MODIFIED BAR RECURSION AND CLASSICAL DEPENDENT CHOICE

ULRICH BERGER AND PAULO OLIVA

Abstract. We introduce a variant of Spector's bar recursion in finite types (which we call "modified bar recursion") to give a realizability interpretation of the classical axiom of dependent choice allowing for the extraction of witnesses from proofs of $\forall\exists$-formulas in classical analysis. As another application, we show that the fan functional can be defined by modified bar recursion together with a version of bar recursion due to Kohlenbach. We also show that the type structure $\mathcal{M}$ of strongly majorizable functionals is a model for modified bar recursion.

§**1. Introduction.** In [22], Spector extended Gödel's Dialectica Interpretation of Peano Arithmetic [10] to classical analysis using bar recursion in finite types. Although considered questionable from an intuitionistic point of view ([1], 6.6), there has been considerable interest in bar recursion, and several variants of this definition scheme and their interrelations have been studied by, e.g., Schwichtenberg [19], Bezem [8] and Kohlenbach [14]. In this paper we add another variant of bar recursion and use it to give a realizability interpretation of the negatively translated axiom of dependent choice that can be used to extract witnesses from proofs of $\forall\exists$-formulas in full classical analysis. Our interpretation is inspired by a paper by Berardi, Bezem and Coquand [2] who use a similar kind of recursion in order to interpret dependent choice. The main difference to our paper is that in [2] a rather ad-hoc infinitary term calculus and a non-standard notion of realizability are used whereas we work with a straightforward combination of negative translation, A-translation, modified realizability, and Plotkin's adequacy result for the partial continuous functional semantics of PCF [18].

As a second application of bar recursion, we show that the definition of the fan functional within PCF given in [3] and [17] can be derived from Kohlenbach's and our variant of bar recursion. Furthermore, we prove that our version of bar recursion exists in the model of majorizable functions. The relation between modified bar recursion and Spector's original definition is established in [5].

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§2. **Bar recursion in finite types.** We work in a suitable extension of Heyting Arithmetic in finite types, HA^ω, with equality in all types. For convenience, we enrich the type system by the formation of finite sequences. So, our *Types* are $\mathbb{N}$, function types $\rho \to \sigma$, product types $\rho \times \sigma$, and finite sequences ρ^*. We set $\rho^\omega :\equiv \mathbb{N} \to \rho$. The *level* of a type is defined by $\mathsf{level}(\mathbb{N}) = 0$, $\mathsf{level}(\rho \times \sigma) = \mathsf{max}(\mathsf{level}(\rho), \mathsf{level}(\sigma))$, $\mathsf{level}(\rho^*) = \mathsf{level}(\rho)$, $\mathsf{level}(\rho \to \sigma) = \mathsf{max}(\mathsf{level}(\rho) + 1, \mathsf{level}(\sigma))$. By o we will denote an arbitrary but fixed type of level 0, and by ρ, τ, σ arbitrary types. The terms of our version of HA^ω are a suitable extension of the terms of Gödel's system T [10] in lambda calculus notation. We use the variables $i, j, k, l, m, n \colon \mathbb{N}$ and $s, t \colon \rho^*$; $\alpha, \beta \colon \rho^\omega$, where ρ is an arbitrary type. Other letters will be used for different types in different contexts. By $\overset{\tau}{=}$ we denote equality of type τ for which we assume the usual equality axioms. However, equality between functions is *not* assumed to be extensional. We also do *not* assume decidability for $\overset{\tau}{=}$, when $\mathsf{level}(\tau) > 0$ (if $\mathsf{level}(\tau) = 0$ one can, of course, *prove* decidability). Type information will be frequently omitted when it is irrelevant or inferable from the context. We let k^ρ denote the canonical lifting of a number $k \in \mathbb{N}$ to type ρ, e.g., $k^{\rho \to \sigma} :\equiv \lambda x^\rho . k^\sigma$. By an $\exists$-*formula* respectively $\forall\exists$-*formula* we mean a formula of the form $\exists y^\tau\, B$ respectively $\forall z^\sigma\, \exists y^\tau\, B$, where B is provably equivalent to an atomic formula. We will also use the following notations:

$$\begin{aligned}
\langle x_0, \dots, x_{n-1} \rangle &:\equiv \text{the finite sequence with elements } x_0, \dots, x_{n-1}, \\
|s| &:\equiv \text{the length of } s \text{, i.e., } |\langle x_0, \dots, x_{n-1} \rangle| = n, \\
s_k &:\equiv \text{the } k\text{-th element of } s \text{ for } k < |s|, \\
&\ \text{i.e., } \langle x_0, \dots, x_{n-1} \rangle_k = x_k, \\
s * t &:\equiv \text{the concatenation of } s \text{ and } t, \\
s * x &:\equiv s * \langle x \rangle, \\
s * \alpha &:\equiv \text{appending } \alpha \text{ to } s \text{, i.e.,} \\
&\ s * \alpha :\equiv \lambda k . \left[\mathsf{if}\ k {<} |s|\ \mathsf{then}\ s_k\ \mathsf{else}\ \alpha(k {-} |s|)\right], \\
s \,@\, \alpha &:\equiv \text{overwriting } \alpha \text{ with } s \text{, i.e.,} \\
&\ s \,@\, \alpha :\equiv \lambda k . \left[\mathsf{if}\ k < |s|\ \mathsf{then}\ s_k\ \mathsf{else}\ \alpha(k)\right], \\
\overline{\alpha}k &:\equiv \langle \alpha(0), \dots, \alpha(k-1) \rangle, \\
\beta \in \overline{\alpha}k &:\equiv \overline{\beta}k \overset{\rho^*}{=} \overline{\alpha}k.
\end{aligned}$$

Definition 1. *Spector's definition of bar recursion* [22] *reads in our notation as follows*:

$$\Phi(Y, G, H, s) \overset{\tau}{=} \begin{cases} G(s) & \mathsf{if}\ Y\big(s \,@\, 0^{\rho^\omega}\big) < |s|, \\ H(s, \lambda x^\rho . \Phi(Y, G, H, s * x)) & \mathsf{otherwise}. \end{cases} \tag{1}$$

In his thesis [14] *Kohlenbach introduced the following kind of bar recursion which differs from Spector's only in the stopping condition*:

(2)
$$\Phi(Y, G, H, s) \overset{\tau}{=} \begin{cases} G(s) & \text{if } Y(s \mathbin{@} 0^{\rho^\omega}) \overset{o}{=} Y\big(s \mathbin{@} 1^{\rho^\omega}\big), \\ H(s, \lambda x^\rho . \Phi(Y, G, H, s * x)) & \text{otherwise.} \end{cases}$$

Finally, we define Modified bar recursion at type ρ:

$$\Phi(Y, H, s) \overset{o}{=} Y(s \mathbin{@} H(s, \lambda x^\rho . \Phi(Y, H, s * x))). \tag{3}$$

Note that each of the equations above defines a family of functionals $\Phi_{\rho,\tau}$ (Φ_ρ *in the case of modified bar recursion*) *as* ρ *and* τ *range over arbitrary finite types. We shall often omit the parameters* Y, G *and* H *when defining a functional* Φ *using the equations above. We say a model* $\mathcal{S}$ *satisfies one of the respective variants of bar recursion if in* $\mathcal{S}$ *a functional exists satisfying the corresponding equation* (1), (2), *or* (3) *for all possible values of* Y, G, H *and* s.

Recursive definitions similar to (3) occur in [2], and, in a slightly different form, in [3] and [17] in connection with the fan functional (cf. Section 4).

Remark. Note that replacing in equation (3) the operation @ by $*$ would be an inessential change. However it *is* essential that the type of $\Phi(s)$ is of level 0. If, for example, the type of $\Phi(s)$ were $\mathbb{N} \to \mathbb{N}$ we could set $Y(\alpha)(m) \overset{\mathbb{N}}{:\equiv} \alpha(m) + 1$ and $H(s, F)(k) \overset{\mathbb{N}}{:\equiv} F(0)(|s| + 1)$, and obtain the equation

$$\Phi(s)(m) \overset{\mathbb{N}}{=} (s \mathbin{@} \lambda k . \Phi(s * 0)(|s| + 1))(m) + 1$$

implying

$$\Phi(\langle\,\rangle)(0) \overset{\mathbb{N}}{=} \Phi(\langle 0 \rangle)(1) + 1 \overset{\mathbb{N}}{=} \Phi(\langle 0, 0 \rangle)(2) + 2 \overset{\mathbb{N}}{=} \cdots$$

which is unsatisfiable in $\mathbb{N}$.

