Modified Bar Recursion

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Abstract

Modified bar recursion has been used to give a realizability interpretation of the classical axioms of countable and dependent choice. In this paper we survey the main results concerning this scheme of bar recursion and its relations to well-known functionals. In particular we show that Spector's bar recursion can be defined primitive recursively via the scheme of modified bar recursion, but not the other way around.

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1 Introduction

In 62, Spector [21] introduced the scheme of bar recursion in order to extended Gödel's functional interpretation of Peano Arithmetic [10] to classical analysis (i.e. classical arithmetic $\mathbf{PA}^\omega$ plus the scheme of countable choice $\mathtt{AC}$). Although considered questionable from an intuitionistic point of view ([1], 6.6) there has been considerable interest in bar recursion, and several variants of this definition scheme and their interrelations have been studied by, e.g. Schwichtenberg [19], Bezem [7] and Kohlenbach [14]. In [4] a new form of bar recursion (so-called modified bar recursion) was introduced which allows

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to replace Spector's use of functional interpretation by that of a variation of modified realizability. This provides a different approach for extracting witness from proofs of Σ_1-formulas in full classical analysis. This interpretation was inspired by a paper by Berardi, Bezem and Coquand [2] who use a similar kind of recursion in order to interpret dependent choice.

In this paper we survey the main results obtained thus far concerning modified bar recursion. Some of the results are:

- The definition of the fan functional within **PCF** given in [3] and [17] can be derived from Kohlenbach's and our variant of bar recursion.
- Modified bar recursion of the lowest type is primitive recursively equivalent to the functional Γ.
- The type structure $\mathcal{M}$ of strongly majorizable functionals is a model of modified bar recursion. [1]
- Spector's bar recursion is primitive recursively definable in modified bar recursion.
- Modified bar recursion is not S1-S9 computable over the total continuous functionals.

2 Bar recursion in finite types

We work in a suitable extension of Heyting Arithmetic in finite types, $\mathbf{HA}^\omega$. For convenience we enrich the type system by the formation of finite sequences. So, our *Types* are $\mathbb{N}$, function types $\rho \to \sigma$, product types $\rho \times \sigma$, and finite sequences ρ^*. We set $\rho^\omega :\equiv \mathbb{N} \to \rho$. The *level* of a type is defined by $\mathsf{level}(\mathbb{N}) = 0$, $\mathsf{level}(\rho \times \sigma) = \max(\mathsf{level}(\rho), \mathsf{level}(\sigma))$, $\mathsf{level}(\rho^*) = \mathsf{level}(\rho)$, $\mathsf{level}(\rho \to \sigma) = \max(\mathsf{level}(\rho) + 1, \mathsf{level}(\sigma))$. By o we will denote an arbitrary but fixed type of level 0, and by ρ, τ, σ arbitrary. The terms of our version of $\mathbf{HA}^\omega$ are a suitable extension of the terms of Gödel's system T [10] in lambda calculus notation.

We use the variables $i, j, k, l, m, n \colon \mathbb{N}$; $s, t \colon \rho^*$; $\alpha, \beta \colon \rho^\omega$ unless the type of these variables is stated explicitly otherwise. Other letters will be used for different types in different contexts.

By $\stackrel{\tau}{=}$ we denote equality of type τ for which we assume the usual equality axioms. However, equality between functions is *not* assumed to be extensional. We also do *not* assume decidability for $\stackrel{\tau}{=}$, when $\mathsf{level}(\tau) > 0$. Type information

[1] Nonetheless, the realizers for the countable and dependent choice presented by the authors does not necessarily exist in $\mathcal{M}$ since continuity is assumed for the proof of the soundness of the interpretation.

will be frequently omitted, when it is irrelevant or inferable from the context. We let k^ρ denote the canonical lifting of a number $k \in \mathbb{N}$ to type ρ, e.g. $k^{\rho\to\sigma} :\equiv \lambda x^\rho.k^\sigma$. By an $\exists$-*formula* respectively $\forall\exists$-*formula* we mean a formula of the form $\exists y^\tau B$ respectively $\forall z^\sigma \exists y^\tau B$, where B is provably equivalent to an atomic formula and contains only decidable predicates.

We shall use the symbol $*$ as a concatenation operator. Hence, for objects x, finite sequences t and infinite sequences α, the results of $s * x$, $s * t$ and $s * \alpha$ are as expected. The length of a sequence s is denoted by $|s|$ and the k-th element of s by s_k. Moreover, we shall make use the following operations:

$$
\begin{aligned}
s @ \alpha &:\equiv \text{overwriting } \alpha \text{ with } s, \text{ i.e.} \\
&\quad (s @ \alpha)(k) = [\mathsf{if}\ k < |s|\ \mathsf{then}\ s_k\ \mathsf{else}\ \alpha(k)] \\
s @ x &:\equiv s @ \lambda k.x, \text{ i.e.} \\
&\quad (s @ x)(k) = [\mathsf{if}\ k < |s|\ \mathsf{then}\ s_k\ \mathsf{else}\ x] \\
\hat{s} &:\equiv s @ 0^\rho \\
\overline{\alpha}k &:\equiv \langle \alpha(0), \ldots, \alpha(k-1) \rangle \\
\overline{\alpha, k} &:\equiv \overline{\alpha}k @ 0^\rho
\end{aligned}
$$

Finally, $\beta \in s$ abbreviates $\overline{\beta}k = s$.

Definition 2.1 *The scheme of bar recursion introduced by Spector [21] consists of a family of functional symbols $\{\mathsf{SBR}_{\rho,\tau}\}_{\rho,\tau}$ with defining equations:*

$$
\mathsf{SBR}_{\rho,\tau}(Y, G, H, s) \stackrel{\tau}{=} \begin{cases} G(s) & \mathsf{if}\ Y(\hat{s}) \stackrel{o}{<} |s| \\ H(s, \lambda x^\rho.\mathsf{SBR}_{\rho,\tau}(Y, G, H, s * x)) & \mathsf{otherwise}, \end{cases} \tag{1}
$$

where $\stackrel{\tau}{=}$ denotes equality of type τ. In his thesis [14] Kohlenbach introduced the following kind of bar recursion which differs from Spector's only in the stopping condition:

$$
\mathsf{KBR}_{\rho,\tau}(Y, G, H, s) \stackrel{\tau}{=} \begin{cases} G(s) & \mathsf{if}\ Y(\hat{s}) \stackrel{o}{=} Y(s @ 1) \\ H(s, \lambda x^\rho.\mathsf{KBR}_{\rho,\tau}(Y, G, H, s * x)) & \mathsf{otherwise}. \end{cases} \tag{2}
$$

Finally, we define modified bar recursion at type *ρ:*

$$
\mathsf{MBR}_\rho(Y, H, s) \stackrel{o}{=} Y(s @ H(s, \lambda x^\rho.\mathsf{MBR}_\rho(Y, H, s * x))). \tag{3}
$$

In the three schemes of bar recursion the functional Y has type $\rho^\omega \to o$. The functional H has type $\rho^* \to (\rho \to \tau) \to \tau$ in the first two cases and type $\rho^* \to (\rho \to \tau) \to \rho^\omega$ in the last.

By SBR *we mean the family of symbols* $\{\mathsf{SBR}_{\rho,\tau}\}_{\rho,\tau}$ *together with their defining equations; similarly with* KBR *and* MBR. *We say a model* $\mathcal{S}$ *satisfies one of the respective variants of bar recursion if in* $\mathcal{S}$ *(for any given types* τ *and* ρ*) a functional exists satisfying* $\mathsf{SBR}_{\rho,\tau}$, $\mathsf{KBR}_{\rho,\tau}$, *or* MBR_{ρ} *for all possible values of* Y, G, H, s.

Recursive definitions similar to MBR occur in [2], and, in a slightly different form in [3] and [17] in connection with the fan functional (cf. Section 4).

The structures of primary interest to interpret bar recursion are the model $\mathcal{C}$ of *total continuous functionals* of Kleene [13] and Kreisel [15], the model $\widehat{\mathcal{C}}$ of *partial continuous functionals* of Scott [20] and Ershov [8], and the model $\mathcal{M}$ of (strongly) *majorizable functionals* introduced by Howard [11] and Bezem [6].

Theorem 2.1 *The models* $\mathcal{C}$ *and* $\widehat{\mathcal{C}}$ *satisfy all three variants of bar recursion.*

Proof. In the model $\widehat{\mathcal{C}}$ all three forms of bar recursion can simply be defined as the least fixed points of suitable continuous functionals. For $\mathcal{C}$ we use Ershov's result in [8] according to which the model $\mathcal{C}$ can be identified with the total elements of $\widehat{\mathcal{C}}$. Therefore it suffices to show that all three versions of bar recursion are total in $\widehat{\mathcal{C}}$. For Spector's version this has been shown by Ershov [8], and for the other versions similar argument apply. For example, in order to see that $\Phi(s)$ defined recursively by MBR is total for given total Y, H and s one uses bar induction on the bar

$$P(s) :\equiv Y(s \mathbin{@} \perp_\rho) \text{ is total}$$

where $\perp_\rho$ denotes the undefined element of type ρ. $P(s)$ is a bar because Y is continuous. $\Box$

Theorem 2.2 ([6], [14]) $\mathcal{M}$ *satisfies* SBR *but not* KBR.

We show in Section 6 that $\mathcal{M}$ satisfies MBR.

3 Modified bar recursion and the axiom of countable choice

The aim of this section is to show how modified bar recursion can be used to extract witnesses from proofs of Σ^0_1-formulas in classical arithmetic plus the axiom (scheme) of countable choice

$$\mathtt{AC} \quad : \quad \forall n\, \exists y\, A(n, y) \to \exists f\, \forall n\, A(n, f(n)).$$

Actually we will need only the following *weak modified bar recursion* which is a special case of MBR where H is constant:

$$\mathsf{WMBR}_\rho(Y,H,s) \overset{o}{=} Y(s @ \lambda k.H(s,\lambda x.\mathsf{WMBR}_\rho(Y,H,s*x))). \tag{4}$$

Note that in WMBR the returning type of H is ρ, i.e., the argument of Y consists of s followed by an infinite sequence with constant value of type ρ. We shall show in Section 3.3 though, that for $\rho \neq o$ the scheme of weak modified bar recursion is as strong as modified bar recursion.

