} H(\overline{\alpha}n{*}s, F)) \subseteq K$. Hence, it suffices to show that equations of the form (7) (with the mentioned restriction on G) always have a solution Ψ. That is what we will do now.

Consider the set of mappings $\mathcal{T} :\equiv T \to 2^K \backslash \{\emptyset\}$ which can be viewed as the set of labelled trees whose labels range over non-empty subsets of K. We define a partial order $\sqsubseteq$ on $\mathcal{T}$ by

$$f \sqsubseteq g :\equiv \forall s(f(s) \subseteq g(s))$$

and an operation $\chi : \mathcal{T} \to \mathcal{T}$,

$$\chi(f)(s) := \mathsf{img}(\lambda F \in \mathsf{Cons}^f_s.G(s, F)),$$

where $\mathsf{Cons}^f_s :\equiv \{F \in \mathcal{M}_{\rho \to \mathbb{N}} : \forall x^\rho(F(x) \in f(s * x))\}$. Note that any (set-theoretic) function $F\colon \mathcal{M}_\rho \to \mathbb{N}$ satisfying $\forall x^\rho(F(x) \in f(s * x))$ is bounded (by n) and hence lives in $\mathcal{M}$. Therefore Cons^f_s is non-empty and χ is well-defined. We first observe the following.

Lemma 5.5. $(\mathcal{T}, \sqsubseteq)$ is a downwards directed complete semi-lattice with largest element.

Proof. Let S be a downwards directed subset of $\mathcal{T}$. Since we assign non-empty finite sets to the nodes of T, it is easy to see that $\lambda s.\bigcap\{f(s) \mid s \in S\}$ belongs to $\mathcal{T}$ and is the greatest lower bound of S. Furthermore, $\lambda s.K$ is the largest element of $\mathcal{T}$. $\square$

Lemma 5.6. $\chi : \mathcal{T} \to \mathcal{T}$ is monotone.

Proof. Let $f \sqsubseteq g$ and s be fixed. We get that $\mathsf{Cons}^f_s \subseteq \mathsf{Cons}^g_s$, which implies $\chi(f)(s) \subseteq \chi(g)(s)$. ☐

By (a generalisation of) the Knaster-Tarski fixed point theorem we obtain an $f \in \mathcal{T}$ such that $\chi(f) = f$, i.e. $f(s) = \mathsf{img}(\lambda F \in \mathsf{Cons}^f_s.G(s, F))$, for all s. Hence, for every s and $c \in f(s)$ we may choose $F_{s,c} \in \mathsf{Cons}^f_s$ such that $c = G(s, F_{s,c})$. Define the functional $\Psi(s)$ recursively as follows,

$$\Psi(\langle\ \rangle) := \text{the least element of } f(\langle\ \rangle);$$
$$\Psi(s * x) := F_{s,\Psi(s)}(x).$$

Lemma 5.7. The functional Ψ is total and satisfies equation (7).

Proof. We have just shown that Ψ is total. Moreover, note that, for all s, the values assigned to $\Psi(s * x)$ are such that $\Psi(s) = G(s, \lambda x.\Psi(s * x))$. ☐

Proposition 5.8. There exists a functional

$$\Phi : \mathcal{M}_{\rho^\omega \to \mathbb{N}} \times \mathcal{M}_{\rho^* \times (\rho \to \mathbb{N}) \to \rho^\omega} \to \mathcal{M}_{\rho^* \to \mathbb{N}}$$

satisfying modified bar recursion.

Proof. Define

$$G(s) := \begin{cases} \phi_{\hat{s}}(s) & \text{if } \Upsilon(Y)(\hat{s}) < |s| \\ 0 & \text{otherwise.} \end{cases}$$

where $\phi_\alpha : \mathcal{M}_{\rho^*} \to \mathcal{M}_{\mathbb{N}}$ is defined above. We first show that G lives in $\mathcal{M}$, by showing that $\lambda s.\Upsilon(Y^*)((\hat{s})^+)$ maj G (Y^* is a majorant of Y). Assuming s^* maj s we must prove that $\Upsilon(Y^*)((\hat{s^*})^+) \geq G(s)$. If $\Upsilon(Y)(\hat{s}) \geq |s|$ (i.e. s is not in $B_{\hat{s}}$) we are done, since $G(s) = 0$. If $\Upsilon(Y)(\hat{s}) < |s|$ then $G(s) = \phi_{\hat{s}}(s) \leq \Upsilon(Y)(\hat{s})$, and the result follows from Lemma 5.4 (i) by the fact that $(\hat{s^*})^+$ maj $\hat{s}$.

Let $H_Y(s, F) := Y(s \mathbin{@} H(s, F))$. We define $\Phi_{H,Y} : \mathcal{M}_{\rho^* \to \mathbb{N}}$ as

$$\Phi_{H,Y}(s) := \mathsf{SBR}(G, \Upsilon(Y), H_Y, s).$$

By the definition of SBR (and G) we get

$$\Phi_{H,Y}(s) = \begin{cases} \phi_{\hat{s}}(s) & \text{if } \Upsilon(Y)(\hat{s}) < |s| \\ Y(s \mathbin{@} H(s, \lambda x^\rho.\Phi_{H,Y}(s * x))) & \text{otherwise.} \end{cases}$$

Therefore, by the definition of $\phi_\alpha(s)$ we actually have

$$\Phi_{H,Y}(s) = Y(s \mathbin{@} H(s, \lambda x^\rho.\Phi_{H,Y}(s * x))).$$

☐

5.2. *Finding a majorant for* Φ

Now we show that any functional $\Phi \in \mathcal{M}_{\rho^\omega \to \mathbb{N}} \times \mathcal{M}_{\rho^* \times (\rho \to \mathbb{N}) \to \rho^\omega} \to \mathcal{M}_{\rho^* \to \mathbb{N}}$ satisfying MBR (and in particular the one defined in Proposition 5.8) has a majorant, and therefore belongs to $\mathcal{M}$.