The structures of primary interest to interpret bar recursion are the model $\mathcal{C}$ of *total continuous functionals* of Kleene [13] and Kreisel [15], the model $\widehat{\mathcal{C}}$ of *partial continuous functionals* of Scott [20] and Ershov [9] (see also [17]), and the model $\mathcal{M}$ of (strongly) *majorizable functionals* introduced by Howard [11] and Bezem [7].

Theorem 1. *The models* $\mathcal{C}$ *and* $\widehat{\mathcal{C}}$ *satisfy all three variants of bar recursion.*

Proof. In the model $\widehat{\mathcal{C}}$ all three forms of bar recursion can simply be defined as the least fixed points of suitable continuous functionals. For $\mathcal{C}$ we use Ershov's result in [9] according to which the model $\mathcal{C}$ can be identified with the total elements of $\widehat{\mathcal{C}}$. Therefore it suffices to show that all three versions of bar recursion are total in $\widehat{\mathcal{C}}$. For Spector's version this has been shown by Ershov [9], and for the other versions similar argument apply. For example, in order to see that $\Phi(s)$ defined recursively by equation (3) is total for given total Y, H and s one uses bar induction on the bar

$$P(s) :\Leftrightarrow Y(s \mathbin{@} \bot_\rho) \text{ is total}$$

where $\bot_\rho$ denotes the undefined element of type ρ. $P(s)$ is a bar because Y is continuous. ⊣

THEOREM 2. *$\mathcal{M}$ satisfies Spector's bar recursion* (1), *but not Kohlenbach's* (2).

PROOF. See [7] and [14]. ⊣

In Section 5 we will show that $\mathcal{M}$ satisfies modified bar recursion (3).

§3. Using bar recursion to realize classical dependent choice.

The aim of this section is to show how modified bar recursion can be used to extract witnesses from proofs of $\forall\exists$-formulas in classical arithmetic plus the axiom (scheme) of dependent choice [12]

$$\mathbf{DC} \quad \forall n, x^\rho \, \exists y^\rho \, A(n, x, y) \to \forall x \exists f \, (f(0) = x \wedge \forall n \, A(n, f(n), f(n+1))).$$

Actually we will need only the following *weak modified bar recursion* which is the special case of equation (3) where H is constant:

$$\Phi(Y, H, s) \overset{o}{=} Y(s \mathbin{@} \lambda k. H(s, \lambda x. \Phi(Y, H, s * x))). \tag{4}$$

Note that in (4) the returning type of H is ρ, i.e., the argument of Y consists of s followed by an infinite sequence with constant value of type ρ.

Before dealing with dependent choice we discuss our extraction method in general and then give a realizer for the (simpler) classical axiom of countable choice.

3.1. Witnesses from classical proofs.

The method we use to extract witnesses from classical proofs is a combination of Gödel's negative translation (translation P^o in [16] page 42, see also [23]), the Dragalin/Friedman/Leivant trick, also called A-translation [25], and Kreisel's (formalized) modified realizability [24]. The method works in general for proofs in PA^ω, the classical variant of HA^ω. In order to extend it to PA^ω plus extra axioms Γ (e.g., $\Gamma \equiv \mathbf{DC}$) one has to find realizers for Γ^N, the negative translation of Γ [1], where $\bot$ is replaced by an $\exists$-formula (regarding negation, $\neg C$, is defined by $C \to \bot$). However, it is more direct and technically simpler to follow [6] and combine the Dragalin/Friedman/Leivant trick and modified realizability: instead of replacing $\bot$ by a $\exists$-formula we slightly change the definition of modified realizability by regarding $y \,\mathbf{mr}\, \bot$ as an (uninterpreted) atomic formula. More formally we define

$$y^\tau \,\mathbf{mr}_\tau\, \bot :\equiv P_\bot(y),$$

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[1] The negative translation double-negates atomic formulas, replaces $\exists x$ by $\neg\forall x \neg$ and $A \vee B$ by $\neg(\neg A \wedge \neg B)$.

where $P_\perp$ is a new unary predicate symbol and τ is the type of the witness to be extracted. Therefore, we have a modified realizability for each type τ, according to the type of the existential quantifier in the $\forall\exists$-formula we are realizing. The other clauses of modified realizability are as usual, e.g.,

$$f\,\mathbf{mr}_\tau(A \to B) :\equiv \forall x\,(x\,\mathbf{mr}_\tau\,A \to f x\,\mathbf{mr}_\tau\,B).$$

In the following proposition Δ is an axiom system possibly containing $P_\perp$ and further constants, which has the following closure property: If $D \in \Delta$ and B is a quantifier free formula with decidable predicates, then also the universal closure of $D[\lambda y^\tau.B/P_\perp]$ is in Δ, where $D[\lambda y^\tau.B/P_\perp]$ is obtained from D by replacing any occurrence of a formula $P_\perp(L)$ in D by $B[L/y]$.

Proposition 1. *Assume there is a vector Φ of closed terms such that*

$$\mathsf{HA}^\omega + \Delta \vdash \Phi\,\boldsymbol{mr}_\tau\,\Gamma^N.$$

Then from any proof

$$\mathsf{PA}^\omega + \Gamma \vdash \forall z^\sigma\,\exists y^\tau\,B(z,y),$$

where $\forall z^\sigma\,\exists y^\tau\,B(z,y)$ is a $\forall\exists$-formula in the language of HA^ω, one can extract a closed term $M^{\sigma\to\tau}$ such that

$$\mathsf{HA}^\omega + \Delta \vdash \forall z\,B(z, Mz).$$

Proof. The proof is folklore. The main steps are as follows. Assuming w.l.o.g. that $B(z,y)$ is atomic, we obtain from the hypothesis $\mathsf{PA}^\omega + \Gamma \vdash \forall z^\sigma\,\exists y^\tau\,B(z,y)$ via negative translation

$$\mathsf{HA}^\omega + \Gamma^N \vdash_m \forall y\,(B(z,y) \to \bot) \to \bot,$$

where $\vdash_m$ denotes derivability in minimal logic, i.e., ex-falso-quodlibet is not used. Now, soundness of modified realizability (which holds for our abstract version of modified realizability and minimal logic [6]), together with the assumption on Φ allows us to extract from this proof a closed term M such that

$$\mathsf{HA}^\omega + \Delta \vdash Mz\,\mathbf{mr}_\tau(\forall y\,(B(z,y) \to \bot) \to \bot)$$

i.e.,

$$\mathsf{HA}^\omega + \Delta \vdash \forall f^{\tau\to\tau}\,(\forall y\,(B(z,y) \to P_\perp(fy)) \to P_\perp(Mzf)).$$

Replacing $P_\perp$ by $\lambda y.B(z,y)$ respectively, and instantiating f by the identity function it follows

$$\mathsf{HA}^\omega + \Delta \vdash \forall z\,B(z, Mz(\lambda y.y)). \qquad \dashv$$

We will apply this proposition with $\tau :\equiv o$ (writing $\mathbf{mr}$ instead of $\mathbf{mr}_o$), $\Gamma :\equiv \mathbf{DC}$, or $\Gamma :\equiv \mathbf{AC}$ (countable choice, see below), and an axiom system Δ consisting of the defining equation (3) for modified bar recursion, where the defined functionals Φ are new constants, together with the axiom of continuity and the scheme of relativized quantifier free bar induction which are defined as follows:

Continuity $\qquad \forall F^{\rho^\omega\to o}, \alpha\,\exists n\,\forall\beta\,(\overline{\alpha}n = \overline{\beta}n \to F(\alpha) = F(\beta)).$

We call any n such that $\forall\beta\,(\overline{\alpha}n = \overline{\beta}n \to F(\alpha) = F(\beta))$ a point of continuity of F at α.