Remark 3.1 *In [4] the authors have shown how the same idea for the realizer of* AC *can be extended to give a realizer for the dependent choice [12]*

$$\mathtt{DC} \quad : \quad \forall n\,\forall x\,\exists y\,A(n,x,y) \to \forall x \exists f\,(f(0) = x \wedge \forall n\,A(n,f(n),f(n+1))).$$

3.1 Witnesses from classical proofs

The method we use to extract witnesses from classical proofs is a combination of Gödel's negative translation (translation P^o in [16] page 42, see also [22]), the Dragalin/Friedman/Leivant trick, also called A-translation [24], and Kreisel's (formalized) modified realizability [23]. The method works in general for proofs in $\mathbf{PA}^\omega$, the classical variant of $\mathbf{HA}^\omega$. In order to extend it to $\mathbf{PA}^\omega$ plus extra axioms Γ (e.g. $\Gamma \equiv \mathtt{DC}$) one has to find realizers for Γ^N, the negative translation of Γ [2], where $\perp$ is replaced by an $\exists$-formula (regarding negation, $\neg C$, is defined by $C \to \perp$). However, it is more direct and technically simpler to follow [5] and combine the Dragalin/Friedman/Leivant trick and modified realizability: instead of replacing $\perp$ by a $\exists$-formula we slightly change the definition of modified realizability by regarding $y\,\mathbf{mr}\perp$ as an (uninterpreted) atomic formula. More formally we define

$$y^\tau\,\mathbf{mr}_\tau \perp :\equiv P_\perp(y),$$

where $P_\perp$ is a new unary predicate symbol and τ is the type of the witness to be extracted. Therefore, we have a modified realizability for each type τ, according to the type of the existential quantifier in the $\forall\exists$-formula we are realizing. The other clauses of modified realizability are as usual, e.g.

$$f\,\mathbf{mr}_\tau\,(A \to B) :\equiv \forall x\,(x\,\mathbf{mr}_\tau\,A \to fx\,\mathbf{mr}_\tau\,B).$$

In the following proposition Δ is an axiom system possibly containing $P_\perp$ and further constants, which has the following closure property: If $D \in \Delta$ and B

[2] The negative translation double-negates atomic formulas, replaces $\exists x$ by $\neg\forall x\neg$ and $A \vee B$ by $\neg(\neg A \wedge \neg B)$.

is a quantifier free formula with decidable predicates, then also the universal closure of $D[\lambda y^\tau.B/P_\perp]$ is in Δ, where $D[\lambda y^\tau.B/P_\perp]$ is obtained from D by replacing any occurrence of a formula $P_\perp(L)$ in D by $B[L/y]$.

Proposition 3.1 *Assume there is a vector Φ of closed terms such that*

$$\mathbf{HA}^\omega + \Delta \vdash \Phi\, \mathbf{mr}_\tau\, \Gamma^N.$$

Then from any proof

$$\mathbf{PA}^\omega + \Gamma \vdash \forall z^\sigma\, \exists y^\tau\, B(z,y),$$

where $\forall z^\sigma\, \exists y^\tau\, B(z,y)$ is a $\forall\exists$-formula in the language of $\mathbf{HA}^\omega$, one can extract a closed term $M^{\sigma\to\tau}$ such that

$$\mathbf{HA}^\omega + \Delta \vdash \forall z\, B(z, Mz).$$

Proof. The proof is folklore. The main steps are as follows. Assuming w.l.o.g. that $B(z,y)$ is atomic, we obtain from the hypothesis $\mathbf{PA}^\omega + \Gamma \vdash \forall z^\sigma\, \exists y^\tau\, B(z,y)$ via negative translation

$$\mathbf{HA}^\omega + \Gamma^N \vdash_m \forall y\,(B(z,y) \to \perp) \to \perp,$$

where $\vdash_m$ denotes derivability in minimal logic, i.e. ex-falso-quodlibet is not used. Now, soundness of modified realizability (which holds for our abstract version of modified realizability and minimal logic [5]), together with the assumption on Φ allows us to extract from this proof a closed term M such that

$$\mathbf{HA}^\omega + \Delta \vdash Mz\, \mathbf{mr}_\tau\, (\forall y\,(B(z,y) \to \perp) \to \perp)$$

i.e.

$$\mathbf{HA}^\omega + \Delta \vdash \forall f^{\tau\to\tau}\, (\forall y\,(B(z,y) \to P_\perp(fy)) \to P_\perp(Mzf)).$$

Replacing $P_\perp$ by $\lambda y.B(z,y)$ respectively, and instantiating f by the identity function it follows

$$\mathbf{HA}^\omega + \Delta \vdash \forall z\, B(z, Mz(\lambda y.y)).$$

$\square$

We will apply this proposition with $\tau :\equiv o$ (writing $\mathbf{mr}$ instead of $\mathbf{mr_o}$), $\Gamma :\equiv \mathtt{AC}$ (countable choice, see below), and an axiom system Δ consisting of the defining equations for modified bar recursion, where the defined functionals Φ are new constants, together with the axiom of continuity and the scheme of relativized quantifier free pointwise bar induction which are defined as follows:

Continuity : $\forall F^{\rho^\omega \to o}, \alpha\, \exists n\, \forall \beta\, (\overline{\alpha}n = \overline{\beta}n \to F(\alpha) = F(\beta))$, [3]

Relativized qf pointwise BI : $(A) \wedge (B) \to P(\langle\ \rangle)$,

where

$(A) \equiv \forall \alpha \in S\, \exists n\, P(\overline{\alpha}n)$,
$(B) \equiv \forall s \in S\, (\forall x\, [S(x, |s|) \to P(s * x)] \to P(s))$.

Here $S(x, n)$ is an arbitrary predicate, $P(s)$ a quantifier free predicate in the language of $\mathbf{HA}^\omega[P_\perp]$, and $\alpha \in S$ and $s \in S$ are shorthands for $\forall n\, S(\alpha(n), n)$ and $\forall i < |s| S(s_i, i)$, respectively. Clearly the condition on Δ in Proposition 3.1 is satisfied.

In order to make sure that realizers can indeed be used to compute witnesses one needs to know that,

- the axioms of $\mathbf{HA}^\omega + \Delta$ hold in a suitable model, and
- every closed term of type level 0 (e.g. of type $\mathbb{N}$) can be reduced to a numeral in an effective and provably correct way.

For the first point we can choose the model $\mathcal{C}$ of total continuous functionals, for instance. In [2] the second point is solved by building the notion of reducibility to normal form into the definition of realizability. In our case we solve this problem by applying Plotkin's adequacy result [18] as follows: each term in the language of $\mathbf{HA}^\omega$ plus the bar recursive constants can be naturally viewed as a term in the language **PCF** [18], by defining the bar recursors by means of the general fixed point combinator. In this way our term calculus also inherits **PCF**'s call-by-name reduction, i.e. if M is bar recursive and M reduces to M' then M' is bar recursive. Furthermore reduction is provably correct in our system, i.e. if M reduces to M' then $M = M'$ is provable. Now let M be a closed term of type $\mathbb{N}$. By Theorem 2.1, M has a total value, which is a natural number n, in the model of partial continuous functionals. Hence, by Plotkin's adequacy theorem M reduces to the numeral denoting n.

3.2 Realizing $\mathtt{AC}^N$

We now construct a realizer of the classical (i.e. negatively translated) axiom of countable choice,

$$\mathtt{AC}^N \quad \forall n\, (\forall y\, (A(n, y)^N \to \perp) \to \perp) \to \forall f\, (\forall n\, A(n, f(n))^N \to \perp) \to \perp.$$

[3] We call any n such that $\forall \beta\, (\overline{\alpha}n = \overline{\beta}n \to F(\alpha) = F(\beta))$ a point of continuity of F at α.

Following Spector [21] we reduce $\mathtt{AC}^N$ to the double negation shift

$$\mathtt{DNS} \qquad \forall n\,((B(n) \to \bot) \to \bot) \to (\forall n\, B(n) \to \bot) \to \bot$$

observing that $\mathtt{AC} + \mathtt{DNS} \vdash_m \mathtt{AC}^N$, where DNS is used with the formula $B(n) :\equiv (\exists y\, A(n,y))^N$. Therefore it suffices to show that this instance of DNS is realizable. The following lemma, whose proof is trivial, is necessary to see that the weak form of modified bar recursion WMBR suffices in the interpretation of AC and DC.

Lemma 3.1 *Let B be a formula such that all of its atomic sub-formulas occur in negated form. Then there is a closed term H such that $\forall \vec{z}\, H\, \mathbf{mr}\, (\bot \to B)$ is provable (in minimal logic), where $\vec{z}$ are the free variables of B (it is important here that H is closed, i.e. does not depend on $\vec{z}$).*

Note that the formula $B(n) :\equiv (\exists y\, A(n,y))^N$ to which we apply DNS is of the form specified in Lemma 3.1.

Theorem 3.1 *The double negation shift* DNS *for a formula $B(n)$ is realizable using* WMBR *provided $B(n)$ is of the form specified in Lemma 3.1.*

Proof. In order to realize the formula

$$\forall n((B(n) \to \bot) \to \bot) \to (\forall n B(n) \to \bot) \to \bot$$

we assume we are given realizers

$$Y^{\rho^\omega \to o}\, \mathbf{mr}\, (\forall n B(n) \to \bot)$$
$$G^{\mathbb{N} \to (\rho \to o) \to o}\, \mathbf{mr}\, \forall n((B(n) \to \bot) \to \bot)$$

and try to build a realizer for $\bot$. Using weak WMBR we define

$$\Psi(s) = Y(s \mathbin{@} \lambda k.H(G(|s|, \lambda x^\rho.\Psi(s * x))))$$

where $H^{o \to \rho}$ is a closed term such that $\forall k\, H\, \mathbf{mr}\, (\bot \to B(k))$ is provable, according to Lemma 3.1. We set

$$S(x,n) :\equiv x\, \mathbf{mr}\, B(n),$$
$$P(s) :\equiv \Psi(s)\, \mathbf{mr}\, \bot,$$

and, by quantifier free pointwise bar induction relativized to S, we show $P(\langle\,\rangle)$, i.e. $\Psi(\langle\,\rangle)\, \mathbf{mr}\, \bot$.

(A) $\forall \alpha \in S\, \exists n\, P(\overline{\alpha}n)$. Let $\alpha \in S$ be fixed, and let n be the point of continuity of Y at α, according to the continuity axiom. By assumptions on α and Y we get $\forall \beta\, Y(\overline{\alpha}n \mathbin{@} \beta)\, \mathbf{mr}\, \bot$, which implies $\Psi(\overline{\alpha}n)\, \mathbf{mr}\, \bot$.