Lemma 5.9. Let Y^* maj Y (of type $\rho^\omega \to \mathbb{N}$) and α be fixed, and $n = \Upsilon(Y^*)(\alpha^+)$. If $\overline{\alpha}n$ maj s and $|s| = n$, then for all infinite sequences β we have

$$\Upsilon(Y^*)(s @ \beta), \Upsilon(Y)(s @ \beta), Y(s @ \beta) \leq n.$$

Proof. Let β be fixed (and β^* a majorant for β). Since $\overline{\alpha}n$ maj s we get $(\overline{\alpha}n @ \beta^*)^+$ maj $s @ \beta$. Therefore, by Lemma 5.4 (i), (ii) and (iii) we have

$$n = \Upsilon(Y^*)((\overline{\alpha}n @ \beta^*)^+) \geq \Upsilon(Y^*)(s @ \beta), \Upsilon(Y)(s @ \beta), Y(s @ \beta).$$

□

In the following we extend the $(\cdot)^+$ operator of Lemma 5.1 for functionals F of type $\rho^* \to \mathbb{N}$ as

$$F^+ := \lambda s. \max_{s' \prec s} F(s'),$$

where $s' \prec s$ denotes that s' is a prefix of s.

Lemma 5.10. Let F and G be of type $\rho^* \to \mathbb{N}$. If

$$\forall s^*, s \; (s^* \text{ maj } s \wedge |s^*| = |s| \to F(s^*) \geq F(s), G(s))$$

then F^+ maj G^+, G.

Proof. Let s^* maj s be fixed. For all prefixes t^* (of s^*) and t (of s) of the same length, by the assumption of the lemma, we have $F(t^*) \geq F(t), G(t)$, and hence

$$\max_{s' \prec s^*} F(s') \geq \max_{s' \prec s} F(s'), \max_{s' \prec s} G(s').$$

Therefore, F^+ maj G^+, G. □

Proposition 5.11. If Φ is a functional of type

$$\mathcal{M}_{\rho^\omega \to \mathbb{N}} \times \mathcal{M}_{\rho^* \times (\rho \to \mathbb{N}) \to \rho^\omega} \to \mathcal{M}_{\rho^* \to \mathbb{N}},$$

which for any given $Y \in \mathcal{M}_{\rho^\omega \to \mathbb{N}}$ and $H \in \mathcal{M}_{\rho^* \times (\rho \to \mathbb{N}) \to \rho^\omega}$ satisfies MBR, i.e.

$$\forall s (\Phi(s) = Y(s @ H(s, \lambda x.\Phi(s)))),$$

then $\Phi \in \mathcal{M}$.

Proof. Our proof is based on the proof of the main result of (Bez85). The idea is that, if Φ satisfies MBR then the functional

$$\Phi^* := \lambda Y, H.(\Phi(\widehat{Y}, H))^+$$

majorizes Φ, where $\widehat{Y}(\alpha) := \Upsilon(Y)(\alpha^+)$. Let Y^* maj Y and H^* maj H be fixed. Using the abbreviations

$$\Phi_1 \equiv \Phi(\widehat{Y^*}, H^*),$$
$$\Phi_2 \equiv \Phi(\widehat{Y}, H),$$
$$\Phi_3 \equiv \Phi(Y, H),$$

we must show

$$(\Phi_1)^+ \text{ maj } (\Phi_2)^+, \Phi_3.$$

By Lemma 5.10, it is enough to show $\forall s^* \, P(s^*)$ where,

$$P(s^*) \equiv \forall s \; \big((s^* \text{ maj } s \wedge |s^*| = |s|) \to \Phi_1(s^*) \geq \Phi_1(s), \Phi_2(s), \Phi_3(s)\big).$$

We prove this by bar induction:

i) $\forall\alpha\exists n\, P(\overline{\alpha}n)$. Let α be fixed and $n := \widehat{Y^*}(\alpha) = \Upsilon(Y^*)(\alpha^+)$. By Lemma 5.4 (iii) and our assumption that Φ satisfies MBR we get $\Phi_1(\overline{\alpha}n) = n$. Let s be such that $|s| = n$ and $\overline{\alpha}n$ maj s. By Lemma 5.9, $n \geq \Phi_1(s), \Phi_2(s), \Phi_3(s)$.

ii) $\forall s^*(\forall x^*\, P(s^* * x^*) \to P(s^*))$. Let s^* be fixed. Assume that $\forall x^*\, P(s^* * x^*)$, i.e.

$$\forall x^*, x, s\ \big(s^* * x^* \text{ maj } s * x \to \Phi_1(s^* * x^*) \geq \Phi_1(s * x), \Phi_2(s * x), \Phi_3(s * x)\big).$$