Relativized quantifier free bar induction

$$\forall\alpha \in S\,\exists n\,P(\overline{\alpha}n)\wedge\forall s \in S\,\big(\forall x\,[S(s{*}x) \to P(s{*}x)] \to P(s)\big)\wedge S(\langle\rangle) \to P(\langle\rangle).$$

Here $S(s)$ is an arbitrary, and $P(s)$ a quantifier free predicate in the language of $\mathsf{HA}^\omega[P_\perp]$, and $\alpha \in S$ and $s \in S$ are shorthands for $\forall n\,S(\overline{\alpha}n)$ and $S(s)$ respectively. Clearly the condition on Δ in Proposition 1 is satisfied.

In order to make sure that realizers can indeed be used to compute witnesses one needs to know that, 1. the axioms of $\mathsf{HA}^\omega + \Delta$ hold in a suitable model—here we can choose the model $\mathcal{C}$ of continuous functionals—and, 2. every closed term of type level 0 (e.g., of type $\mathbb{N}$) can be reduced to a numeral in an effective and provably correct way. In [2] this is solved by building the notion of reducibility to normal form into the definition of realizability. In our case we solve this problem by applying Plotkin's adequacy result [18] as follows: each term in the language of HA^ω plus the bar recursive constants can be naturally viewed as a term in the language PCF [18], by defining the bar recursors by means of the general fixed point combinator. In this way our term calculus also inherits PCF's call-by-name reduction, i.e., if M is bar recursive and M reduces to M' then M' is bar recursive. Furthermore reduction is provably correct in our system, i.e., if M reduces to M' then $M = M'$ is provable. Now let M be a closed term of type $\mathbb{N}$. By Theorem 1, M has a total value, which is a natural number n, in the model of partial continuous functionals. Hence, by Plotkin's adequacy theorem M reduces to the numeral denoting n.

3.2. Realizing $\mathbf{AC}^N$. We now construct a realizer of the negatively translated axiom of countable choice

AC $$\forall n^{\mathbb{N}}\,\exists y^{\rho}\,A(n,y) \to \exists f\,\forall n\,A(n,f(n)).$$

The realizer for $\mathbf{AC}^N$ is similar to the one for $\mathbf{DC}^N$, but technically simpler, so that the essential idea underlying the construction is more visible. Moreover we only need the following special case of relativized quantifier free bar induction:

Relativized quantifier free pointwise bar induction

$$\forall\alpha \in S\,\exists n\,P(\overline{\alpha}n) \wedge \forall s \in S\,\big(\forall x\,\big[S(x,|s|) \to P(s*x)\big] \to P(s)\big) \to P(\langle\,\rangle),$$

where $S(x,n)$ is arbitrary, $P(s)$ is quantifier free, and $\alpha \in S$, $s \in S$ are shorthands for $\forall n\,S(\alpha(n),n)$ and $\forall i < |s|\,S(s_i,i)$, respectively. The principles of relativized quantifier free bar induction respectively pointwise bar induction are similar to Luckhardt's general bar induction over species for quantifier free formulas, $(\mathrm{aBI})^{\rho}_{\mathrm{D}}$, respectively higher bar induction over species, $(\mathrm{hBI})^{\rho}_{\mathrm{D}}$ ([16], page 144).

The negative translation of **AC** is $\mathbf{AC}^N$

$\mathbf{AC}^N$ $$\forall n\,(\forall y\,(A(n,y)^N \to \perp) \to \perp) \to \forall f\,(\forall n\,A(n,f(n))^N \to \perp) \to \perp.$$

Following Spector [22] we reduce $\mathbf{AC}^N$ to the double negation shift

$$\textbf{DNS} \qquad \forall n\,((B(n) \to \bot) \to \bot) \to (\forall n\, B(n) \to \bot) \to \bot$$

observing that $\mathbf{AC} + \mathbf{DNS} \vdash_m \mathbf{AC}^N$, where **DNS** is used with the formula $B(n) :\equiv \exists y\, A(n,y)^N$ [2]. Therefore it suffices to show that this instance of **DNS** is realizable. The following lemma, whose proof is trivial, is necessary to see that the weak form (4) of modified bar recursion suffices to realize **AC** and **DC**.

Lemma 1. *Let B be a formula such that all of its atomic subformulas occur in negated form. Then there is a closed term H such that $\forall \vec{z}\, H\, \boldsymbol{mr}(\bot \to B)$ is provable (in minimal logic), where $\vec{z}$ are the free variables of B (it is important here that H is closed, i.p. does not depend on $\vec{z}$).*

Note that the formula $B(n) :\equiv \exists y\, A(n,y)^N$ to which we apply **DNS** is of the form specified in Lemma 1.

Theorem 3. *The double negation shift **DNS** for a formula $B(n)$ is realizable using the weak form (4) of modified bar recursion provided $B(n)$ is of the form specified in Lemma 1.*

Proof. In order to realize the formula

$$\forall n((B(n) \to \bot) \to \bot) \to (\forall n B(n) \to \bot) \to \bot$$

we assume we are given realizers

$$Y^{\rho^\omega \to o}\, \mathbf{mr}(\forall n B(n) \to \bot)$$
$$G^{\mathbb{N} \to (\rho \to o) \to o}\, \mathbf{mr}\, \forall n((B(n) \to \bot) \to \bot)$$

and try to build a realizer for $\bot$. Using weak modified bar recursion (4) we define

$$\Psi(s) = Y(s\, @\, \lambda n.H(G(|s|, \lambda x^\rho.\Psi(s * x))))$$

where $H^{o \to \rho}$ is a closed term such that $\forall n\, H\, \mathbf{mr}(\bot \to B(n))$ is provable, according to Lemma 1. We set

$$S(x,n) :\equiv x\, \mathbf{mr}\, B(n),$$
$$P(s) :\equiv \Psi(s)\, \mathbf{mr}\, \bot,$$

and, by quantifier free pointwise bar induction relativized to S, we show $P(\langle\,\rangle)$, i.e., $\Psi(\langle\,\rangle)\, \mathbf{mr}\, \bot$.

(i) $\forall \alpha \in S\; \exists n\; P(\overline{\alpha}n)$. Let $\alpha \in S$, i.e., $\alpha\, \mathbf{mr}\, \forall n B(n)$. Let n be the point of continuity of Y at α, according to the continuity axiom. By assumption on Y, we get $\forall \beta\; (Y(\overline{\alpha}n\, @\, \beta)\, \mathbf{mr}\, \bot)$, which implies $\Psi(\overline{\alpha}n)\, \mathbf{mr}\, \bot$.

(ii) $\forall s \in S(\forall x\, [S(x,|s|) \to P(s*x)] \to P(s))$. Let $s \in S$ be fixed. Suppose $\forall x\; [S(x,|s|) \to P(s * x)]$, i.e., $\forall x\; [x\, \mathbf{mr}\, B(|s|) \to \Psi(s * x)\, \mathbf{mr}\, \bot]$, in other words

$$\lambda x^\rho.\Psi(s * x)\, \mathbf{mr}(B(|s|) \to \bot).$$

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[2] The reduction is obvious because $\mathbf{AC}^N$ is equivalent in minimal logic to $\forall n\, \neg\neg\exists y\, A(n,y)^N \to \neg\neg\exists f\, \forall n\, A(n, f(n))^N$.

Using the assumption on G we obtain

$$G(|s|, \lambda x^{\rho}.\Psi(s * x)) \,\mathbf{mr}\, \bot,$$

and from that, setting $w :\overset{\rho}{\equiv} H(G(|s|, \lambda x^{\rho}.\Psi(s * x)))$, we obtain $w \,\mathbf{mr}\, B(n)$, for all n. Because $s \in S$ it follows that $s \mathbin{@} \lambda n.w \,\mathbf{mr}\, \forall n\, B(n)$ and therefore

$$Y(s \mathbin{@} \lambda n.w) \,\mathbf{mr}\, \bot.$$

Since $\Psi(s) = Y(s \mathbin{@} \lambda n.w)$ we have $P(s)$. ⊣

As explained above Theorem 3 yields

Corollary 1. *The negative translation of the countable axiom of choice, $\boldsymbol{AC}^N$ is realizable using the weak form* (4) *of modified bar recursion.*

3.3. Realizing $\mathbf{DC}^N$. With a similar but technically more involved construction we now prove

Theorem 4. *The negative translation of the axiom of dependent choice, $\boldsymbol{DC}^N$, is realizable using the weak form* (4) *of modified bar recursion.*