(B) $\forall s \in S(\forall x\, [S(x, |s|) \to P(s * x)] \to P(s))$. Let $s \in S$ be fixed. Suppose $\forall x\, [S(x, |s|) \to P(s * x)]$, i.e. $\forall x\, [x\, \mathbf{mr}\, B(|s|) \to \Psi(s * x)\, \mathbf{mr}\, \bot]$, i.e.

$$\lambda x^{\rho}.\Psi(s * x)\, \mathbf{mr}\, (B(|s|) \to \bot).$$

Using the assumption on G we obtain

$$G(|s|, \lambda x^{\rho}.\Psi(s * x))\, \mathbf{mr}\, \bot,$$

and from that, setting $z :\equiv H(G(|s|, \lambda x^{\rho}.\Psi(s * x)))$, we obtain $z\, \mathbf{mr}\, B(k)$, for all k. Because $s \in S$ it follows that $s \,@\, \lambda k.z\, \mathbf{mr}\, \forall k\, B(k)$ and therefore

$$Y(s \,@\, \lambda k.z)\, \mathbf{mr}\, \bot.$$

Since $\Psi(s) = Y(s \,@\, \lambda k.z)$ we have $P(s)$. $\square$

As explained above, Theorem 3.1 yields

Corollary 3.1 *The negative translation of the countable axiom of choice,* $\mathtt{AC}^N$ *is realizable using* WMBR.

3.3 Remarks on weak modified bar recursion

As shown above, for obtaining Theorem 3.1 only a "weak" form of modified bar recursion is required. We can show, however, that for higher types WMBR is as strong as MBR.

Theorem 3.2 $\mathsf{WMBR}_{\rho^\omega}$ *defines* MBR_ρ *primitive recursively.*

Proof. We show that MBR_ρ can be defined using $\mathsf{WMBR}_{\rho^\omega}$. For an element x of type ρ and each natural number i we define the function $[x]_i : \rho^\omega$

$$[x]_i(k) :\equiv \begin{cases} x & \text{if } k = i \\ 0^\rho & \text{otherwise.} \end{cases}$$

For a sequence $s = \langle s_0, \ldots, s_n \rangle$ of type ρ^* we define

$$\mathsf{up}(s) :\equiv \langle [s_0]_0, \ldots, [s_n]_n \rangle,$$

(note that $\mathsf{up}(s)$ has type $(\rho^\omega)^*$) and for a sequence $s = \langle \alpha_0, \ldots, \alpha_n \rangle$ of type $(\rho^\omega)^*$ we define

$$\mathsf{down}(s) :\equiv \langle \alpha_0(0), \ldots, \alpha_n(n) \rangle,$$

which has type ρ^*. Observe that

(i) $\mathsf{down}(\mathsf{up}(s)) = s$,
(ii) $\mathsf{up}(s) * [x]_{|s|} = \mathsf{up}(s * x)$,
(iii) $\lambda k.(\mathsf{up}(s) \mathbin{@} \lambda k.\alpha)(k)(k) = s \mathbin{@} \alpha$.

Given functionals $Y : \rho^\omega \to o$ and $H : \rho^* \times (\rho \to o) \to \rho^\omega$ we define $\tilde{Y}$ of type $(\rho^\omega)^\omega \to o$ and $\tilde{H}$ of type $(\rho^\omega)^* \times (\rho^\omega \to o) \to \rho^\omega$ as follows

(iv) $\tilde{Y}(\alpha) :\equiv Y(\lambda k.\alpha(k)(k))$, (i.e. Y gets the diagonal of $\alpha : (\rho^\omega)^\omega$)
(v) $\tilde{H}(s, F) :\equiv H(\mathsf{down}(s), \lambda x^\rho.F([x]_{|s|}))$.

We show that $\mathsf{MBR}_\rho(Y, H, s)$ can be defined as $\mathsf{WMBR}_{\rho^\omega}(\tilde{Y}, \tilde{H}, \mathsf{up}(s))$, i.e.

$$\mathsf{MBR}_\rho(Y, H, s) :\equiv \mathsf{WMBR}_{\rho^\omega}(\tilde{Y}, \tilde{H}, \mathsf{up}(s)) =$$

$$\tilde{Y}(\mathsf{up}(s) \mathbin{@} \lambda k.\tilde{H}(\mathsf{up}(s), \lambda x.\mathsf{WMBR}_{\rho^\omega}(\tilde{Y}, \tilde{H}, \mathsf{up}(s) * x))) \overset{(v),(i)}{=}$$

$$\tilde{Y}(\mathsf{up}(s) \mathbin{@} \lambda k.H(s, \lambda x.\mathsf{WMBR}_{\rho^\omega}(\tilde{Y}, \tilde{H}, \mathsf{up}(s) * [x]_{|s|}))) \overset{(ii)}{=}$$

$$\tilde{Y}(\mathsf{up}(s) \mathbin{@} \lambda k.H(s, \lambda x.\mathsf{WMBR}_{\rho^\omega}(\tilde{Y}, \tilde{H}, \mathsf{up}(s * x)))) \overset{(iv),(iii)}{=}$$

$$Y(s \mathbin{@} H(s, \lambda x.\mathsf{WMBR}_{\rho^\omega}(\tilde{Y}, \tilde{H}, \mathsf{up}(s * x)))) =$$

$$Y(s \mathbin{@} H(s, \lambda x.\mathsf{MBR}_\rho(Y, H, s * x))).$$

□

Note that for $\rho \neq \mathbb{N}$ we get WMBR_ρ defines MBR_ρ.

Corollary 3.2 WMBR *defines* MBR *primitive recursively.*

Note, however, that the results above does not imply that $\mathsf{WMBR}_\mathbb{N}$ already defines $\mathsf{MBR}_\mathbb{N}$ primitive recursively. That is still an open question. Nonetheless, we have the following results, which follows from the proof of Theorem 3.2.

Corollary 3.3 *For any functional Y given with its modulus of continuity ω, the functional $\lambda H, s.\mathsf{MBR}_\mathbb{N}(Y, H, s)$ is primitive recursively definable in $\lambda H, s.\mathsf{WMBR}_\mathbb{N}(\tilde{Y}, H, s)$, where $\tilde{Y}$ is primitive recursively definable in Y and ω.*

4 The fan functional

A functional $\Psi^{(\mathbb{N}^\omega \to o) \to \mathbb{N}}$ is called *fan functional* if it computes a modulus of uniform continuity for every functional $Y^{\mathbb{N}^\omega \to o}$ restricted to infinite $0, 1$-sequences, i.e.

$$\texttt{FAN}(\Psi) \;:\equiv\; \forall Y; \alpha, \beta \leq \lambda x.1(\overline{\alpha}(\Psi(Y)) = \overline{\beta}(\Psi(Y)) \to Y\alpha \overset{o}{=} Y\beta)$$

(note that $\rho = \mathbb{N}$ here). A recursive fan functional which is presented [3] and [17] uses two procedures, (we shall often omit the parameter Y in the functionals Φ and Ψ for better readability)

$$\Phi(s, v) = s \mathbin{@} [\mathsf{if}\ Y(\Phi(s * 0, v)) \neq v\ \mathsf{then}\ \Phi(s * 0, v)\ \mathsf{else}\ \Phi(s * 1, v)] \tag{5}$$

$$\Psi(s) \overset{\mathbb{N}}{=} \begin{cases} 0 & \mathsf{if}\ Y(\alpha) = Y(s \mathbin{@} 0), \\ & \text{ where } \alpha = \Phi(s, Y(s \mathbin{@} 0)) \\ 1 + \max\{\Psi(s * 0), \Psi(s * 1)\} & \mathsf{otherwise.} \end{cases} \tag{6}$$

The first functional, $\Phi(Y, s, v)$, returns an infinite path α having s as a prefix, such that $Y(s \mathbin{@} \alpha) \neq v$, if such a path exists, and returns s extended by $\lambda x.1$, otherwise, i.e. if Y is constant v on all paths extending s. The second functional, $\Psi(Y, s)$, returns the maximum point of continuity for Y on all extension of s. We show that the functional $\lambda Y.\Psi(Y, \langle\ \rangle)$ is a fan functional and is primitive recursive in MBR and KBR. The proof of the following two lemmas (which can be formalized in $\mathbf{HA}^\omega$+ rBI) can be found in [4].

Lemma 4.1 MBR *is equivalent to*

$$\Phi(s^{\rho^*}) \overset{\rho^\omega}{=} s \mathbin{@} H(s, \lambda t^{\rho^*} \lambda x^\rho . Y^{\rho^\omega \to o}(\Phi(s * t * x))). \tag{7}$$

Lemma 4.2 KBR *is equivalent to,*

$$\Phi(s) \overset{\tau}{=} \begin{cases} G(s) & \mathsf{if}\ Y(s \mathbin{@} 0^\rho) \overset{o}{=} Y(s \mathbin{@} J(s)) \\ H(s, \lambda x^\rho . \Phi(s * x)) & \mathsf{otherwise,} \end{cases} \tag{8}$$

where the new parameter J is of type $\rho^* \to \rho^\omega$ and, as usual, $\Phi(s)$ is shorthand for the more accurate $\Phi(Y, G, H, J, s)$.

Theorem 4.1 *The functional $\lambda Y.\Psi(Y, \langle\ \rangle)$ is primitive recursively definable in* MBR *and* KBR *together.*

Proof. We show that procedures Φ and Ψ satisfying the equations (5) and (6) respectively can be defined using MBR and KBR.

For defining the functional $\Phi(s, v)$ we use equation (7) of Lemma 4.1.

$$\Phi(s, v) \overset{\mathbb{N}^\omega}{=} s \mathbin{@} H(s, v, \lambda t \lambda x . Y(\Phi(s * t * x)))$$

where H is defined by course of value primitive recursion as

$$H(s,v,F)(n) \stackrel{o}{=} \begin{cases} s_n \text{ if } n < |s| \\ 0 \quad \text{if } n \geq |s| \wedge F(c,0) \neq v \\ 1 \quad \text{if } n \geq |s| \wedge F(c,0) = v, \end{cases}$$

with $c :\equiv \langle H(s,v,F)(|s|), \dots, H(s,v,F)(n-1) \rangle$. Clearly Φ satisfies equation (5) at all $n \leq |s|$. For $n > |s|$ we first observe that

$$\Phi(s,v)(n) \stackrel{o}{=} \begin{cases} 0 \text{ if } Y(\Phi(s * c_{s,n} * 0) \neq v \\ 1 \text{ if } Y(\Phi(s * c_{s,n} * 0) = v, \end{cases}$$

where $c_{s,n} :\equiv \langle \Phi(s,v)(|s|), \dots, \Phi(s,v)(n-1) \rangle$. Now if $Y(\Phi(s * 0)) \neq v$ then $\Phi(s,v)(|s|) = 0$ and therefore $s * c_{s,n} = s * 0 * c_{s*0,n}$. Hence $\Phi(s)(n) = \Phi(s*0)(n)$ as required by (5). The case $Y(\Phi(s * 0)) = v$ is similar.