We prove $P(s^*)$. Assume s is such that s^* maj s. If x^* maj x then $s^* * x^*$ majorizes $s^* * x$ and $s * x$. By our assumption $\forall x^*\, P(s^* * x^*)$ we have

$$\Phi_1(s^* * x^*) \geq \Phi_1(s^* * x), \Phi_1(s * x), \Phi_2(s * x), \Phi_3(s * x)$$

which implies

$$\lambda x.\Phi_1(s^* * x) \text{ maj } \lambda x.\Phi_1(s * x), \lambda x.\Phi_2(s * x), \lambda x.\Phi_3(s * x),$$

and by the definition of majorizability

$$H^*(s^*, \lambda x.\Phi_1(s^* * x)) \text{ maj } H^*(s, \lambda x.\Phi_1(s * x)), H(s, \lambda x.\Phi_2(s * x)), H(s, \lambda x.\Phi_3(s * x)),$$

which implies

$$\begin{aligned}&(s^* \mathbin{@} H^*(s^*, \lambda x.\Phi_1(s^* * x)))^+ \text{ maj}\\ &\quad (s \mathbin{@} H^*(s, \lambda x.\Phi_1(s * x)))^+, (s \mathbin{@} H(s, \lambda x.\Phi_2(s * x)))^+, s \mathbin{@} H(s, \lambda x.\Phi_3(s * x)).\end{aligned}$$

And finally, by Lemma 5.4 (i) and (ii),

$$\begin{aligned}\overbrace{\widehat{Y^*}(s^* \mathbin{@} H^*(s^*, \lambda x.\Phi_1(s^* * x)))}^{\Phi_1(s^*)} \geq \overbrace{\widehat{Y^*}(s \mathbin{@} H^*(s, \lambda x.\Phi_1(s * x)))}^{\Phi_1(s)},&\\ \underbrace{\widehat{Y}(s \mathbin{@} H(s, \lambda x.\Phi_2(s * x)))}_{\Phi_2(s)}, \underbrace{Y(s \mathbin{@} H(s, \lambda x.\Phi_3(s * x)))}_{\Phi_3(s)}&.\end{aligned}$$

□

Theorem 5.12. There exists a $\Phi \in \mathcal{M}$ (not unique) satisfying MBR.

Proof. In Section 5.1 we have constructed a

$$\Phi \in \mathcal{M}_{\rho^\omega \to \mathbb{N}} \times \mathcal{M}_{\rho^* \to (\rho \to \mathbb{N}) \to \rho^\omega} \to \mathcal{M}_{\rho^* \to \mathbb{N}}$$

satisfying MBR. By Proposition 5.11, $\Phi \in \mathcal{M}$. The fact that Φ is not unique follows by taking, e.g.,

$$Y(\alpha) = \begin{cases} 1 & \text{if } \exists n \forall m \geq n(\alpha(m) = 1) \\ 0 & \text{otherwise,} \end{cases}$$

and $H(s, F) = F(0)$. Both $\Phi(Y, H, s) = 0$ and $\Phi(Y, H, s) = 1$ are valid solutions. □

Corollary 5.13. KBR is not definable from MBR.

Although we have shown that modified bar recursion exists in the model of strongly majorizable functionals, the use of continuity principles in the soundness proof (Theorem 2.5) of the interpretation means that the bar recursive extracted term will not in general be a realizer in Bezem's model (which is the case for Spector's interpretation).

6. Definability of SBR from MBR

In this section we show that SBR is definable from MBR. This definability relies on a nested use of MBR. On a first stage MBR is used to define the following search operator.

Definition 6.1. $\tilde{\mu}(Y, \alpha^{\rho^\omega}, k) := \min n \geq k(Y(\overline{\alpha, n}) < n)$.

Kohlenbach (Koh90) has shown that $\tilde{\mu}$ is definable from SBR.

Proposition 6.2. $\tilde{\mu}$ is definable from MBR.

Proof. Let n be the value of $\tilde{\mu}(Y, \alpha, k)$. The case when $n = k$ is simple and will be treated at the end of the proof. We will assume that $n > k$. In this case we note that, by the minimality condition, $Y(\overline{\alpha, n-1}) \geq n - 1$. Hence, $Y(\overline{\alpha, n-1}) + 1$ can be used (for bounded search) as an upper bound for the value of n. What we show is that $Y(\overline{\alpha, n-1}) + 1$ is definable from MBR, by defining an appropriate H which computes $\overline{\alpha, n-1}$. We now indicate how this intuition can be formalised.
By MBR we can define a functional Φ satisfying $\Phi(\alpha, Y, j) = Y(\overline{\alpha, m-1})$, where

$$(*) \quad m \overset{\mathbb{N}}{=} \begin{cases} j & \text{if } Y(\overline{\alpha, j}) < j \\ \tilde{\mu}^b(Y, \alpha, k, \Phi(\alpha, Y, j+1) + 1) & \text{otherwise} \end{cases}$$

and $\tilde{\mu}^b$ is the (primitive recursive) bounded version of $\tilde{\mu}$ which uses an extra argument as an upper bound for the search. We then define

$$\tilde{\mu}(Y, \alpha, k) := \begin{cases} k & \text{if } Y(\overline{\alpha, k}) < k \\ \tilde{\mu}^b(Y, \alpha, k, \Phi(\alpha, Y, k) + 1) & \text{otherwise.} \end{cases}$$