Proof. Let σ be the type of realizers of $A(n, x, y)^N$. Given x_0^{ρ} and realizers

$$G^{\mathbb{N}\to\rho\to(\rho\to\sigma\to o)\to o} \,\mathbf{mr}\, \forall n, x\, (\forall y\, (A(n, x, y)^N \to \bot) \to \bot),$$
$$Y^{\rho^{\omega}\to\sigma^{\omega}\to o} \,\mathbf{mr}\, \forall f\, (f(0) = x_0 \wedge \forall n\, A(n, f(n), f(n+1))^N \to \bot),$$

we have to construct a realizer of $\bot$. In the rest of this proof the variables β and t have the types $(\rho \times \sigma)^{\omega}$ and $(\rho \times \sigma)^{*}$ respectively. First we perform a trivial transformation on Y defining

$$\tilde{Y}^{(\rho\times\sigma)^{\omega}\to o}(\beta) :\equiv Y(x_0 * (\pi_0 \circ \beta), \pi_1 \circ \beta),$$

where π_0, π_1 are the left and right projection and $\circ$ is composition of functions. Using weak bar recursion (4) we now define

$$\Psi(t) = \tilde{Y}\big(t \mathbin{@} \lambda n.\pi\big(0^{\rho}, H\big(G\big(|t|, (x_0 * (\pi_0 \circ t))_{|t|}, \lambda y^{\rho} \lambda z^{\sigma}.\Psi(t * \pi(y, z))\big)\big)\big)\big),$$

where $\forall n, x, y\, H \,\mathbf{mr}(\bot \to A(n, x, y)^N)$ according to Lemma 1, $\pi(.,.)$ is pairing, and $\pi_0 \circ t :\equiv \langle \pi_0(t_0), \dots, \pi_0(t_{|t|-1}) \rangle$ (hence $(\pi_0 \circ t)_i = \pi_0(t_i)$ for $i < |t|$). We define predicates

$$S(t) :\equiv \forall i < |t|\ \ (\pi_1(t_i) \,\mathbf{mr}\, A(i, (\langle x_0 \rangle * (\pi_0 \circ t))_i, (\pi_0 \circ t)_i)^N)$$
$$P(t) :\equiv \Psi(t) \,\mathbf{mr}\, \bot.$$

We show $P(\langle\, \rangle)$ by quantifier free bar induction relativized to S. Obviously $S(\langle\, \rangle)$ holds.

(i) $\forall \beta \in S\, \exists n\, P(\overline{\beta} n)$. Let $\beta \in S$. Set $f^{\rho^{\omega}} :\equiv \langle x_0 \rangle * (\pi_0 \circ \beta)$ and $\gamma^{\sigma^{\omega}} :\equiv \pi_1 \circ \beta$. Then $f(0) = x_0$ and $\forall n\, \gamma(n) \,\mathbf{mr}\, A(n, f(n), f(n+1))^N$. Therefore $Y(f, \gamma) \,\mathbf{mr}\, \bot$. Let n be a point of continuity of $\tilde{Y}$ at β. Then

$$\Psi(\overline{\beta} n) = \tilde{Y}(\beta) = Y(f, \gamma)$$

and therefore $\Psi(\overline{\beta} n) \,\mathbf{mr}\, \bot$, i.e., $P(\overline{\beta} n)$.

(ii) $\forall t \in S\,(\forall q^{\rho\times\sigma}[S(t*q) \to P(t*q)] \to P(t))$. Let $t \in S$ where, say, $t = \langle \pi(x_1,z_0),\dots,\pi(x_n,z_{n-1})\rangle$. Assume further $\forall q\,[S(t*q) \to P(t*q)]$, i.e.,

$$\forall x_{n+1}, z_n \big[\forall i \le n\; z_i\,\mathbf{mr}\,A(i,x_i,x_{i+1})^N \to \\ \Psi\big(\langle \pi(x_1,z_0),\dots,\pi(x_{n+1},z_n)\rangle\big)\,\mathbf{mr}\,\bot\big].$$

Because $t \in S$ it follows that

$$\forall x_{n+1}, z_n \big[z_n\,\mathbf{mr}\,A(n,x_n,x_{n+1})^N \to \Psi\big(\langle \pi(x_1,z_0),\dots,\pi(x_{n+1},z_n)\rangle\big)\,\mathbf{mr}\,\bot\big]$$

i.e.,

$$\lambda y \lambda z.\Psi(t*\pi(y,z))\,\mathbf{mr}\,\forall y\,(A(n,x_n,y)^N \to \bot).$$

By the assumption on G it follows $G(n,x_n,\lambda y\lambda z.\Psi(t*\pi(y,z)))\,\mathbf{mr}\,\bot$. Hence, for $w \overset{\sigma}{:\equiv} H(G(n,x_n,\lambda y\lambda z.\Psi(t*\pi(y,z))))$, we have $\forall n,x,x'$ $(w\,\mathbf{mr}\,A(n,x,x')^N)$. Now we set $f^{\rho^\omega} :\equiv \langle x_0,x_1,\dots,x_n\rangle @ 0^\rho$ and $\gamma^{\sigma^\omega} :\equiv \langle z_0,\dots,z_{n-1}\rangle @ w$. Then $\forall n\,\gamma(n)\,\mathbf{mr}\,A(n,f(n),f(n+1))^N$ and therefore $Y(f,\gamma)\,\mathbf{mr}\,\bot$. But, because $x_n = (x_0 * (\pi_0 \circ t))_{|t|}$ we have

$$\Psi(t) = \tilde{Y}(t @ \pi(0^\rho,a)) = Y(f,\gamma).$$

Hence $\Psi(t)\,\mathbf{mr}\,\bot$, i.e., $P(t)$. ⊣

§4. **Bar recursion and the fan functional.** A functional $\mathrm{FAN}^{(\mathbb{N}^\omega\to o)\to\mathbb{N}}$ is called *fan functional* if it computes a modulus of uniform continuity for every continuous functional $Y^{\mathbb{N}^\omega\to o}$ restricted to infinite 0, 1-sequences, i.e., if FAN satisfies

$$\forall Y\,\forall \alpha,\beta \le \lambda x.1\big(\overline{\alpha}(\mathrm{FAN}(Y)) = \overline{\beta}(\mathrm{FAN}(Y)) \to Y\alpha \overset{o}{=} Y\beta\big).$$

A recursive algorithm for $\mathrm{FAN}(Y)$ that was given in [3] and [17] uses two procedures,

$$(5)\quad \Phi(s^{\mathbb{N}^*}, v^o) \overset{\mathbb{N}^\omega}{=} \\ s @ \big[\text{if } Y(\Phi(s*0,v)) \neq v \text{ then } \Phi(s*0,v) \text{ else } \Phi(s*1,v)\big]$$

$$(6)\quad \Psi(Y,s) \overset{\mathbb{N}}{=} \begin{cases} 0 & \text{if } Y(\alpha) = Y(s @ \lambda k.0), \\ & \text{where } \alpha = \Phi(s,\, Y(s @ \lambda k.0)), \\ 1 + \max\{\Psi(Y,s*0), \Psi(Y,s*1)\} & \text{otherwise.} \end{cases}$$

The first functional, $\Phi(s,v)$, returns an infinite path α having s as a prefix, such that $Y(s @ \alpha) \neq v$, if such a path exists, and returns s extended by $\lambda x.1$, otherwise, i.e., if Y is constant v on all paths extending s. The second functional, $\Psi(Y,s)$, returns the least point of uniform continuity for Y on all extension of s. Therefore, a fan functional can be defined as $\mathrm{FAN}(Y) :\equiv$

$\Psi(Y, \langle\,\rangle)$. A more formal proof that $\lambda Y.\Psi(Y, \langle\,\rangle)$ is indeed a fan functional can be found in [3] and [17] [3].

THEOREM 5. *The functional* FAN *can be defined using bar recursions* (3) *and* (2) *together.*

Before we give the proof of the theorem we prove two lemmas.