One immediately sees that a functional Ψ satisfying (6) can be defined from an instance of equation (8) using the functional Φ above. □

We next prove that $\lambda Y.\Psi(Y, \langle\, \rangle)$ is indeed a fan functional. First we need the following lemma.

Lemma 4.3 *For any functional $Y : \mathbb{N}^\omega \to \mathbb{N}$, sequence $s \in \{0,1\}^*$ and number v, if there exists a β such that $Y(s * \beta) \neq v$ then $Y(\Phi(Y,s,v)) \neq v$.*

Proof. Let Y, s and v be fixed. We show by bar induction on 0,1-sequences that for all binary sequences t extending s

$$P(t) :\equiv \exists \beta (Y(t * \beta) \neq v) \to Y(\Phi(Y,t,v)) \neq v.$$

(A) $\forall \alpha \exists n P(\overline{\alpha} n)$. For a fixed α let n be the point of continuity of Y on α. If there exists a β such that $Y(\overline{\alpha} n * \beta) \neq v$, since n is a point of continuity of Y on α, we have $Y(\Phi(Y, \overline{\alpha} n, v)) \neq v$.

(B) $\forall t (P(t * 0) \wedge P(t * 1) \to P(t))$. Fix t and assume (i) $P(t * 0)$ and (ii) $P(t * 1)$. Moreover, assume that $\exists \beta (Y(t * \beta) \neq v)$. If $\beta(0) = 0$, by (i), $Y(\Phi(Y, t * 0, v)) \neq v$. Hence, by Equation (5), $\Phi(Y,t,v) = s \mathbin{@} \Phi(Y, t * 0, v)$. Similarly when $\beta(0) = 1$. □

Theorem 4.2 *$\lambda Y.\Psi(Y, \langle\, \rangle)$ is a fan functional. More precisely,*

$$\mathbf{HA}^\omega + \mathsf{MBR} + \mathsf{KBR} \vdash \mathtt{FAN}(\lambda Y.\Psi(Y, \langle\, \rangle)).$$

Proof. We show by induction on the point of uniform continuity k of the functional Y that $\Psi(Y, \langle\, \rangle) = k$. If Y has point of uniform continuity 0 it is clear that, for any function α, $Y(\alpha) = Y(\lambda x.0)$ and $\Psi(Y, \langle\, \rangle) = 0$, by Equation

(6). Assume the point of uniform continuity of Y is $k+1$. That means that the two functionals $\lambda\alpha.Y(0*\alpha)$ and $\lambda\alpha.Y(1*\alpha)$ have point of uniform continuity at most k, and moreover, at least one of them has point of uniform continuity exactly k. We just need to show that (let $v :\equiv Y(s \mathbin{@} 0)$) $Y(\Phi(s,v)) \neq v$, but this follows from Lemma 4.3. $\Box$

Remark 4.1 *Kohlenbach [14] has shown that* KBR *is primitive recursively definable in* SBR *and* $\hat{\mu}$ *(where* $\hat{\mu}$ *is the functional defined as,*

$$\hat{\mu}(Y, \alpha^{\rho^\omega}) :\equiv \min n\ [Y(\overline{\alpha}n \mathbin{@} 0) = Y(\overline{\alpha}n \mathbin{@} 1)].$$

Since in Section 7 we show that MBR *defines* SBR *primitive recursively, the theorem above can be strengthened by replacing* $\mathbf{HA}^\omega + \mathsf{MBR} + \mathsf{KBR} + \mathrm{rBI}$ *with* $\mathbf{HA}^\omega + \mathsf{MBR} + \hat{\mu} + \mathrm{rBI}$.

5 The functional Γ

The functional Γ (introduced in [9]) is defined as

$$\Gamma(Y, s) \stackrel{\mathbb{N}}{=} Y(s * 0 * \lambda n^{\mathbb{N}}.\Gamma(Y, s * (n+1))). \tag{9}$$

It is easy to see that equation (9), in the model of continuous functionals, specifies a unique functional. Gandy and Hyland's purpose for defining the functional Γ was to show that there exists a functional having a recursive associate but not being S1-S9 computable in the total functionals, even with the fan functional as an oracle. In the following we show that modified bar recursion of the lowest type is primitive recursively equivalent to the functional Γ. Hence, one can view MBR as an extension of the functional Γ to higher types.

Theorem 5.1 *The functional* Γ *is primitive recursively equivalent to* $\mathsf{MBR}_{\mathbb{N}}$*.*

Proof. It is easy to see that $\mathsf{MBR}_{\mathbb{N}}$ defines the functional Γ. For the other direction the intuition is as follows. We first use Γ to compute the values of $\mathsf{MBR}_{\mathbb{N}}(Y, H, s) + 1$. The advantage of doing this is that, if the sequence s contains only positive numbers, the functional Y will get a sequence α containing only one zero, namely the one introduced by the functional Γ. Say α has the form $s * 0 * \beta$. Therefore, it is easy to transform α (primitive recursively) into a the sequence $s * H(s, \beta)$. Moreover, since β is exactly $\lambda x.\mathsf{MBR}_{\mathbb{N}}(s * x)$ we are done. Now we give the formal proof. Define

(i) $\mathsf{MBR}_{\mathbb{N}}(Y, H, s) :\equiv \Gamma(\tilde{Y}, \tilde{s}) - 1$, where

$\tilde{s} :\equiv \langle s_0 + 1, \ldots, s_{|s|-1} + 1 \rangle$

and (ii) $\tilde{Y}(\alpha) :\equiv Y(\hat{\alpha}) + 1$. Moreover, let c be a shorthand for

$\mu m \leq k\, [\alpha(m) = 0]$,

and

$$\hat{\alpha}(k) :\equiv \begin{cases} \alpha(k) - 1 & \text{if } \alpha(c) \neq 0 \\ H(\overline{\alpha}c, \lambda n.(\alpha(n + c + 1) - 1))(k - c) & \text{otherwise.} \end{cases}$$

We only have to notice that if α has the form $\tilde{s} * 0 * \beta$ then

(iii) $\hat{\alpha} = s * H(s, \lambda n.(\beta(n) - 1))$.

We then have the following:

$$\begin{aligned} \mathsf{MBR}_{\mathbb{N}}(Y, H, s) &\overset{(i)}{=} \Gamma(\tilde{Y}, \tilde{s}) - 1 \\ &\overset{(9)}{=} \tilde{Y}(\tilde{s} * 0 * \lambda n.\Gamma(\tilde{Y}, \tilde{s} * (n + 1))) - 1 \\ &= \tilde{Y}(\tilde{s} * 0 * \lambda n.\Gamma(\tilde{Y}, \widetilde{s * n})) - 1 \\ &\overset{(ii)}{=} Y(\tilde{s} * 0 * \widehat{\lambda n.\Gamma(\tilde{Y}}, \widetilde{s * n})) \\ &\overset{(iii)}{=} Y(s * H(s, \lambda n.(\Gamma(\tilde{Y}, \widetilde{s * n}) - 1))) \\ &\overset{(i)}{=} Y(s * H(s, \lambda n.\mathsf{MBR}_{\mathbb{N}}(Y, H, s * n))) \end{aligned}$$

□

6 The model $\mathcal{M}$ of strongly majorizable functionals

The model $\mathcal{M}$ $(= \bigcup \mathcal{M}_\rho)$ of strongly majorizable functionals (introduced in [6] as a variation of Howard's majorizable functionals [11]) and the strongly majorizability relation $\text{s-maj}_\rho \subseteq \mathcal{M}_\rho \times \mathcal{M}_\rho$ are defined by induction on types as follows:

$$\begin{aligned} &n \text{ s-maj}_o\, m :\equiv n \geq m, \\ &\mathcal{M}_o :\equiv \mathbb{N}, \\ &F^* \text{ s-maj}_{\rho\to\tau}\, F :\equiv F^*, F \in \mathcal{M}_\rho \to \mathcal{M}_\tau \wedge \\ &\qquad\qquad \forall G^*, G \in \mathcal{M}_\rho\, [G^* \text{ s-maj}_\rho\, G \to F^*G^* \text{ s-maj}_\tau\, F^*G, FG], \\ &\mathcal{M}_{\rho\to\tau} :\equiv \{F \in \mathcal{M}_\rho \to \mathcal{M}_\tau : \exists F^* \in \mathcal{M}_\rho \to \mathcal{M}_\tau\; F^* \text{ s-maj}_{\rho\to\tau}\, F\}. \end{aligned}$$

In the following we abbreviate s-maj_ρ by maj_ρ and by "majorizable" always mean "strongly majorizable". We often omit the type in the relation maj_ρ.

In [14] it is shown that KBR is provably not primitive recursively definable from SBR, since SBR yields a well defined functional in the model of (strongly) majorizable functionals $\mathcal{M}$ (cf. [6]) and KBR does not (in the following we will by "majorizable" always mean "strongly majorizable"). SBR, however, can be primitive recursively defined from KBR (cf. [14]).[4] In this section we show that a functional satisfying MBR exists in $\mathcal{M}$. We first show that there exists a functional [5]

$$\Phi : \mathcal{M}_{\rho^\omega \to o} \times \mathcal{M}_{\rho^* \times (\rho \to o) \to \rho^\omega} \times \mathcal{M}_{\rho^*} \to \mathcal{M}_o$$

satisfying MBR, then we show that any such Φ has a majorant and therefore belongs to $\mathcal{M}$.

Most of our results in this section rely on Lemma 6.2 which can be viewed as a *weak continuity property* of functionals Y (of type $\rho^\omega \to o$) in $\mathcal{M}$. It says that a bound on the value of $Y(\alpha)$ can be determined from an initial segment of α. For the rest of this section all variables (unless stated otherwise) are assumed to range over the type structure $\mathcal{M}$.

Lemma 6.1 ([6], 1.4, 1.5) *Let* $\max^\rho$ *be inductively defined as,*

$$\max_{i\le n}{}^{o} m_i :\equiv \max\{m_0, \ldots, m_n\},$$
$$\max_{i\le n}{}^{\tau\to\rho} X_i :\equiv \lambda Y^\tau . \max_{i\le n}{}^{\rho} X_i\, Y,$$

and for α^{ρ^ω}, *define* $\alpha^+(n) :\equiv \max_{i\le n}{}^{\rho}\alpha(i)$. *Then,*

$$\forall n(\alpha(n) \text{ maj } \beta(n)) \to \alpha^+ \text{ maj } \beta^+, \beta.$$

We also use addition in all types, which is done pointwise, e.g. if x, y are of type $\tau \to \rho$ then $x +_{\tau\to\rho} y :\equiv \lambda z^\tau\, (x(z) +_\rho y(z))$.