We show that this is a good definition of $\tilde{\mu}$ by showing that $\Phi(\alpha, Y, k) + 1$ is a good upper bound on $\tilde{\mu}(Y, \alpha, k)$, assuming that $\tilde{\mu}(Y, \alpha, k) > k$. In fact, assuming $\tilde{\mu}(Y, \alpha, k) = n$, we show by induction on j that, for $k \leq j \leq n$, $n \leq \Phi(\alpha, Y, j) + 1$.

i) $j = n$. We see that the first case of $(*)$ will be satisfied, m is equal n and $\Phi(\alpha, Y, j) + 1 = Y(\overline{\alpha, n-1}) + 1 \geq n$.

ii) $j < n$. By induction hypothesis $\Phi(\alpha, Y, j+1) + 1$ is a bound for n. Therefore, m (see second case of $(*)$) has value n, and as above we get $\Phi(\alpha, Y, j) + 1 \geq n$. □

Proposition 6.3. $\mathsf{SBR}_{\rho,\mathbb{N}}$ is definable from MBR.

Proof. We show how to define (primitive recursively in MBR) a Ψ satisfying the equation ($\mathsf{SBR}_{\rho,\mathbb{N}}$),

$$(i) \quad \Psi(Y, G, H, s) \overset{\mathbb{N}}{=} \begin{cases} G(s) & \text{if } Y(\hat{s}) < |s| \\ H(s, \lambda x^\rho . \Psi(Y, G, H, s * x)) & \text{otherwise.} \end{cases}$$

Let Φ be a functional satisfying MBR. In the following π_0 and π_1 will denote projections, i.e. $\pi_i(\langle x_0^\tau, x_1^\rho \rangle) = x_i$, $i \in \{0, 1\}$. If $s^{(\tau \times \rho)^*} = \langle s_0, \ldots, s_n \rangle$, $\pi_i(s)$ also denotes $\langle \pi_i(s_0), \ldots, \pi_i(s_n) \rangle$. In the same way we define $\pi_i(\alpha^{(\tau \times \rho)^\omega})$. We also write $\langle 0, s \rangle$ for $\langle \langle 0, s_0 \rangle, \ldots, \langle 0, s_{|s|-1} \rangle \rangle$. Finally, let $n^\uparrow$ denote the standard lifting from type $\mathbb{N}$ to ρ, and $x^\downarrow$ the reverse operation. We first define two operations,

$$(ii) \quad \tilde{H}(s, F) \overset{(\mathbb{N} \times \rho)^\omega}{:=} \lambda n . \langle 1, (H(\pi_1(s), \lambda x^\rho . F(\langle 0, x \rangle)))^\uparrow \rangle$$

and

$(iii)\tilde{Y}_{G,k}(\alpha) \stackrel{\mathbb{N}}{:=} \begin{cases} G(\pi_1(\overline{\alpha}n)) & \text{if } \bigwedge_{i=0}^{n-1} (\pi_0(\alpha(i)) = 0) \\ (\pi_1(\alpha(n)))^{\downarrow} & \text{otherwise,} \end{cases}$

where $n := \tilde{\mu}(Y, \pi_1(\alpha), k)$ in the definition of $\tilde{Y}_{G,k}$. Note that the operation $\tilde{H}$ is primitive recursive in H, s and F; and the operation $\tilde{Y}$ is primitive recursive in Y, G, k, α and MBR (since it uses $\tilde{\mu}$). It is easy to show that

$(*)$ if $Y(\hat{s}) \geq |s|$ (hence $n > |s|$) then $\tilde{Y}_{G,|s|}(\langle 0, s\rangle * \beta) = \tilde{Y}_{G,|s|+1}(\langle 0, s\rangle * \beta)$, for all β.

Define

(iv) $\Psi(Y, G, H, s) := \Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s\rangle))$.

We show that Ψ satisfies equation (i), i.e.

(v) $\Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s\rangle) = \begin{cases} G(s) & \text{if } Y(\hat{s}) < |s| \\ H(s, \lambda x^{\rho}.\Phi(\tilde{Y}_{G,|s|+1}, \tilde{H}, \langle 0, s * x\rangle)) & \text{otherwise.} \end{cases}$

We first note that, by the definition of MBR (and (ii)),

(vi) $\Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s\rangle) = \tilde{Y}_{G,|s|}(\langle 0, s\rangle \;@\; \lambda n.\langle 1, (H(s, \lambda x^{\rho}.\Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s * x\rangle)))^{\uparrow}\rangle)$.

We will show that (v) holds. Assume $Y(\hat{s}) < |s|$, we have,

$$\Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s\rangle) \stackrel{(vi)}{=} \tilde{Y}_{G,|s|}(\langle 0, s\rangle \;@\; \ldots) \stackrel{(iii)}{=} G(s).$$

On the other hand, if $Y(\hat{s}) \geq |s|$ then,

$$\begin{aligned} \Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s\rangle) &\stackrel{(vi)}{=} \tilde{Y}_{G,|s|}(\langle 0, s\rangle \;@\; \lambda n.\langle 1, (H(s, \lambda x.\Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s * x\rangle)))^{\uparrow}\rangle) \\ &\stackrel{(iii)}{=} H(s, \lambda x.\Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s * x\rangle)) \\ &\stackrel{(*)}{=} H(s, \lambda x.\Phi(\tilde{Y}_{G,|s|+1}, \tilde{H}, \langle 0, s * x\rangle)), \end{aligned}$$

and the proof is concluded. □

In the following we show that nothing is lost by restricting the type level of τ in $\mathsf{SBR}_{\rho,\tau}$ to 0. As pointed out in Remark 2.3, for simplicity we identify types of level 0 with $\mathbb{N}$.