LEMMA 2. *Modified bar recursion* (3) *is equivalent to*

$$\Phi(s^{\rho^*}) \overset{o}{=} Y\big(s \,@\, H\big(s, \lambda t^{\rho^*} \lambda x^{\rho}.\Phi(s * t * x)\big)\big) \tag{7}$$

and also to

$$\Phi(s^{\rho^*}) \overset{\rho^\omega}{=} s \,@\, H\big(s, \lambda t^{\rho^*} \lambda x^{\rho}.Y^{\rho^\omega \to o}(\Phi(s * t * x))\big). \tag{8}$$

PROOF. Obviously equation (7) subsumes modified bar recursion. It is also easy to see that equations (7) and (8) are equivalent: Given Φ satisfying (7) we define $\Phi'(s) :\equiv s \,@\, H(s, \lambda t \lambda x.\Phi(s * t * x))$ which satisfies (8), provably by relativized bar induction. Conversely, if Φ' satisfies (8) then Φ defined by $\Phi(s) :\equiv Y(\Phi'(s))$ satisfies (7). Furthermore it is clear that we can replace the operation @ in each of the equations (3), (7) and (8) by $*$, i.e., we prefix with s instead of overwriting (see the definitions at the beginning of Section 2). Hence it suffices to show that we can define a functional Φ satisfying

$$\Phi(s^{\rho^*}) \overset{o}{=} Y\big(s * H\big(s, \lambda t^{\rho^*} \lambda x^{\rho}.\Phi(s * t * x)\big)\big) \tag{9}$$

by modified bar recursion. To this end we will use equation (3) (where @ is replaced by $*$) at type ρ^*. We define $\mathsf{freeze}\colon \rho^* \to \rho^{**}$ and $\mathsf{melt}\colon \rho^{**} \to \rho^*$ by $\mathsf{freeze}(\langle x_0, \dots, x_{n-1}\rangle) = \langle\langle x_0\rangle, \dots, \langle x_{n-1}\rangle\rangle$, $\mathsf{melt}(\langle s_0, \dots, s_{n-1}\rangle) = s_0 * \dots * s_{n-1}$, so that $\mathsf{melt}(\mathsf{freeze}(s)) = s$. Given $Y^{\rho^\omega \to o}$ and $H^{\rho^* \to (\rho^* \times \rho \to o) \to \rho^\omega}$ we define using modified bar recursion (3)

$$\Psi(q) = Y(\mathsf{melt}(q) * H(\mathsf{melt}(q), \lambda t \lambda x.\Psi(q * (t * x)))).$$

By relativized bar induction one easily proves

$$\forall q, q'\, (\mathsf{melt}(q) = \mathsf{melt}(q') \to \Psi(q) = \Psi(q')),$$

which implies, again by relativized bar induction, that Φ, defined by $\Phi(s) :\equiv \Psi(\mathsf{freeze}(s))$, satisfies (9). ⊣

LEMMA 3. *Kohlenbach's bar recursion* (2) *is equivalent to*

$$\Phi(s) \overset{\tau}{=} \begin{cases} G(s) & \text{if } Y(s \,@\, 0^{\rho^\omega}) \overset{o}{=} Y(s \,@\, J(s)), \\ H(s, \lambda x^{\rho}.\Phi(s * x)) & \text{otherwise,} \end{cases} \tag{10}$$

where the new parameter J is of type $\rho^* \to \rho^\omega$ and, as usual, $\Phi(s)$ is shorthand for the more accurate $\Phi(Y, G, H, J, s)$.

---

[3] The authors were informed that Robin Gandy knew a recursive definition of the fan functional in $\widehat{\mathcal{C}}$ already around 1973.

Proof. Our proof is based on the proof of Theorem 3.66 in [14]. The fact that (2) can be defined from (10) is trivial. To define (10) from (2) one uses the following trick. For s^{ρ^*}, $s + (\dot{-})k$ denotes pointwise addition (cut-off subtraction) of appropriate type, and $\kappa(n) :\equiv n$, $\kappa(f^{\rho\to\sigma}) :\equiv \kappa(f(0^\rho))$, $\kappa(z^{\rho\times\sigma}) :\equiv \kappa(\pi_0(z))$, so $\kappa(x^\rho + 2) > 1$ and $\kappa(n^\rho) = n$. Define

$$\eta(\beta^{\rho^\omega})(n) :\equiv \begin{cases} \beta(n)\dot{-}2 & \text{if } \kappa(\beta(n)) > 1, \\ J(\phi(\overline{\beta}n))(n) & \text{if } \kappa(\beta(n)) = 1, \\ 0 & \text{if } \kappa(\beta(n)) = 0, \end{cases}$$

where $\phi(s) :\equiv \langle s_0, \dots, s_{k-1} \rangle$ with $k < |s|$ minimal such that $\kappa(s_k) = 1$ (if $s = \langle\,\rangle$ then k is zero). Clearly

$$\eta((s+2) @ 0^{\rho^\omega}) = s @ 0^{\rho^\omega},$$
$$\eta((s+2) @ 1^{\rho^\omega}) = s @ J(s).$$

Now we can define using Kohlenbach's bar recursion (2)

$$\tilde{\Phi}(s) \overset{\tau}{=} \begin{cases} G(s\dot{-}2) & \text{if } Y(\eta(s @ 0^{\rho^\omega})) = Y(\eta(s @ 1^\rho)), \\ H(s\dot{-}2, \lambda x^\rho.\tilde{\Phi}(s * (x+2))) & \text{otherwise.} \end{cases}$$

Then clearly $\Phi(s) :\equiv \tilde{\Phi}(s+2)$ satisfies (10). ⊣

Proof of Theorem 5. We show that procedures Φ and Ψ satisfying the equations (5) and (6) respectively can be defined using equations (3) and (2).

For defining the functional $\Phi(s, v)$ we use equation (8) of Lemma 2.

$$\Phi(s, v) \overset{o^\omega}{=} s @ H(s, v, \lambda t \lambda x. Y(\Phi(s * t * x, v)))$$

where H is defined by course of value primitive recursion as

$$H(s, v, F)(n) \overset{o}{=} \begin{cases} s_n & \text{if } n < |s|, \\ 0 & \text{if } n \geq |s| \wedge F(c, 0) \neq v, \\ 1 & \text{if } n \geq |s| \wedge F(c, 0) = v, \end{cases}$$

with $c :\equiv \langle H(s, v, F)(|s|), \dots, H(s, v, F)(n-1) \rangle$. Clearly Φ satisfies equation (5) at all $n < |s|$. For $n \geq |s|$ we first observe that

$$\Phi(s, v)(n) \overset{o}{=} \begin{cases} 0 & \text{if } Y(\Phi(s * c_{s,n} * 0, v)) \neq v, \\ 1 & \text{if } Y(\Phi(s * c_{s,n} * 0, v)) = v, \end{cases}$$

where $c_{s,n} :\equiv \langle \Phi(s, v)(|s|), \dots, \Phi(s, v)(n-1) \rangle$. Now if $Y(\Phi(s * 0, v)) \neq v$ then $\Phi(s, v)(|s|) = 0$ and therefore $s * c_{s,n} = s * 0 * c_{s*0,n}$. Hence $\Phi(s, v)(n) = \Phi(s * 0, v)(n)$ as required by (5). The case $Y(\Phi(s * 0, v)) = v$ is similar.

One immediately sees that a functional Ψ satisfying (6) can be defined from an instance of equation (10) using the functional Φ above. ⊣

§5. Modified bar recursion and the model $\mathcal{M}$.

The model $\mathcal{M}$ $(= \bigcup \mathcal{M}_\rho)$ of strongly majorizable functionals (introduced in [7] as a variation of Howard's majorizable functionals [11]) and the strongly majorizability relation s-maj$_\rho \subseteq \mathcal{M}_\rho \times \mathcal{M}_\rho$ are defined simultaneously by induction on types as follows [4]

$$
\begin{aligned}
n \,\text{s-maj}_{\mathbb{N}}\, m &:\equiv n, m \in \mathbb{N} \wedge n \geq m, \qquad\qquad \mathcal{M}_{\mathbb{N}} :\equiv \mathbb{N},\\
F^* \,\text{s-maj}_{\rho\to\tau}\, F &:\equiv F^*,\ F \in \mathcal{M}_\rho \to \mathcal{M}_\tau \wedge\\
&\qquad \forall G^*,\ G \in \mathcal{M}_\rho \left[G^* \,\text{s-maj}_\rho\, G \to F^*G^* \,\text{s-maj}_\tau\, F^*G, FG\right],\\
\mathcal{M}_{\rho\to\tau} &:\equiv \left\{F \in \mathcal{M}_\rho \to \mathcal{M}_\tau \colon \exists F^* \in \mathcal{M}_\rho \to \mathcal{M}_\tau\ F^* \,\text{s-maj}_{\rho\to\tau}\, F\right\}.
\end{aligned}
$$

In the following we abbreviate s-maj$_\rho$ by maj$_\rho$ and by "majorizable" we always mean "strongly majorizable". We often omit the type in the relation maj$_\rho$. We shall sometimes write "$F\colon \rho \to \sigma$" for "$F \in \mathcal{M}_{\rho\to\sigma}$" (as opposed to "$F\colon \mathcal{M}_\rho \to \mathcal{M}_\sigma$" which just means that F is a set-theoretic function from $\mathcal{M}_\rho$ to $\mathcal{M}_\sigma$, i.e., $F \in \mathcal{M}_\rho \to \mathcal{M}_\sigma$).