Lemma 6.2 (Weak continuity for $\mathcal{M}$) $\forall Y^{\rho^\omega \to \mathbb{N}}, \alpha\ \exists n^{\mathbb{N}}\ \forall \beta \in \overline{\alpha}n\, (Y(\beta) \le n)$.

Proof. Let Y and α be fixed, α^* maj α and Y^* maj Y. From the assumption

$(a)\ \forall n\ \exists \beta \in \overline{\alpha}n(Y(\beta) > n)$

[4] For the rest of the paper "s_1 is primitive recursively definable in s_2" should be understood as "there exists a closed term t such that $\mathbf{E\text{-}HA}^\omega \vdash s_1 = t(s_2)$"

[5] By $\mathcal{M}_\rho \to \mathcal{M}_\tau$ we mean an arbitrary function from $\mathcal{M}_\rho$ to $\mathcal{M}_\tau$. By $\mathcal{M}_{\rho\to\tau}$ we mean a functional from $\mathcal{M}_\rho$ to $\mathcal{M}_\tau$ which belongs to $\mathcal{M}$.

we derive a contradiction. For any n, let β_n be the function whose existence we are assuming in (a). Let

$$\beta_n^*(i) :\equiv \begin{cases} 0^\rho & i < n \\ [\beta_n(i)]^* & i \geq n, \end{cases}$$

where $[\beta_n(i)]^*$ denotes some majorant of $\beta_n(i)$. Having defined the functional β_n^* we note two of its properties,

i) $\forall i < n(\beta_n^*(i) = 0^\rho)$,

ii) $(\alpha^* +_\rho \beta_n^*)^+ \text{ maj } \beta_n$. (by Lemma 6.1)

Consider the functional $\hat{\alpha}$ defined as, $\hat{\alpha}(n) :\equiv \alpha^*(n) +_\rho \sum_{i \in \mathbb{N}} \beta_i^*(n)$. Since at each point n only finitely many β_i^* are non-zero, α^* is well defined. Let $Y^*(\hat{\alpha}^+) = m$. Note that $\hat{\alpha}^+$ maj β_i, for all $i \in \mathbb{N}$, and from (a) we should have $m < Y(\beta_m) \leq m$, a contradiction. $\Box$

We extend, for convenience, the definition of majorability for finite sequences. For s^*, s of type ρ^* we define

$$s^* \text{ maj }^{\rho^*} s :\equiv (s^* \mathbin{@} \lambda i.s^*_{|s^*|-1}) \text{ maj }^{\rho^\omega} (s \mathbin{@} \lambda i.s_{|s|-1}),$$

where $s_{|s|-1} = 0^\rho$ if $s = \langle\ \rangle$.

6.1 Finding $\Phi \in \mathcal{M}_{\rho^\omega \to o} \times \mathcal{M}_{\rho^* \times (\rho \to o) \to \rho^\omega} \times \mathcal{M}_{\rho^*} \to \mathcal{M}_o$ *satisfying* MBR

For any type ρ, the elements s of $\mathcal{M}_{\rho^*}$ (finite sequences of elements in ρ) can be viewed as nodes of an infinite tree which we call T. The infinite paths of T are the elements of $\mathcal{M}_{\rho^\omega}$ (which is just $\mathcal{M}_\rho^\omega$ as shown in [6]). For fixed Y and H, the functional Φ we are looking for should assign values to the nodes of T according to MBR. For each node s the set of nodes s' extending s is denoted by B_s.

Let $Y, H \in \mathcal{M}$ be fixed. We show that at each infinite path α there exists a point n such that a functional $\Phi_{\alpha,n} : \mathcal{M}_{\rho^*} \to \mathcal{M}_o$ can be defined satisfying MBR for all $s \in B_{\overline{\alpha}n}$. Then, by bar induction, a functional Φ can be defined for all nodes of T.

Let $\alpha \in \mathcal{M}_\rho^\omega$ be fixed, n the number whose existence is stated in Lemma 6.2, and $K :\equiv \{0, 1, \dots, n\}$. We show how to define a functional $\Phi_{\alpha,n}(s)$ such that, for $s \in B_{\overline{\alpha}n}$, equation

$$\Phi_{\alpha,n}(s) = Y(s @ H(s, \lambda x.\Phi_{\alpha,n}(s * x))$$

holds. Here we note that, for $s \in B_{\overline{\alpha}n}$, by Lemma 6.2, $\Phi_{\alpha,n}(s)$ must belong to K. Therefore, for those $s \in B_{\overline{\alpha}n}$, what we have is an instance of the more general equation,

$$\Psi(s) = G(s, \lambda x.\Psi(s * x)), \tag{10}$$

where $Img(G) \subseteq K$. To see that modified bar recursion becomes an instance of (10), let

$$G(s, F) :\equiv Y(\overline{\alpha}n * s @ H(\overline{\alpha}n * s, F)),$$

and, clearly, $Img(G) = Img(\lambda s, F.Y(\overline{\alpha}n * s @ H(\overline{\alpha}n * s, F)) \subseteq K$. Hence, it suffices to show that equations of the form (10) (with the mentioned restriction on G) always have a solution Ψ. That is what we will do now.

Consider the set $\mathcal{T} = \mathrm{T} \to 2^K \backslash \{\emptyset\}$. The set $\mathcal{T}$ can be viewed as the set of labelled trees whose labels range over non-empty subsets of K. We define a partial order $\sqsubseteq$ on $\mathcal{T}$ as follows

$$f \sqsubseteq g :\equiv \forall s.(f(s) \subseteq g(s)).$$

Finally, we define an operation $\chi : \mathcal{T} \to \mathcal{T}$,

$$\chi(f)(s) :\equiv Img(\lambda F \in \mathsf{Cons}^f_s.G(s, F)),$$

where $\mathsf{Cons}^f_s :\equiv \{F : \forall x^\rho.F(x) \in f(s * x)\}$. We first observe the following.

Lemma 6.3 $(\mathcal{T}, \sqsubseteq)$ *is a directed complete semi-lattice.*

Proof. Let S be a directed subset of $\mathcal{T}$. Since we assign non-empty finite sets to the nodes of T, it is easy to see that $\bigcap S$ belongs to $\mathcal{T}$ and it is smaller than any element in S. $\Box$

Lemma 6.4 $\chi : \mathcal{T} \to \mathcal{T}$ *is monotone.*

Proof. Let $f \sqsubseteq g$ and s be fixed. We get that $\mathsf{Cons}^f_s \subseteq \mathsf{Cons}^g_s$, which implies $\chi(f)(s) \subseteq \chi(g)(s)$. $\Box$

By the Knaster-Tarski fixed point theorem we obtain an $f \in \mathcal{T}$ such that $\chi(f) = f$, i.e. $f(s) = Img(\lambda F \in \mathsf{Cons}^f_s.G(s, F))$, for all s. Let F_s be a functional from $f(s)$ to Cons^f_s such that $c = G(s, F_s(c))$, for all $c \in f(s)$. Define the functional $\Phi(s)$ recursively as follows,

$\Psi(\langle\,\rangle) :\equiv$ arbitrary element of $f(\langle\,\rangle)$;
$\Psi(s * x) :\equiv F_s(\Psi(s))(x)$.

Lemma 6.5 *The functional Ψ is total and satisfies equation (10).*

Proof. We have just shown that Φ is total. Moreover, note that, for all s, the values assigned to $\Phi(s * x)$ are such that $\Phi(s) = G(s, \lambda x.\Phi(s * x))$. $\square$

Corollary 6.1 *There exists a functional*

$$\Phi : \mathcal{M}_{\rho^\omega \to o} \times \mathcal{M}_{\rho^* \times (\rho \to o) \to \rho^\omega} \times \mathcal{M}_{\rho^*} \to \mathcal{M}_o$$

satisfying modified bar recursion.

6.2 Finding a majorant for Φ

Now we show that the Φ (whose existence is proven in Corollary 6.1) has a majorant, and therefore belongs to $\mathcal{M}$.

Lemma 6.6 *Let s^* and s s.t. $|s^*| = |s|$ be fixed. If s^* maj s then*

$$\forall \beta \in s \exists \beta^* \in s^* \, (\beta^* \text{ maj } \beta).$$

Proof. Let s^*, s and $\beta \in s$ be fixed. Moreover, assume $|s^*| = |s| = n$ and s^* maj s. Define β^* as,

$$\beta^*(i) :\equiv \begin{cases} s^*_i & \text{if } i < n \\ \max^\rho\{\max_{j<i}{}^\rho \beta^*(j), [\beta(i)]^*\} & \text{otherwise,} \end{cases}$$

where $[\beta(i)]^*$ is some majorant of $\beta(i)$. First note that, for all i, $\beta^*(i)$ maj $\beta(i)$. We show that β^* maj β. Let $k \geq i$.

If $k < n$ then $\beta^*(k) = s^*_k$ maj s^*_i maj $s_i = \beta(i)$.

If $k \geq n$ then $\beta^*(k) = \max^\rho\{\max_{j<k}{}^\rho \beta^*(j), [\beta(k)]^*\}$ maj $\beta^*(i)$ maj $\beta(i)$. $\square$

Lemma 6.7 *Define $\Upsilon : \mathcal{M}_{\rho^\omega \to \mathbb{N}} \to \mathcal{M}_{\rho^\omega} \to \mathcal{M}_{\mathbb{N}}$ as,*

$$\Upsilon(Y)(\alpha) :\equiv \min n \, [\forall \beta \in \overline{\alpha}n (Y(\beta) \leq n)].$$

Then,

i) If Y maj Y *then* $\Upsilon(Y)$ maj Y*,*
ii) $\Upsilon(Y)$ *is continuous and* $\Upsilon(Y)(\alpha)$ *is a point of continuity for* $\Upsilon(Y)$ *on* α*,*
iii) Υ maj Υ*. (therefore,* $\Upsilon \in \mathcal{M}$*)*

Proof. First of all, we note that, by Lemma 6.2, the functional Υ is well defined.

i) Assume that Y maj Y and α^* maj α. By the definition of Υ we have $\Upsilon(Y)(\alpha^*) \geq Y(\alpha^*)$, and by the self majorability of Y we have $Y(\alpha^*) \geq Y(\alpha)$. It is only left to show that $\Upsilon(Y)(\alpha^*) \geq \Upsilon(Y)(\alpha)$. Let $n = \Upsilon(Y)(\alpha^*)$ and $m = \Upsilon(Y)(\alpha)$, and assume $n < m$. By the definition of Υ there must exist a $\beta \in \overline{\alpha}m$ such that $Y(\beta) = m$. But since $n < m$, by Lemma 6.6 there exists a $\beta^* \in \overline{\alpha^*}n$ which majorizes β. Since Y majorizes itself we have $Y(\beta^*) \geq Y(\beta)$, a contradiction.