Proposition 6.4. $\mathsf{SBR}_{\rho,\tau}$ is definable from $\mathsf{SBR}_{\rho',\mathbb{N}}$, where if $\tau = \tau_1 \to \ldots \to \tau_n \to \mathbb{N}$ then $\rho' = \rho \times \tau_1 \times \ldots \times \tau_n$.

Proof. Let $\tau = \tau_1 \to \ldots \to \tau_n \to \mathbb{N}$ and $\pi_i^{n+1}(\langle x_0, \ldots, x_n\rangle^{(\rho \times \tau_1 \times \ldots \times \tau_n)}) = x_i$. We will show that $\mathsf{SBR}_{\rho,\tau}$ can be defined from $\mathsf{SBR}_{\rho \times \tau_1 \times \ldots \times \tau_n, \mathbb{N}}$. Let G, H and Y be given, we have to define a functional Φ such that,

(i) $\Phi(Y, G, H, s) \stackrel{\tau}{=} \begin{cases} G(s) & \text{if } Y(\hat{s}) \stackrel{\mathbb{N}}{<} |s| \\ H(s, \lambda x^{\rho}.\Phi(s * x)) & \text{otherwise.} \end{cases}$

Notice that in the recursive call of Φ we, for simplicity, omit the arguments Y, G and H. From Y, G and H we define,

(ii) $\tilde{Y}(\alpha) := Y(\pi_0^{n+1}(\alpha))$;
$(iii)\tilde{G}(t) := G(\pi_0^{n+1}(t))(y)$;
(iv) $\tilde{H}(t, F) := H(\pi_0^{n+1}(t), \lambda x^{\rho}, z_1^{\tau_1}, \ldots, z_n^{\tau_n}.F(\langle x, z_1, \ldots, z_n\rangle))(y)$;

where y denotes $\pi_1^{n+1}(t_{|t|-1}), \ldots, \pi_{n+1}^{n+1}(t_{|t|-1})$ and the types are,

$$\alpha \quad : \quad (\rho \times \tau_1 \times \ldots \times \tau_n)^{\omega}$$

$$
\begin{array}{rcl}
y & : & \tau_1 \times \ldots \times \tau_n \\
t & : & (\rho \times \tau_1 \times \ldots \times \tau_n)^* \\
F & : & (\rho \times \tau_1 \times \ldots \times \tau_n) \to \mathbb{N}
\end{array}
$$

and we define (using $\mathsf{SBR}_{\rho \times \tau_1 \times \ldots \times \tau_n, \mathbb{N}}$),

$$
(v)\ \ \Psi(\tilde{Y}, \tilde{G}, \tilde{H}, t) \stackrel{\mathbb{N}}{=} \begin{cases} \tilde{G}(t) & \text{if } \tilde{Y}(\hat{t}) \stackrel{\mathbb{N}}{<} |t| \\ \tilde{H}(t, \lambda x^{\rho \times \tau_1 \times \ldots \times \tau_n}.\Psi(t * x)) & \text{otherwise.} \end{cases}
$$

Finally we set, ($\langle s, \mathbf{y} \rangle$ abbreviates $\langle \langle s_0, \mathbf{y} \rangle, \ldots, \langle s_{|s|-1}, \mathbf{y} \rangle \rangle$)

$(vi)\ \Phi(Y, G, H, s) :\stackrel{\mathcal{T}}{=} \lambda \mathbf{y}.\Psi(\tilde{Y}, \tilde{G}, \tilde{H}, \langle s, \mathbf{y} \rangle)$.

We show that equation (i) is satisfied by Φ. One easily verifies that

$(vii) \Psi(\tilde{Y}, \tilde{G}, \tilde{H}, \langle s, \mathbf{y} \rangle) = \Psi(\tilde{Y}, \tilde{G}, \tilde{H}, \langle \langle s_0, \mathbf{z} \rangle, \ldots, \langle s_{|s|-2}, \mathbf{z} \rangle, \langle s_{|s|-1}, \mathbf{y} \rangle \rangle)$,

for arbitrary $\mathbf{z}$. Let Y, G, H and s be fixed and t abbreviate $\langle s, \mathbf{y} \rangle$. By (ii), $Y(\hat{s}) < |s|$ if and only if $\tilde{Y}(\hat{t}) < |t|$. Therefore, if $Y(\hat{s}) < |s|$ then

$$
\begin{array}{rcl}
\Phi(Y, G, H, s) & \stackrel{(vi)}{=} & \lambda \mathbf{y}.\Psi(\tilde{Y}, \tilde{G}, \tilde{H}, \langle s, \mathbf{y} \rangle) \\
& \stackrel{(v)}{=} & \lambda \mathbf{y}.\tilde{G}(\langle s, \mathbf{y} \rangle) \stackrel{(iii)}{=} \lambda \mathbf{y}.G(s)(\mathbf{y}) = G(s).
\end{array}
$$