In [14] it is shown that the scheme of bar recursion (2) is provably not primitive recursively definable from (1), since (1) yields a well defined functional in the model of (strongly) majorizable functionals $\mathcal{M}$ (cf. [7]) and (2) does not. Equation (1), however, can be primitive recursively defined from (2) (cf. [14]). In [5] it is shown that a functional

$$\Phi\colon \mathcal{M}_{\rho^\omega\to\mathbb{N}} \times \mathcal{M}_{\rho^*\times(\rho\to\mathbb{N})\to\rho^\omega} \times \mathcal{M}_{\rho^*} \to \mathcal{M}_{\mathbb{N}},$$

exists satisfying equation (3). We now show that any such Φ indeed lives in $\mathcal{M}$, i.e., we show that there is a functional Φ^* majorizing Φ. Recall that for continuous functionals Y of type $\rho^\omega \to \mathbb{N}$ it is the case that from some initial segment of α the value of $Y(\alpha)$ is determined. For the majorizable functionals this does not hold, but a "weak continuity" property does hold. It says that a bound on the value of $Y(\alpha)$ can be determined from an initial segment of α. We prove this result in Lemma 5. This turned out to be an important tool for proving the main theorem of this section. For the rest of this section all variables (unless stated otherwise) are assumed to range over the type structure $\mathcal{M}$. We first recall from [7] the following lemma:

LEMMA 4 ([7], 1.4, 1.5). *For* $F_0, \dots, F_n\colon \rho$ *we define* $\max^\rho\langle F_0, \dots, F_n\rangle\colon \rho$, *also written* $\max_{i\leq n}{}^{\rho} F_i\colon \rho$, *as*

$$
\begin{aligned}
\max_{i\leq n}{}^{\mathbb{N}}\, m_i &:\equiv \max\{m_0, \dots, m_n\},\\
\max_{i\leq n}{}^{\tau\to\rho} F_i &:\equiv \lambda x^\tau . \max_{i\leq n}{}^{\rho} F_i(x),
\end{aligned}
$$

and for α^{ρ^ω}, *define* $\alpha^+(n) :\equiv \max_{i\leq n}{}^{\rho}\alpha(i)$. *Then,*

$$\forall n(\alpha(n) \operatorname{maj} \beta(n)) \to \alpha^+ \operatorname{maj} \beta^+, \beta.$$

We also use pointwise addition in all types ρ, denoted $x +_\rho y$.

---

[4] For simplicity, we only consider the base type $\mathbb{N}$ and functional types. Later we extend the definition of majorizability for types ρ^*.

Lemma 5 (Weak continuity for $\mathcal{M}$). $\forall Y^{\rho^{\omega} \to \mathbb{N}}, \alpha \, \exists n^{\mathbb{N}} \, \forall \beta \in \overline{\alpha}n \, (Y(\beta) \leq n)$.

Proof. Let Y and α be fixed, $\alpha^* \operatorname{maj} \alpha$ and $Y^* \operatorname{maj} Y$. From the assumption

$$(*) \qquad \forall n \, \exists \beta \in \overline{\alpha}n(Y(\beta) > n)$$

we derive a contradiction. For any n, let β_n be the functional whose existence we are assuming in $(*)$. Let

$$\beta_n^*(i) :\equiv \begin{cases} 0^{\rho} & i < n, \\ \beta_n(i)^* & i \geq n, \end{cases}$$

where $\beta_n(i)^*$ denotes some majorant of $\beta_n(i)$. Having defined the functional β_n^* we note two of its properties,

(i) $\forall i < n(\beta_n^*(i) = 0^{\rho})$,
(ii) $(\alpha^* +_{\rho^{\omega}} \beta_n^*)^+ \operatorname{maj} \beta_n$ (by Lemma 4).

Consider the functional $\hat{\alpha}$ defined as $\hat{\alpha}(n) :\equiv \alpha^*(n) +_{\rho} \sum_{i \in \mathbb{N}} \beta_i^*(n)$. Since at each point n only finitely many β_i^* are non-zero, α^* is well defined. Let $Y^*(\hat{\alpha}^+) = l$. Note that $\hat{\alpha}^+ \operatorname{maj} \beta_i$, for all $i \in \mathbb{N}$, and from $(*)$ we should have $l < Y(\beta_l) \leq l$, a contradiction. ⊣

We extend, for convenience, the definition of majorizability to finite sequences, i.e., for sequences $s^*, s \in \mathcal{M}_{\rho}^*$ we define

$$s^* \operatorname{maj}_{\rho^*} s :\equiv |s^*| \geq |s| \wedge \forall i \leq j < |s^*| \big(s_j^* \operatorname{maj} s_i^* \wedge (i < |s| \to s_j^* \operatorname{maj} s_i)\big).$$

It is clear that for any sequence $s \in \mathcal{M}_{\rho}^*$ we can find an $s^* \in \mathcal{M}_{\rho}^*$ such that $s^* \operatorname{maj} s$. Therefore, we define $\mathcal{M}_{\rho^*}$ as $\mathcal{M}_{\rho}^*$. Majorizability for functionals involving the type ρ^* is extended accordingly, e.g., for $F^*, F \in \mathcal{M}_{\rho^*} \to \mathcal{M}_{\mathbb{N}}$

$$F^* \operatorname{maj}_{\rho^* \to \mathbb{N}} F :\equiv \forall s^*, s \in \mathcal{M}_{\rho^*} \, (s^* \operatorname{maj}_{\rho^*} s \to F^*(s^*) \geq F^*(s), F(s)).$$

Lemma 6. *Let* s^* *and* s *s.t.* $|s^*| = |s|$ *be fixed. If* $s^* \operatorname{maj} s$ *then*

$$\forall \beta \in s \exists \beta^* \in s^* \, (\beta^* \operatorname{maj} \beta).$$

Proof. Let s^*, s and $\beta \in s$ be fixed. Moreover, assume $|s^*| = |s| = n$ and $s^* \operatorname{maj} s$. We define β^* recursively as

$$\beta^*(i) :\equiv \begin{cases} s_i^* & \text{if } i < n, \\ \max^{\rho}(\overline{\beta^*}(i) * \beta(i)^*) & \text{otherwise}, \end{cases}$$

where $\beta(i)^*$ is some majorant of $\beta(i)$. First note that, for all i, $\beta^*(i) \operatorname{maj} \beta(i)$. We show that $\beta^* \operatorname{maj} \beta$. Let $k \geq i$.

If $k < n$ then $\beta^*(k) = s_k^* \operatorname{maj} s_i^* \operatorname{maj} s_i = \beta(i)$.

If $k \geq n$ then $\beta^*(k) = \max^{\rho}\{\max\limits_{j<k}{}^{\rho} \beta^*(j), \beta(k)^*\} \operatorname{maj} \beta^*(i) \operatorname{maj} \beta(i)$. ⊣

In the following we shall make use of two functionals Ω and Γ defined below. The functional Ω was first introduced in [14], 3.40.

Lemma 7 ([14], 3.41). *Define functionals* $\min^{\rho}$ (*from non-empty sets* $X \subseteq \mathcal{M}_{\rho}$ *to elements of* $\mathcal{M}_{\rho}$) *and* $\Omega : \mathcal{M}_{\rho} \to \mathcal{M}_{\rho}$ *as*

$$\begin{aligned} \min{}^{\mathbb{N}} XN &:\equiv \min X, \textit{ for } \emptyset \neq X \subseteq \mathbb{N}, \\ \min{}^{\rho\to\tau} X &:\equiv \lambda y^{\rho}. \min{}^{\tau}\{Fy : F \in X\}, \textit{ for } \emptyset \neq X \subseteq \mathcal{M}_{\rho\to\tau}, \\ \Omega(F) &:\equiv \min{}^{\rho}\{F^{*} : F^{*} \operatorname{maj} F\}. \end{aligned}$$

Then,

(i) *For all* F, $\Omega(F) \operatorname{maj} F$,
(ii) $\Omega \operatorname{maj} \Omega$. (*Therefore,* $\Omega \in \mathcal{M}$.)