ii) Let α be fixed and take $n = \Upsilon(Y)(\alpha)$. Suppose there exists a $\beta \in \overline{\alpha}n$ such that $\Upsilon(Y)(\beta) \neq n$. If $\Upsilon(Y)(\beta) < n$ we get, since $\alpha \in \overline{\beta}n$, that $\Upsilon(Y)(\alpha) < n$, a contradiction. Suppose $\Upsilon(Y)(\beta) > n$. Since $\beta \in \overline{\alpha}n$ we have, $\forall \gamma \in \overline{\beta}n(Y(\gamma) \leq n)$, also a contradiction.

iii) Assume Y^* maj Y and α^* maj α. By i) we have that $\Upsilon(Y^*)(\alpha^*) \geq \Upsilon(Y^*)(\alpha)$. It is left to show that $\Upsilon(Y^*)(\alpha^*) \geq \Upsilon(Y)(\alpha)$. Let $n = \Upsilon(Y^*)(\alpha^*)$ and suppose $m = \Upsilon(Y)(\alpha)$. Assume $n < m$. By the definition of $\Upsilon(Y)$, there exists a $\beta \in \overline{\alpha}m$ s.t. $Y(\beta) \geq m$. But, since $n < m$, by Lemma 6.6, there exists a $\beta^* \in \overline{\alpha^*}n$ s.t. β^* maj β. Hence, $Y^*(\beta^*) \geq Y(\beta)$, a contradiction. □

Lemma 6.8 *Let* Y^* maj Y *(of type* $\rho^\omega \to \mathbb{N}$*) and* α *be fixed, and* $n = \Upsilon(Y^*)(\alpha)$*. If* $\overline{\alpha}n$ maj s *and* $|s| = n$ *then for all sequences* β *we have*

$$\Upsilon(Y^*)(s\ @\ \beta), \Upsilon(Y)(s\ @\ \beta), Y(s\ @\ \beta) \leq n.$$

Proof. We prove just that $\Upsilon(Y^*)(s\ @\ \beta) \leq n$. The other two cases follow similarly. Suppose there exists a β such that $n < \Upsilon(Y^*)(s\ @\ \beta)$. Since $\overline{\alpha}n$ maj s, by Lemma 6.6, there exists a β^* such that $\overline{\alpha}n * \beta^*$ maj $s\ @\ \beta$. Therefore, by Lemma 6.7 (iii), we must have $n < \Upsilon(Y^*)(\overline{\alpha}n * \beta^*)$. And by the fact that n is a point of continuity for $\Upsilon(Y^*)$ on α we get $\Upsilon(Y^*)(\overline{\alpha}n * \beta^*) = n$, a contradiction. □

In the following we extend the $(\cdot)^+$ operator of Lemma 6.1 for functionals F of type $\rho^* \to \mathbb{N}$ as

$$F^+ :\equiv \lambda s. \max_{s' \prec s} F(s'),$$

where $s' \prec s :\equiv |s'| \leq |s| \ \wedge\ \forall i < |s'|\ (s'_i = s_i)$.

Lemma 6.9 *Let F and G be of type $\rho^* \to \mathbb{N}$. If*

$$\forall s^*, s\, [s^* \text{ maj } s \wedge |s^*| = |s| \to F(s^*) \geq F(s), G(s)]$$

then F^+ maj G^+, G.

Proof. Let s^* maj s be fixed. For all prefixes t^* (of s^*) and t (of s) of the same length, by the assumption of the lemma, we have $F(t^*) \geq F(t), G(t)$. Therefore,

$$\max_{s' \prec s^*} F(s') \geq \max_{s' \prec s} F(s'), \max_{s' \prec s} G(s').$$

Therefore, F^+ maj G^+, G. $\Box$

Theorem 6.1 *If Φ is a functional of type*

$$\mathcal{M}_{\rho^\omega \to \mathbb{N}} \times \mathcal{M}_{\rho^* \times (\rho \to \mathbb{N}) \to \rho^\omega} \times \mathcal{M}_{\rho^*} \to \mathcal{M}_{\mathbb{N}},$$

which for any given $Y, H, s \in \mathcal{M}$ (of appropriate types) satisfies MBR*, then $\Phi \in \mathcal{M}$.*

Proof. Our proof is based on the proof of the main result of [6]. The idea is that, if Φ satisfies MBR then the functional

$$\Phi^* :\equiv \lambda Y, H.[\lambda s.\Phi(\hat{Y}, \hat{H}, s)]^+ \text{ maj } \Phi,$$

where

$$\hat{Y}(\alpha) :\equiv \Upsilon(Y)(\alpha^+) \text{ and}$$
$$\hat{H}(s, F) :\equiv H(s, \lambda x.F(\{x\}_s)),$$

and $\{x\}_s$ abbreviates $\max^\rho\{\max_{i<|s|}{}^\rho s_i, x\}$. Let Y^* maj Y and H^* maj H be fixed. The fact that Φ^* maj Φ follows from,

$$[\lambda s.\Phi(\hat{Y^*}, \hat{H^*}, s)]^+ \text{ maj } [\lambda s.\Phi(\hat{Y}, \hat{H}, s)]^+, \lambda s.\Phi(Y, H, s),$$

which follows, by Lemma 6.9, from $\forall s^*\, P(s^*)$ where (For the rest of the proof s^* maj s is a shorthand for s^* maj $s \;\wedge\; |s^*| = |s|$, i.e. majorizability is only considered for sequences of equal length.)

$$\underbrace{\forall s\, [s^* \text{ maj } s \to \Phi(\hat{Y^*}, \hat{H^*}, s^*) \geq \Phi(\hat{Y^*}, \hat{H^*}, s), \Phi(\hat{Y}, \hat{H}, s), \Phi(Y, H, s)]}_{\equiv:\; P(s^*)}.$$

We prove $\forall s^*\, P(s^*)$ by bar induction:

i) $\forall \alpha \exists n\, P(\overline{\alpha}n)$. Let α be fixed and $n :\equiv \hat{Y^*}(\alpha) = \Upsilon(Y^*)(\alpha^+)$. If $\overline{\alpha}n$ does not majorize any sequence s we are done. Let s be such that $\overline{\alpha}n$ maj s. Note that

$\overline{\alpha^+ n} = \overline{(\overline{\alpha}n \;@\; \beta)^+ n}$, for all β. Therefore, by Lemma 6.7 (*ii*) and our assumption that Φ satisfies MBR we get $\Phi(\hat{Y}^*, \hat{H}^*, \overline{\alpha}n) = n$. Since $\overline{\alpha^+ n}$ maj $\overline{(s \;@\; \beta)^+ n}$ (for all β), by Lemma 6.8, we have $n \geq \Phi(\hat{Y}^*, \hat{H}^*, s), \Phi(\hat{Y}, \hat{H}, s), \Phi(Y, H, s)$.

ii) $\forall s^* (\forall x\, P(s^* * x) \to P(s^*))$. Let s^* be fixed. Assume that $\forall x\, P(s^* * x)$, i.e.

$$\forall x, s\, [s^* {*} x \text{ maj } s \to \Phi(\hat{Y}^*, \hat{H}^*, s^* {*} x) \geq \Phi(\hat{Y}^*, \hat{H}^*, s), \Phi(\hat{Y}, \hat{H}, s), \Phi(Y, H, s)].$$

Note that if s^* does not majorize any sequence we are again done. Assume s is such that s^* maj s. If x^* maj x then (by $\forall x\, P(s^* * x)$),

$$\overbrace{\Phi(\hat{Y}^*, \hat{H}^*, s^* * \{x^*\}_{s^*})}^{\equiv:\ \Phi_1(\{x^*\}_{s^*})} \geq \overbrace{\Phi(\hat{Y}^*, \hat{H}^*, s * \{x\}_s)}^{\equiv:\ \Phi_2(\{x\}_s)}, \overbrace{\Phi(\hat{Y}, \hat{H}, s * \{x\}_s)}^{\equiv:\ \Phi_3(\{x\}_s)}, \underbrace{\Phi(Y, H, s * x)}_{\equiv:\ \Phi_4(x)}$$

and also $\Phi_1(\{x^*\}_{s^*}) \geq \Phi_1(\{x\}_{s^*})$, which implies

$$\lambda x.\Phi_1(\{x\}_{s^*}) \text{ maj } \lambda x.\Phi_2(\{x\}_s), \lambda x.\Phi_3(\{x\}_s), \lambda x.\Phi_4(x),$$

and by the definition of majorizability

$$\overbrace{H^*(s^*, \lambda x.\Phi_1(\{x\}_{s^*}))}^{\hat{H}^*(s^*, \lambda x.\Phi_1(x))} \text{ maj } \overbrace{H^*(s, \lambda x.\Phi_2(\{x\}_s))}^{\hat{H}^*(s, \lambda x.\Phi_2(x))}, \overbrace{H(s, \lambda x.\Phi_3(\{x\}_s))}^{\hat{H}(s, \lambda x.\Phi_3(x))}, H(s, \lambda x.\Phi_4(x)),$$

which implies

$$\begin{aligned}(s^* \;@\; \hat{H}^*(s^*, \lambda x.\Phi_1(x)))^+ \text{ maj } & (s \;@\; \hat{H}^*(s, \lambda x.\Phi_2(x)))^+, \\ & (s \;@\; \hat{H}(s, \lambda x.\Phi_3(x)))^+, \\ & s \;@\; H(s, \lambda x.\Phi_4(x)).\end{aligned}$$

And finally, by Lemma 6.7 (*i*) and (*iii*),

$$\overbrace{\hat{Y}^*(s^* \;@\; \hat{H}^*(s^*, \lambda x.\Phi_1(x)))}^{\Phi(\hat{Y}^*, \hat{H}^*, s^*)} \geq \overbrace{\hat{Y}^*(s \;@\; \hat{H}^*(s, \lambda x.\Phi_2(x)))}^{\Phi(\hat{Y}^*, \hat{H}^*, s)}, \underbrace{\hat{Y}(s \;@\; \hat{H}(s, \lambda x.\Phi_3(x)))}_{\Phi(\hat{Y}, \hat{H}, s)}, \underbrace{Y(s \;@\; H(s, \lambda x.\Phi_4(x)))}_{\Phi(Y, H, s)}$$

□

Corollary 6.2 *There exists a $\Phi \in \mathcal{M}$ (not unique) satisfying* MBR.