On the other hand, if $Y(\hat{s}) \geq |s|$ then

$$
\begin{array}{rcl}
\Phi(Y, G, H, s) & \stackrel{(vi)}{=} & \lambda \mathbf{y}.\Psi(\tilde{Y}, \tilde{G}, \tilde{H}, \langle s, \mathbf{y} \rangle) \stackrel{(v)}{=} \lambda \mathbf{y}.\tilde{H}(t, \lambda x.\Psi(t * x)) \\
& \stackrel{(iv)}{=} & \lambda \mathbf{y}.H(s, \lambda x, \mathbf{z}.\Psi(t * \langle x, \mathbf{z} \rangle))(\mathbf{y}) \\
& \stackrel{(vii)}{=} & \lambda \mathbf{y}.H(s, \lambda x, \mathbf{z}.\Psi(\langle s * x, \mathbf{z} \rangle))(\mathbf{y}) \\
& \stackrel{(vi)}{=} & \lambda \mathbf{y}.H(s, \lambda x.\Phi(s * x))(\mathbf{y}) = H(s, \lambda x.\Phi(s * x)).
\end{array}
$$

□

Theorem 6.5. SBR is definable from MBR.

7. S1-S9 Computability

In this section we review Tait's result that the fan functional is not S1-S9 computable in the type structure $\mathcal{C}$ of total continuous functionals (published in (GH77)). The proof given here is an elaboration of an argument sketched by Normann in (Nor99). We will use this result, together with the definability of FAN from MBR and KBR and the fact that KBR is S1-S9 computable in $\mathcal{C}$, to show that MBR is not S1-S9 computable in $\mathcal{C}$ and hence MBR is not definable from SBR or KBR in any theory Δ which has $\mathcal{C}$ as a model.

By Theorem 4.1, the non S1-S9 computability of MBR is subsumed by Hyland's result that the functional Γ is not S1-S9 computable in the fan functional ((Hyl75), published in (GH77)). We nevertheless think that it is worthwhile presenting our proof, which is very direct and makes use of elementary domain-theoretic concepts only.

Definition 7.1 (S1-S9). Given a type structure $\mathcal{S}$ we define a family of relations $\Gamma^{\mathcal{S}}$ (parametrized by their arity and type of arguments) on $\mathcal{S}$ inductively as follows. We abbreviate $y_1, \ldots, y_n$ (of arbitrary type) by $\mathbf{y}$. The variables $e_1, e_2, m, n, i, k, k_1, k_2$ range over natural numbers, $f, x, \mathbf{y}$ over functionals of appropriate types and σ denotes the code of the list of types of $\mathbf{y}$. We write $\{e\}^{\mathcal{S}}(\mathbf{y}) = k$ instead of $(e, \mathbf{y}, k) \in \Gamma^{\mathcal{S}}$. The analogues of the schemes S1 and S5 for lists are omitted. For brevity, we also ignore product types.

S1 $\{e\}^{\mathcal{S}}(m, \mathbf{y}) = m + 1$, where $e = \langle 1, \sigma \rangle$.
S2 $\{e\}^{\mathcal{S}}(\mathbf{y}) = k$, where $e = \langle 2, \sigma, k \rangle$.
S3 $\{e\}^{\mathcal{S}}(m, \mathbf{y}) = m$, where $e = \langle 3, \sigma \rangle$.
S4 If $\{e_1\}^{\mathcal{S}}(\mathbf{y}) = k_1$ and $\{e_2\}^{\mathcal{S}}(k_1, \mathbf{y}) = k_2$, then $\{e\}^{\mathcal{S}}(\mathbf{y}) = k_2$,
where $e = \langle 4, e_1, e_2, \sigma \rangle$.
S5 If $\{e_1\}^{\mathcal{S}}(\mathbf{y}) = k$, then $\{e\}^{\mathcal{S}}(0, \mathbf{y}) = k$;
If $\{e\}^{\mathcal{S}}(n, \mathbf{y}) = k$ and $\{e_2\}^{\mathcal{S}}(n, k, \mathbf{y}) = k_1$, then $\{e\}^{\mathcal{S}}(n + 1, \mathbf{y}) = k_1$,
where $e = \langle 5, e_1, e_2, \sigma \rangle$.
S6 If $\{e_1\}^{\mathcal{S}}(\pi(\mathbf{y})) = k$, then $\{e\}^{\mathcal{S}}(\mathbf{y}) = k$,
where π is a permutation and $e = \langle 6, e_1, \ulcorner \pi \urcorner, \sigma \rangle$.
S7 $\{e\}^{\mathcal{S}}(f, x, \mathbf{y}) = f(x)$, where $e = \langle 7, \sigma \rangle$.
S8 If $\{e_1\}^{\mathcal{S}}(x, \mathbf{y}) = f(x)$, for all x, then $\{e\}^{\mathcal{S}}(\mathbf{y}) = y_1(f)$,
where $e = \langle 8, e_1, \sigma \rangle$.
S9 If $\{e_1\}^{\mathcal{S}}(y_1, \ldots, y_i) = k$, then $\{e\}^{\mathcal{S}}(e_1, \mathbf{y}) = k$,
where $i \leq n$ and $e = \langle 9, i, \sigma \rangle$.

One can prove by induction on S1-S9 that for each e and $\mathbf{y}$ there exists at most one k such that $\{e\}^{\mathcal{S}}(\mathbf{y}) = k$. Therefore, each index e gives rise to a partial function, denoted by $\{e\}^{\mathcal{S}}$, which on input $\mathbf{y}$ takes value k if $\{e\}^{\mathcal{S}}(\mathbf{y}) = k$ and is undefined otherwise. It is important to note that for an arbitrary type structure $\mathcal{S}$ the partial function $\{e\}^{\mathcal{S}}$ need not belong to $\mathcal{S}$ (but see Lemma 7.6 below). When the structure $\mathcal{S}$ is clear from the context we will write simply $\{e\}$, rather than $\{e\}^{\mathcal{S}}$.