Lemma 8. *Define* $\Gamma \colon \mathcal{M}_{\rho^{\omega}\to\mathbb{N}} \to (\mathcal{M}_{\rho^{\omega}} \to \mathcal{M}_{\mathbb{N}})$

$$\Gamma(Y)(\alpha) :\equiv \min n\, [\forall \beta \in \overline{\alpha}n(\Omega(Y)(\beta) \leq n)].$$

Then,

(i) $\Gamma(Y) \operatorname{maj} Y$ (*therefore* $\Gamma(Y) \in \mathcal{M}_{\rho^{\omega}\to\mathbb{N}}$),
(ii) $\Gamma(Y)$ *is continuous and* $\Gamma(Y)(\alpha)$ *is a point of continuity for* $\Gamma(Y)$ *at* α,
(iii) $\Gamma \operatorname{maj} \Gamma$ (*therefore,* $\Gamma \in \mathcal{M}$).

Proof. First of all, we note that, by Lemma 5, the functional Γ is well defined. By Lemma 7 (i), $\Omega(Y) \operatorname{maj} Y$.

(i) Let $\alpha^{*} \operatorname{maj} \alpha$. We have to show $\Gamma(Y)(\alpha^{*}) \geq \Gamma(Y)(\alpha), Y(\alpha)$. By the definition of $\Gamma(Y)$, and Lemma 7 (i), we have $\Gamma(Y)(\alpha^{*}) \geq \Omega(Y)(\alpha^{*}) \geq Y(\alpha)$. It is only left to show that $\Gamma(Y)(\alpha^{*}) \geq \Gamma(Y)(\alpha)$. Suppose that $n = \Gamma(Y)(\alpha^{*}) < \Gamma(Y)(\alpha) = m$. Note that there exists a $\beta \in \overline{\alpha}(m-1)$ such that $\Omega(Y)(\beta) \geq m$ (otherwise we get a contradiction to the minimality in the definition of $\Gamma(Y)$). But since $m > n$, by Lemma 6, there exists a $\beta^{*} \in \overline{\alpha^{*}}n$ such that $\beta^{*} \operatorname{maj} \beta$. Therefore, $\Omega(Y)(\beta^{*}) \leq n < m \leq \Omega(Y)(\beta)$. But by Lemma 7 (i) also $\Omega(Y)(\beta^{*}) \geq \Omega(Y)(\beta)$, a contradiction.

(ii) Let α be fixed and take $n = \Gamma(Y)(\alpha)$. Suppose there exists a $\beta \in \overline{\alpha}n$ such that $\Gamma(Y)(\beta) \neq n$. If $\Gamma(Y)(\beta) < n$ we get, since $\alpha \in \overline{\beta}n$, that $\Gamma(Y)(\alpha) < n$, a contradiction. Suppose $\Gamma(Y)(\beta) > n$. Since $\beta \in \overline{\alpha}n$ we have, $\forall \gamma \in \overline{\beta}n(\Omega(Y)(\gamma) \leq n)$, also a contradiction.

(iii) Assume $Y^{*} \operatorname{maj} Y$ and $\alpha^{*} \operatorname{maj} \alpha$. We show $\Gamma(Y^{*})(\alpha^{*}) \geq \Gamma(Y)(\alpha)$. By the self majorizability of $\Gamma(Y)$ we have $\Gamma(Y)(\alpha^{*}) \geq \Gamma(Y)(\alpha)$. We now show $\Gamma(Y^{*})(\alpha^{*}) \geq \Gamma(Y)(\alpha^{*})$. Let $n = \Gamma(Y^{*})(\alpha^{*})$ and suppose $m = \Gamma(Y)(\alpha^{*}) > n$. By the definition of $\Gamma(Y)$, there exists a $\beta \in \overline{\alpha^{*}}(m-1)$ s.t. $\Omega(Y)(\beta) \geq m$. But, since $m > n$, by Lemma 6, there exists a $\beta^{*} \in \overline{\alpha^{*}}n$ s.t. $\beta^{*} \operatorname{maj} \beta$, and by Lemma 7 (ii), $\Omega(Y^{*})(\beta^{*}) \geq m > n$, a contradiction. ⊣

Lemma 9. *Let* $Y^{*} \operatorname{maj} Y$ *of type* $\rho^{\omega} \to \mathbb{N}$ *and* α *of type* ρ^{ω} *be fixed. Set* $n = \Gamma(Y^{*})(\alpha)$. *If* $\overline{\alpha}n \operatorname{maj} s$ *and* $|s| = n$ *then for all sequences* β *we have*

$$\Gamma(Y^{*})(s \mathbin{@} \beta), \Gamma(Y)(s \mathbin{@} \beta), Y(s \mathbin{@} \beta) \leq n.$$

Proof. We prove just that $\Gamma(Y^{*})(s \mathbin{@} \beta) \leq n$. The other two cases follow similarly. Suppose there exists a β such that $n < \Gamma(Y^{*})(s \mathbin{@} \beta)$. Since

$\overline{\alpha}n \text{ maj } s$, by Lemma 6, there exists a β^* such that $\overline{\alpha}n * \beta^* \text{ maj } s \mathbin{@} \beta$. Therefore, by Lemma 8 (iii), we must have $n < \Gamma(Y^*)(\overline{\alpha}n * \beta^*)$. And by the fact that n is a point of continuity for $\Gamma(Y^*)$ on α we get $\Gamma(Y^*)(\overline{\alpha}n * \beta^*) = n$, a contradiction. $\dashv$

We extend the $(\cdot)^+$ operator of Lemma 4 to functionals $F \colon \mathcal{M}_{\rho^*} \to \mathcal{M}_{\mathbb{N}}$ by

$$F^+ :\equiv \lambda s. \max_{s' \preceq s} F(s'),$$

where $s' \preceq s :\equiv |s'| \leq |s| \wedge \forall i < |s'| \, (s'_i = s_i)$.

Lemma 10. *Let F and G be of type $\mathcal{M}_{\rho^*} \to \mathcal{M}_{\mathbb{N}}$. If*

$$\forall s^*, s \, [s^* \text{ maj } s \wedge |s^*| = |s| \to F(s^*) \geq F(s), G(s)]$$

then $F^+ \text{ maj } G^+, G$.

Proof. Let $s^* \text{ maj } s$ be fixed. For all prefixes t^* (of s^*) and t (of s) of the same length, by the assumption of the lemma, we have $F(t^*) \geq F(t), G(t)$. Therefore,

$$\max_{s' \preceq s^*} F(s') \geq \max_{s' \preceq s} F(s'), \max_{s' \preceq s} G(s').$$

Therefore, $F^+ \text{ maj } G^+, G$. $\dashv$

Theorem 6. *If Φ is a functional of type*

$$\mathcal{M}_{\rho^\omega \to \mathbb{N}} \times \mathcal{M}_{\rho^* \times (\rho \to \mathbb{N}) \to \rho^\omega} \times \mathcal{M}_{\rho^*} \to \mathcal{M}_{\mathbb{N}},$$

which for any given $Y, H, s \in \mathcal{M}$ (of appropriate types) satisfies equation (3), then $\Phi \in \mathcal{M}$.