Proof. In Section 6.1 we have constructed a

$$\Phi \in \mathcal{M}_{\rho^\omega \to o} \times \mathcal{M}_{\rho^* \to (\rho \to o) \to \rho^\omega} \times \mathcal{M}_{\rho^*} \to \mathcal{M}_o$$

satisfying MBR. By Theorem 6.1, $\Phi \in \mathcal{M}$. The fact that Φ is not unique follows by taking, e.g.,

$$F(\alpha) = \begin{cases} 1 \text{ if } \alpha = \lambda x.1 \\ 0 \text{ otherwise,} \end{cases}$$

which means that the sets Set^σ_s (cf. Section 6.1) will not be singletons. □

Corollary 6.3 KBR *is not primitive recursively definable in* MBR.

7 Defining SBR primitive recursively in MBR

Assuming we have a term t satisfying MBR we build a term t' (primitive recursively in t) which satisfies SBR.

Definition 7.1 $\tilde{\mu}(Y, \alpha^{\rho^\omega}, k) :\equiv \min n \geq k\, [Y(\overline{\alpha, n}) < n]$.

Kohlenbach [14] has shown that $\tilde{\mu}$ is primitive recursively definable in SBR.

Theorem 7.1 $\tilde{\mu}$ *is primitive recursively definable in* MBR.

Proof. Let n be the value of $\tilde{\mu}(Y, \alpha, k)$. The case when $n = k$ is simple and will be treated at the end of the proof. We will assume that $n > k$. In this case we note that, by the minimality condition, $Y(\overline{\alpha, n-1}) \geq n - 1$. Hence, $Y(\overline{\alpha, n-1}) + 1$ can be used (for bounded search) as an upper bound for the value of n. Using MBR, in order for F to return the required upper bound we have to give as input the sequence $\overline{\alpha, n-1}$. We show how this sequence can be computed by an appropriate H (in the definition of MBR). By MBR we can define a Φ_α satisfying $\Phi_\alpha(s) = Y(s \mathbin{@} (\overline{\alpha, m-1}))$, where,

$$(*) \;\; m \stackrel{o}{=} \begin{cases} |s| + 1 & \text{if } Y(\overline{\alpha, |s|+1}) < |s| + 1 \\ \tilde{\mu}^b(Y, \alpha, k, \Phi_\alpha(s * \alpha(|s|)) + 1) & \text{otherwise,} \end{cases}$$

and $\tilde{\mu}^b$ is the bounded version of $\tilde{\mu}$ (which is primitive recursive). We then define,

$$\tilde{\mu}(Y, \alpha, k) :\equiv \begin{cases} k & \text{if } Y(\overline{\alpha, k}) < k \\ \tilde{\mu}^b(Y, \alpha, k, \Phi_\alpha(\overline{\alpha}k) + 1) & \text{otherwise.} \end{cases}$$

We show that this is a good definition of $\tilde{\mu}$ by showing that $\Phi_\alpha(\overline{\alpha}k) + 1$ is a good upper bound on the value of $\tilde{\mu}(Y, \alpha, k)$ (assume this value is $n > k$). In fact, we show by induction on j that, for $k \leq j < n$, n is bounded by $\Phi_\alpha(\overline{\alpha}j) + 1$.

i) $j = n - 1$. We see that the first case of $(*)$ will be satisfied, m is equal n and $\Phi_\alpha(\overline{\alpha}j) + 1 = Y(\overline{\alpha}j \mathbin{@} (\overline{\alpha, m-1})) + 1 = Y(\overline{\alpha, n-1}) + 1 \geq n$.

ii) $j < n - 1$. By induction hypothesis $\Phi_\alpha(\overline{\alpha}j * \alpha(j)) + 1$ is a bound for n. Therefore, m (see second case of $(*)$) has value n, and as above we get $\Phi_\alpha(\overline{\alpha}j) + 1 \geq n$. $\Box$

Theorem 7.2 $\mathsf{SBR}_{\rho,o}$ *is primitive recursively definable in* MBR_ρ.

Proof. We show how to define (primitive recursively in MBR) a Ψ satisfying the equation ($\mathsf{SBR}_{\rho,o}$),

$$(i)\ \ \Psi(Y, G, H, s) \stackrel{o}{=} \begin{cases} G(s) & \text{if } Y(s * 0) < |s| \\ H(s, \lambda x.\Psi(Y, G, H, s * x)) & \text{otherwise.} \end{cases}$$

Let Φ be a functional satisfying MBR. In the following π_0 and π_1 will denote the projection functional, i.e. $\pi_i(\langle x_0^\rho, x_1^\rho \rangle) = x_i$, $i \in \{0, 1\}$. If $s^{\langle \rho,\rho \rangle^*} = \langle s_0, \ldots, s_n \rangle$, $\pi_i(s)$ also denotes $\langle \pi_i(s_0), \ldots, \pi_i(s_n) \rangle$. In the same way we define $\pi_i(\alpha^{\langle \rho,\rho \rangle^\omega})$. We first define two tilde operations,

$$(ii)\ \ \tilde{H}(s, F) :\equiv \lambda n.\langle 1, H(\pi_1(s), \lambda x.F(\langle 0, x \rangle)) \rangle$$

and [6]

$$(iii)\ \ \tilde{Y}_{G,k}(\alpha) :\equiv \begin{cases} G(\pi_1(s)) \text{ if } \bigwedge_{i=0}^{n} (\pi_0(s_i) = 0) \\ \pi_1(s_n) \quad \text{otherwise,} \end{cases}$$

where (in the definition of $\tilde{Y}_{G,k}$) $s = \langle s_0, \ldots, s_n \rangle = \overline{\alpha}\ \tilde{\mu}(Y, \pi_1(\alpha), k)$. Note that the first operation is primitive recursive in H, s and F; and the second is primitive recursive in Y, G, k, α and MBR (since it uses $\tilde{\mu}$). Moreover, $(*)$ if $Y(s * 0) \geq |s|$ then $\tilde{Y}_{G,|s|} = \tilde{Y}_{G,|s|+1}$.

We abbreviate $\langle \langle 0, s_0 \rangle, \ldots, \langle 0, s_{|s|-1} \rangle \rangle$ by $\langle 0, s \rangle$. Define

$$(iv)\ \ \Psi(Y, G, H, s) :\equiv \Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s \rangle)).$$

[6] Since, by our construction, the first element of the pair will either be 0^ρ or 1^ρ, the test $\pi_0(s_i) = 0$ in the definition (iii) is primitive recursive.

We show that Ψ satisfies equation (i), i.e.

$$(v)\ \ \Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s\rangle) = \begin{cases} G(s) & \text{if } Y(s * 0) < |s| \\ H(s, \lambda x.\Phi(\tilde{Y}_{G,|s|+1}, \tilde{H}, \langle 0, s * x\rangle)) & \text{otherwise.} \end{cases}$$

We first note that, by the definition of MBR (and (ii)),

$$(vi)\ \ \Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s\rangle) = \tilde{Y}_{G,|s|}(\langle 0, s\rangle \ @\ \lambda n.\langle 1, H(s, \lambda x.\Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s * x\rangle))\rangle).$$

We will show that (v) holds. Assume $Y(s * 0) < |s|$, we have,

$$\Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s\rangle) \stackrel{(vi)}{=} \tilde{Y}_{G,|s|}(\langle 0, s\rangle \ @\ \ldots) \stackrel{(iii)}{=} G(s).$$

On the other hand, if $Y(s * 0) \geq |s|$ then,

$$\begin{aligned} \Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s\rangle) &\stackrel{(vi)}{=} \tilde{Y}_{G,|s|}(\langle 0, s\rangle \ @\ \lambda n.\langle 1, H(s, \lambda x.\Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s * x\rangle))\rangle) \\ &\stackrel{(iii)}{=} H(s, \lambda x.\Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s * x\rangle)) \\ &\stackrel{(*)}{=} H(s, \lambda x.\Phi(\tilde{Y}_{G,|s|+1}, \tilde{H}, \langle 0, s * x\rangle)), \end{aligned}$$

and the proof is concluded. $\Box$

Theorem 7.3 $\mathsf{SBR}_{\rho,\tau}$ *is primitive recursively definable in* $\mathsf{SBR}_{\rho',o}$*, where if* $\tau = \tau_1 \to \ldots \to \tau_n \to o$ *then* $\rho' = \rho \times \tau_1 \times \ldots \times \tau_n$.

Proof. Let $\tau = \tau_1 \to \ldots \to \tau_n \to o$. We will show that $\mathsf{SBR}_{\rho,\tau}$ can be defined from $\mathsf{SBR}_{\rho\times\tau_1\times\ldots\times\tau_n,o}$. Let G, H and Y be given, we have to define a functional Φ such that,

$$(i)\ \ \Phi(Y, G, H, s) \stackrel{\tau}{=} \begin{cases} G(s) & \text{if } Y(s \ @\ 0^\rho) \stackrel{\mathbb{N}}{<} |s| \\ H(s, \lambda x^\rho.\Phi(s * x)) & \text{otherwise.} \end{cases}$$

From Y, G and H we define,

$$\begin{aligned} (ii)\ \ & \tilde{Y}(\alpha) :\equiv Y(\pi_0^{n+1}(\alpha)); \\ (iii)\ \ & \tilde{G}(t) :\equiv G(\pi_0^{n+1}(t))(y); \\ (iv)\ \ & \tilde{H}(t, F) :\equiv H(\pi_0^{n+1}(t), \lambda x^\rho, z_1^{\tau_1}, \ldots, z_n^{\tau_n}.F(\langle x, z_1, \ldots, z_n\rangle))(y); \end{aligned}$$

where y denotes $\pi_1^{n+1}(t_{|t|-1}), \ldots, \pi_{n+1}^{n+1}(t_{|t|-1})$ and the types are,

$$\begin{aligned} \alpha &: (\rho \times \tau_1 \times \ldots \times \tau_n)^\omega \\ y &: \tau_1 \times \ldots \times \tau_n \\ t &: (\rho \times \tau_1 \times \ldots \times \tau_n)^* \end{aligned}$$

$$F : (\rho \times \tau_1 \times \ldots \times \tau_n) \to o$$

and we define (using $\mathsf{SBR}_{\rho\times\tau_1\times\ldots\times\tau_n,o}$),

$$(v)\ \ \Psi(\tilde{Y}, \tilde{G}, \tilde{H}, t) \overset{o}{=} \begin{cases} \tilde{G}(t) & \text{if } \tilde{Y}(t \mathbin{@} 0) \overset{\mathbb{N}}{<} |t| \\ \tilde{H}(t, \lambda x^{\rho\times\tau_1\times\ldots\times\tau_n}.\Psi(t * x)) & \text{otherwise.} \end{cases}$$