Definition 7.2. Let $\mathcal{S}_0$ be a subset of a type structure $\mathcal{S}$. A functional G is $\mathcal{S}_0$ *definable from functionals* $\mathsf{F}_1, \ldots, \mathsf{F}_n$ if there is $\Psi \in \mathcal{S}_0$ such that

$$\mathcal{S} \models \forall \Phi_1, \ldots \Phi_n\, (\mathsf{F}_1(\Phi_1) \wedge \ldots \wedge \mathsf{F}_n(\Phi_n) \rightarrow \mathsf{G}(\Psi(\Phi_1, \ldots, \Phi_n))).$$

Definition 7.3 (S1-S9 computability). Let $\mathcal{S}$ be a type structure. An element $f \in \mathcal{S}$ is S1-S9 *computable* if $f = \{e\}^{\mathcal{S}}$, for some number e. The set of all S1-S9 computable elements of $\mathcal{S}$ is denoted $(\text{S1-S9})^{\mathcal{S}}$. Let F, G be functionals. G is *S1-S9 +* F *computable in* $\mathcal{S}$ if G is $(\text{S1-S9})^{\mathcal{S}}$ definable from F.

Proposition 7.4. KBR and SBR are S1-S9 computable in $\mathcal{C}$.

Proof. $\mathcal{C} \models \exists e\, \mathsf{KBR}(\{e\})$ and $\mathcal{C} \models \exists e\, \mathsf{SBR}(\{e\})$, by the recursion theorem. ☐

Ershov (Ers77) showed that the elements of $\mathcal{C}$ can be viewed as equivalence classes of total elements of $\widehat{\mathcal{C}}$ where two total elements of $\widehat{\mathcal{C}}$ are equivalent if they coincide on all total arguments. We denote these equivalence classes by $[[F]]$, i.e. if $F \in \widehat{\mathcal{C}}$ is total, then $[[F]] \in \mathcal{C}$.

Lemma 7.5 (Transfer principle). If $\{e\}^{\mathcal{C}}([[F]]) = k$, then $\{e\}^{\widehat{\mathcal{C}}}(F) = k$.

Proof. Straightforward induction on S1-S9. $\square$

Lemma 7.6.
(i) $\{e\}^{\widehat{\mathcal{C}}} \in \widehat{\mathcal{C}}$ for all S1-S9 indices e.
(ii) If $\{e\}^{\mathcal{C}}$ is total, then $\{e\}^{\mathcal{C}} \in \mathcal{C}$.

Proof. For part (i) one shows by an easy induction on S1-S9 that if $\{e\}^{\widehat{\mathcal{C}}}(\mathbf{x}) = k$, then there exist compact $\mathbf{x}_0 \sqsubseteq \mathbf{x}$ (in the domain-theoretic sense) with $\{e\}^{\widehat{\mathcal{C}}}(\mathbf{x}_0) = k$ and also $\{e\}^{\widehat{\mathcal{C}}}(\mathbf{y}) = k$ for all $\mathbf{y} \sqsupseteq \mathbf{x}$.

Part (ii) follows from part (i) and the transfer principle (Lemma 7.5). $\square$

Lemma 7.7. If

(i) e is a S1-S9 code of type 3,
(ii) $\mathbf{x}, \mathbf{y} \in \widehat{\mathcal{C}}$ of type 2 coincide at all total recursive arguments (of type 1),
(iii) $\mathbf{x}$ are total S1-S9 computable in $\widehat{\mathcal{C}}$,
(iv) $\{e\}^{\mathcal{C}}([[\mathbf{x}]]) = k$,
then $\{e\}^{\widehat{\mathcal{C}}}(\mathbf{y}) = k$.

Proof. By induction on S1-S9, the critical point being S8. Assume e is of the form $\langle 8, e_1, \sigma\rangle$ and that $(i) - (iv)$ hold. Then there must exist a function $f \in \mathcal{C}$ such that

(v) $f(n) = \{e_1\}^{\mathcal{C}}(n, [[\mathbf{x}]])$, for all $n \in \mathbb{N}$, and
(vi) $[[x_1]](f) = k$,
By (iii) and (v) we get that f is recursive. Let n be fixed and assume that $\{e_1\}^{\mathcal{C}}(n, [[\mathbf{x}]]) = l$. By induction hypothesis we have that

$$\{e_1\}^{\widehat{\mathcal{C}}}(n, \mathbf{y}) = l,$$

i.e. $\lambda n^{\mathbb{N}}.\{e_1\}^{\widehat{\mathcal{C}}}(n, \mathbf{y})$ $(= [[\lambda p^{\mathbb{N}_\perp}.\{e_1\}^{\widehat{\mathcal{C}}}(p, \mathbf{y})]])$ is identical to f. By (vi),

$$[[x_1]]([[\lambda p^{\mathbb{N}_\perp}.\{e_1\}^{\widehat{\mathcal{C}}}(p, \mathbf{y})]]) = k.$$

Hence,

$$x_1(\lambda p.\{e_1\}^{\widehat{\mathcal{C}}}(p, \mathbf{y})) = k.$$

Note that $\lambda p.\{e_1\}^{\widehat{\mathcal{C}}}(p, \mathbf{y})$ is total and recursive. Therefore, by assumption (ii),

$$y_1(\lambda p.\{e_1\}^{\widehat{\mathcal{C}}}(p, \mathbf{y})) = k,$$

and, by S8, $\{e\}^{\widehat{\mathcal{C}}}(\mathbf{y}) = k$. $\square$

Theorem 7.8 (GH77). FAN is not S1-S9 computable in $\mathcal{C}$.