Proof. Our proof is based on the proof of the main result of [7]. The idea is that, if Φ satisfies equation (3) then the functional

$$\Phi^* :\equiv \lambda Y, H. \big[\lambda s. \Phi(\hat{Y}, \hat{H}, s)\big]^+ \text{ maj } \Phi,$$

where

$$\hat{Y}(\alpha) :\equiv \Gamma(Y)(\alpha^+) \text{ and}$$
$$\hat{H}(s, F) :\equiv H(s, \lambda x. F(\{x\}_s)),$$

and $\{x\}_s$ abbreviates $\max^\rho(s * x)$. Let $Y^* \text{ maj } Y$ and $H^* \text{ maj } H$ be fixed. For the rest of the proof $s^* \text{ maj } s$ is a shorthand for $s^* \text{ maj } s \wedge |s^*| = |s|$, i.e., majorizability is only considered for sequences of equal length. The fact that $\Phi^* \text{ maj } \Phi$ follows from,

$$\big[\lambda s. \Phi(\hat{Y}^*, \hat{H}^*, s)\big]^+ \text{ maj } \big[\lambda s. \Phi(\hat{Y}, \hat{H}, s)\big]^+, \lambda s. \Phi(Y, H, s),$$

which follows, by Lemma 10, from $\forall s^* \; P(s^*)$ where

$$P(s^*) :\equiv \forall s \; \big[s^* \text{ maj } s \to \Phi(\hat{Y}^*, \hat{H}^*, s^*) \geq \Phi(\hat{Y}^*, \hat{H}^*, s), \Phi(\hat{Y}, \hat{H}, s), \Phi(Y, H, s)\big].$$

We prove $\forall s^* \; P(s^*)$ by bar induction:

(i) $\forall\alpha\exists n\, P(\overline{\alpha}n)$. Let α be fixed and $n :\equiv \hat{Y}^*(\alpha) = \Gamma(Y^*)(\alpha^+)$. If $\overline{\alpha}n$ does not majorize any sequence s we are done. Let s be such that $\overline{\alpha}n \operatorname{maj} s$. Note that $\overline{\alpha^+}n = \overline{(\overline{\alpha}n \mathbin{@} \beta)^+}n$, for all β. Therefore, by Lemma 8 (ii) and our assumption that Φ satisfies (3) we get $\Phi(\hat{Y}^*, \hat{H}^*, \overline{\alpha}n) = n$. Since $\overline{\alpha^+}n \operatorname{maj} \overline{(s \mathbin{@} \beta)^+}n$ (for all β), by Lemma 9, we have $n \geq \Phi(\hat{Y}^*, \hat{H}^*, s), \Phi(\hat{Y}, \hat{H}, s), \Phi(Y, H, s)$.

(ii) $\forall s^*(\forall x\; P(s^* * x) \rightarrow P(s^*))$. Let s^* be fixed. Assume that $\forall x\; P(s^* * x)$, i.e.,

$$\forall x, s\; \big[s^* * x \operatorname{maj} s \rightarrow \Phi(\hat{Y}^*, \hat{H}^*, s^* * x) \geq \\ \Phi(\hat{Y}^*, \hat{H}^*, s), \Phi(\hat{Y}, \hat{H}, s), \Phi(Y, H, s)\big].$$

We derive $P(s^*)$. Note that if s^* does not majorize any sequence we are again done. Assume s is such that $s^* \operatorname{maj} s$. If $x^* \operatorname{maj} x$ then (by $\forall x\; P(s^* * x)$),

$$\underbrace{\Phi(\hat{Y}^*, \hat{H}^*, s^* * \{x^*\}_{s^*})}_{\equiv:\, \Phi_1(\{x^*\}_{s^*})} \geq \\ \underbrace{\Phi(\hat{Y}^*, \hat{H}^*, s * \{x\}_s)}_{\equiv:\, \Phi_2(\{x\}_s)}, \underbrace{\Phi(\hat{Y}, \hat{H}, s * \{x\}_s)}_{\equiv:\, \Phi_3(\{x\}_s)}, \underbrace{\Phi(Y, H, s * x)}_{\equiv:\, \Phi_4(x)}.$$

and also $\Phi_1(\{x^*\}_{s^*}) \geq \Phi_1(\{x\}_{s^*})$, which implies

$$\lambda x.\Phi_1(\{x\}_{s^*}) \operatorname{maj} \lambda x.\Phi_2(\{x\}_s), \lambda x.\Phi_3(\{x\}_s), \lambda x.\Phi_4(x),$$

and by the definition of majorizability

$$\underbrace{H^*(s^*, \lambda x.\Phi_1(\{x\}_{s^*}))}_{\hat{H}^*(s^*, \lambda x.\Phi_1(x))} \operatorname{maj} \\ \underbrace{H^*(s, \lambda x.\Phi_2(\{x\}_s))}_{\hat{H}^*(s, \lambda x.\Phi_2(x))}, \underbrace{H(s, \lambda x.\Phi_3(\{x\}_s))}_{\hat{H}(s, \lambda x.\Phi_3(x))}, H(s, \lambda x.\Phi_4(x)),$$

which implies

$$\big(s^* \mathbin{@} \hat{H}^*(s^*, \lambda x.\Phi_1(x))\big)^+ \operatorname{maj} \big(s \mathbin{@} \hat{H}^*(s, \lambda x.\Phi_2(x))\big)^+, \\ \big(s \mathbin{@} \hat{H}(s, \lambda x.\Phi_3(x))\big)^+, \\ s \mathbin{@} H(s, \lambda x.\Phi_4(x)).$$

And finally, by Lemma 8 (i) and (iii),

$$(\Phi(\hat{Y}^*, \hat{H}^*, s^*) =)\; \hat{Y}^*(s^* \mathbin{@} \hat{H}^*(s^*, \lambda x.\Phi_1(x))) \geq \\ \hat{Y}^*(s \mathbin{@} \hat{H}^*(s, \lambda x.\Phi_2(x)))\; (= \Phi(\hat{Y}^*, \hat{H}^*, s)), \\ \hat{Y}(s \mathbin{@} \hat{H}(s, \lambda x.\Phi_3(x)))\; (= \Phi(\hat{Y}, \hat{H}, s)), \\ Y(s \mathbin{@} H(s, \lambda x.\Phi_4(x)))\; (= \Phi(Y, H, s)). \qquad \dashv$$

In [5] we show that there exists a functional

$$\Phi : \mathcal{M}_{\rho^\omega \to \mathbb{N}} \times \mathcal{M}_{\rho^* \times (\rho \to \mathbb{N}) \to \rho^\omega} \times \mathcal{M}_{\rho^*} \to \mathcal{M}_{\mathbb{N}}$$

which, for parameters Y, H, s in $\mathcal{M}$, satisfies equation (3). Therefore, by the theorem above, we obtain that $\mathcal{M}$ satisfies modified bar recursion.

§6. Conclusion. In this paper, we discussed modified bar recursion a variant of Spector's bar recursion that seems to be of some significance in proof theory and the theory and higher type recursion theory. Our main result was an abstract modified realizability interpretation (where realizability for falsity is uninterpreted) of the axioms of countable and dependent choice that can be used to extract programs from non-constructive proofs using these axioms. A similar result can be found in [2], however we claim that our solution is more accessible, since it builds on the well-known model of continuous functionals and the notion of modified realizability instead of an ad-hoc model and realizability as in [2]. It can be noted here that the weak form of modified bar recursion (4) used for the realization of dependent choice can be implemented quite efficiently by equipping the functional with an internal memory that records the value of $H(s, \lambda x.\Phi(s * x))$ and thus avoids its repeated computation. Such an optimization does not seem to be possible for the solution given in [2]. In order to make the realizability interpretation of dependent choice useful for program synthesis, it seems necessary to combine it with optimizations of the A-translation as development e.g., in [6] and [4]. To find out whether this is possible, will be a subject of further research.

Another important result was a definition of the fan functional using modified bar recursion and a version of bar recursion due to Kohlenbach, improving [3] and [17] where a PCF definition of the fan functional was given. In [21] this definition of the fan functional has been applied to give a purely functional algorithm for exact integration of real functions.

The paper concluded with some new results on the model $\mathcal{M}$ of strongly majorizable functionals, in particular, the fact that modified bar recursion exists in $\mathcal{M}$. In [5], further results on the relation between modified bar recursion and other bar recursive definitions can be found. One important result of [5] is that modified bar recursion defines Spector bar recursion primitive recursively and that the converse does not hold.

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DEPARTMENT OF COMPUTER SCIENCE
UNIVERSITY OF WALES SWANSEA
SINGLETON PARK
SWANSEA, SA2 8PP, UNITED KINGDOM
E-mail: u.berger@swansea.ac.uk
URL: http://www-compsci.swan.ac.uk/~csulrich/

BRICS
DEPARTMENT OF COMPUTER SCIENCE
UNIVERSITY OF AARHUS
AARHUS C, 8000, DENMARK
E-mail: pbo@brics.dk
URL: http://www.brics.dk/~pbo/