Proof. (Nor99) Assume e is such that $\mathcal{C} \models \mathsf{FAN}(\{e\})$. Let O be a total (S1-S9 computable) element of $\widehat{\mathcal{C}}$ which is constant zero. Assume $\{e\}^{\mathcal{C}}([[O]]) = k$. Let F be another type two functional in $\widehat{\mathcal{C}}$ such that $F(f) = 0$ whenever f is total and recursive, but which is $\perp$ for some other total (non recursive) f. Such a functional F can be defined using Kleene's well-known non-wellfounded tree which has no infinite recursive path. By Lemma 7.7 $\{e\}^{\widehat{\mathcal{C}}}(F) = k$. By $(+)$ there is a compact $G \sqsubseteq F$ in $\widehat{\mathcal{C}}$ such that $\{e\}^{\widehat{\mathcal{C}}}(G) = k$. It is easy to see that one can extend G to a total G' such that k is not a modulus of uniform continuity for G'. Assume $\{e\}^{\mathcal{C}}([[G']]) = l$. By the transfer principle $\{e\}^{\widehat{\mathcal{C}}}(G') = l$ and l must equal k, i.e. $\{e\}^{\mathcal{C}}([[G']]) = k$, a contradiction. $\square$

Proposition 7.9. FAN is S1-S9 + MBR computable in $\mathcal{C}$.

Proof. By Lemma 7.4, KBR is S1-S9 computable in $\mathcal{C}$. The proof of Theorem 2.7 in (BO05) gives a definition of FAN from MBR and KBR which clearly can be carried out by a S1-S9 computation. □

Theorem 7.10 (essentially (Hyl75)). MBR is not S1-S9 computable in $\mathcal{C}$.

Proof. Theorem 7.8 and Proposition 7.9. □

This result apparently contradicts the fact that MBR is defined by recursion, which, due to Kleene's recursion theorem, is permitted in S1-S9. The point is that in Scheme S8 the object named f is required to live in the model we are working with. This is no problem in $\widehat{\mathcal{C}}$ (see Lemma 7.6), but in model $\mathcal{C}$ it means that f has to be total. Therefore, in $\mathcal{C}$, the recursive definition of MBR (given by an index e) does not define a functional, since $(e, \mathbf{y}, k) \notin \Gamma^{\mathcal{C}}$, for any $\mathbf{y}$ and k.

Theorem 7.11. MBR is neither definable from SBR nor from KBR in any theory that is validated by $\mathcal{C}$.

Proof. This follows from Theorem 7.10, Proposition 7.4, Theorem 2.2 and the fact that the set of S1-S9 computable functionals is closed under Gödel primitive recursion. □

Gandy and Hyland (GH77) also showed that the functional Γ (see Section 4) is not S1-S9 + FAN computable in $\mathcal{C}$. From Theorem 4.1 we obtain the following theorem.

Theorem 7.12. $\mathsf{MBR}_{\mathbb{N}}$ is not S1-S9 + FAN computable in $\mathcal{C}$.

8. Summary

We have given a complete explanation of the relation between modified bar recursion MBR and Spector's original definition SBR as well as Kohlenbach's variant KBR. The relation between MBR and Gandy's Γ functional is also settled. The results are summarised in Figure 1.

An arrow from functional F to functional G indicates that G is definable from F. The lack of an arrow represents the fact that the definability is not possible. The definabilities are shown constructively through the presentation of a term satisfying the required equations, the non-definabilities are obtained model-theoretically via Bezem's model $\mathcal{M}$ and Kleene's S1-S9 computability in the model $\mathcal{C}$ of partial continuous functionals.

While MBR and KBR together define the fan functional (BO05), neither MBR nor KBR alone can define FAN, because, firstly, MBR exists in $\mathcal{M}$ (Theorem 5.12), but FAN does not (there are discontinuous functionals in $\mathcal{M}$ at type 2), and, secondly, KBR is S1-S9 computable in $\mathcal{C}$ (Koh90), but FAN is not (GH77; Nor99).

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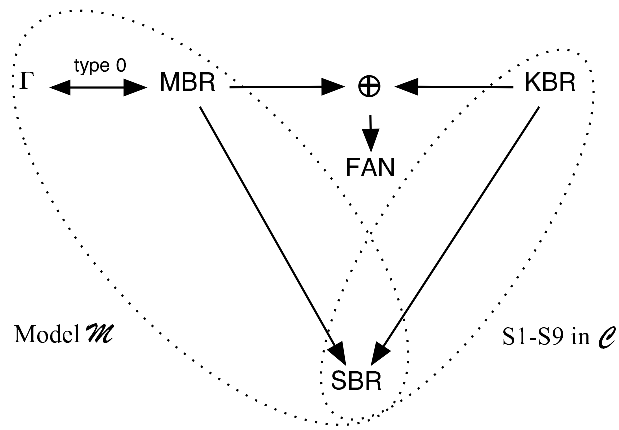

Fig. 1. **Summary of results**

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