Finally we set, ($\langle s, \mathbf{y}\rangle$ abbreviates $\langle\langle s_0, \mathbf{y}\rangle, \ldots, \langle s_{|s|-1}, \mathbf{y}\rangle\rangle$)

$$(vi)\ \ \Phi(Y, G, H, s) :\overset{\tau}{\equiv} \lambda \mathbf{y}.\Psi(\tilde{Y}, \tilde{G}, \tilde{H}, \langle s, \mathbf{y}\rangle).$$

We show that equation (i) is satisfied by Φ. One easily verifies that

$$(vii)\ \ \Psi(\tilde{Y}, \tilde{G}, \tilde{H}, \langle s, \mathbf{y}\rangle) = \Psi(\tilde{Y}, \tilde{G}, \tilde{H}, \langle\langle s_0, \mathbf{z}\rangle, \ldots, \langle s_{|s|-2}, \mathbf{z}\rangle, \langle s_{|s|-1}, \mathbf{y}\rangle\rangle),$$

for arbitrary $\mathbf{z}$. Let Y, G, H and s be fixed and t abbreviate $\langle s, \mathbf{y}\rangle$. By (ii), $Y(s \mathbin{@} 0) < |s|$ iff $\tilde{Y}(t \mathbin{@} 0) < |t|$. Therefore, if $Y(s \mathbin{@} 0) < |s|$ then

$$\begin{aligned} \Phi(Y, G, H, s) &\overset{(vi)}{=} \lambda \mathbf{y}.\Psi(\tilde{Y}, \tilde{G}, \tilde{H}, \langle s, \mathbf{y}\rangle) \\ &\overset{(v)}{=} \lambda \mathbf{y}.\tilde{G}(\langle\langle s, \mathbf{y}\rangle) \overset{(iii)}{=} \lambda \mathbf{y}.G(s)(\mathbf{y}) = G(s). \end{aligned}$$

On the other hand, if $Y(s \mathbin{@} 0) \geq |s|$ then

$$\begin{aligned} \Phi(Y, G, H, s) &\overset{(vi)}{=} \lambda \mathbf{y}.\Psi(\tilde{Y}, \tilde{G}, \tilde{H}, \langle s, \mathbf{y}\rangle) \overset{(v)}{=} \lambda \mathbf{y}.\tilde{H}(t, \lambda x.\Psi(t * x)) \\ &\overset{(iv)}{=} \lambda \mathbf{y}.H(s, \lambda x, \mathbf{z}.\Psi(t * \langle x, \mathbf{z}\rangle))(\mathbf{y}) \\ &\overset{(vii)}{=} \lambda \mathbf{y}.H(s, \lambda x, \mathbf{z}.\Psi(\langle s, \mathbf{z}\rangle))(\mathbf{y}) \\ &\overset{(vi)}{=} \lambda \mathbf{y}.H(s, \lambda x.\Phi(s * x))(\mathbf{y}) = H(s, \lambda x.\Phi(s * x)) \end{aligned}$$

□

Corollary 7.1 SBR *is primitive recursively definable in* MBR.

8 S1-S9 Computability

Definition 8.1 (Axioms S1-S9) *In any applicative type structure $\mathcal{S}$ (containing $\mathbb{N}$) we define a set of relations Γ (parametrized by their arity and type* *of arguments) on $\mathcal{S}$ inductively as follows,* [7]

S1 $\{e\}^{\mathcal{S}}(m, \overrightarrow{y}) = m + 1$, *where* $e = \langle 1, \sigma \rangle$.
S2 $\{e\}^{\mathcal{S}}(\overrightarrow{y}) = k$, *where* $e = \langle 2, \sigma, k \rangle$.
S3 $\{e\}^{\mathcal{S}}(m, \overrightarrow{y}) = m$, *where* $e = \langle 3, \sigma \rangle$.
S4 *If* $\{e_1\}^{\mathcal{S}}(\overrightarrow{y}) = k_1$ *and* $\{e_2\}^{\mathcal{S}}(k_1, \overrightarrow{y}) = k_2$ *then* $\{e\}^{\mathcal{S}}(\overrightarrow{y}) = k_2$, *where* $e = \langle 4, e_1, e_2, \sigma \rangle$.
S5 *Can be omitted in the presence of* S9,
S6 *If* $\{e_1\}^{\mathcal{S}}(\tau(\overrightarrow{y})) = k$ *then* $\{e\}^{\mathcal{S}}(\overrightarrow{y}) = k$, *where* $e = \langle 6, e_1, \tau, \sigma \rangle$.
S7 $\{e\}^{\mathcal{S}}(f, x, \overrightarrow{y}) = f(x)$, *where* $e = \langle 7, \sigma \rangle$.
S8 *If* $\{e_1\}^{\mathcal{S}}(x, \overrightarrow{y}) = f(x)$, *for all* x, *then* $\{e\}^{\mathcal{S}}(\overrightarrow{y}) = y_1(f)$, *where* $e = \langle 8, e_1, \sigma \rangle$.
S9 *If* $\{e_1\}^{\mathcal{S}}(y_1, \ldots, y_i) = k$ *then* $\{e\}^{\mathcal{S}}(e_1, \overrightarrow{y}) = k$, *where* $i \leq n$ *and* $e = \langle 9, i, \sigma \rangle$.

One can prove by induction on S1-S9 *that for each* e *and* $\overrightarrow{y}$ *there exists at most one* k *such that* $\{e\}^{\mathcal{S}}(\overrightarrow{y}) = k$. *Therefore, each index* e *gives rise to a partial functional (denoted by* $\{e\}^{\mathcal{S}}$*) which on input* $\overrightarrow{y}$ *takes value* k *if* $\{e\}^{\mathcal{S}}(\overrightarrow{y}) = k$ *and is undefined otherwise. It is important to note that the functional* $\{e\}^{\mathcal{S}}$ *yielded by an index* e *need not belong to* $\mathcal{S}$. *The set of all indices* e *such that* $\{e\}^{\mathcal{S}} \in \mathcal{S}$ *is denoted by* $\mathcal{R}ec^{\mathcal{S}}$. *If* $\{e\}^{\mathcal{S}}$ *is a functional of the form* $\lambda\Psi, \overrightarrow{y}.\{e\}^{\mathcal{S}}(\Psi, \overrightarrow{y})$ *then* $\{e\}^{\mathcal{S}}_{\Psi}$ *denotes the functional* $\lambda\overrightarrow{y}.\{e\}(\Psi, \overrightarrow{y})$.

Definition 8.2 *A formula* P *in the language of* $\mathbf{HA}^{\omega}$ *having a unique free variable is called an* specification of a functional *or just* functional. *(e.g.* SBR *having variables* Y, G, H *and* s *universally quantified is an specification for Spector's bar recursor.)*

Definition 8.3 (S1-S9 computability) *Let* P, Q *be specifications and* $\mathcal{S}$ *any applicative type structure (containing* $\mathbb{N}$*). Then,*

P *is* S1-S9 computable in $\mathcal{S}$ *if* $\mathcal{S} \models \exists e \in \mathcal{R}ec^{\mathcal{S}}.\mathsf{P}(\{e\}^{\mathcal{S}})$.
P *is* S1-S9 + Q computable in $\mathcal{S}$ *if* $\mathcal{S} \models \exists\Psi\ (\mathsf{Q}(\Psi) \wedge \exists e \in \mathcal{R}ec^{\mathcal{S}}.\mathsf{P}(\{e\}^{\mathcal{S}}_{\Psi}))$.

Lemma 8.1 KBR *and* SBR *are* S1-S9 *computable in* $\mathcal{C}$.

Proof. One shows $\mathcal{C} \models \exists e \in \mathcal{R}ec^{\mathcal{C}}.\mathsf{KBR}(\{e\}^{\mathcal{C}})$ and $\mathcal{C} \models \exists e \in \mathcal{R}ec^{\mathcal{C}}.\mathsf{SBR}(\{e\}^{\mathcal{C}})$ using the recursion theorem. □

The total elements of $\mathcal{C}$ can be viewed as equivalence classes of elements of $\hat{\mathcal{C}}$. We denote these equivalence classes by $[F]$, i.e. if $F \in \hat{\mathcal{C}}$ is total then

[7] We abbreviate $y_1, \ldots, y_n$ (of arbitrary type) by $\overrightarrow{y}$. The variables $e_1, e_2, m, n, i, k, k_1, k_2$ range over natural numbers, σ ranges over codes for finite types and $f, x, \overrightarrow{y}$ over functionals of appropriate types. We write $\{e\}^{\mathcal{S}}(\overrightarrow{y}) = k$ instead of $\mathcal{S} \models \Gamma(e, \overrightarrow{y}, k)$.

$[F] \in \mathcal{C}$. We have a *transfer principle* which says that if $\{e\}^{\mathcal{C}}([F]) = k$ then $\{e\}^{\hat{\mathcal{C}}}(F) = k$. Moreover, (+) if $\{e\}^{\hat{\mathcal{C}}}(F) = k$ then there exists a compact element $G \in \hat{\mathcal{C}}$ such that $G \sqsubseteq F$ and $\{e\}^{\hat{\mathcal{C}}}(G) = k$.

Lemma 8.2 *If*

(*i*) *e is a* S1-S9 *code of type 3,*
(*ii*) *$\overrightarrow{x}, \overrightarrow{y} \in \hat{\mathcal{C}}$ (of type 2) coincide in all total recursive arguments,*
(*iii*) *$\overrightarrow{x}$ total* S1-S9 *computable in $\hat{\mathcal{C}}$,*
(*iv*) *$\{e\}^{\mathcal{C}}([\overrightarrow{x}]) = k$,*

then $\{e\}^{\hat{\mathcal{C}}}(\overrightarrow{y}) = k$.

Proof. By induction on S1-S9 codes, the critical point being S8. Assume e is of the form $\langle 8, e_1, \sigma \rangle$ and that $(i) - (iv)$ hold. From (iv), by the definition of S1-S9, there must exist a function $f \in \mathcal{C}$ such that

(*v*) $f(n) = \{e_1\}^{\mathcal{C}}(n, [\overrightarrow{x}])$, for all $n \in \mathbb{N}$, and
(*vi*) $[x_1](f) = k$,

By (iii) and (v) we get that f is recursive. Let n be fixed and assume that $\{e_1\}^{\mathcal{C}}(n, [\overrightarrow{x}]) = l$. By induction hypothesis we have that

$$\{e_1\}^{\hat{\mathcal{C}}}(n, \overrightarrow{y}) = l,$$

i.e. $\lambda n^{\mathbb{N}}.\{e_1\}^{\hat{\mathcal{C}}}(n, \overrightarrow{y})$ $(= [\lambda p^{\mathbb{N}_\perp}.\{e_1\}^{\hat{\mathcal{C}}}(p, \overrightarrow{y})])$ is identical to f. By (vi),

$$[x_1]([\lambda p^{\mathbb{N}_\perp}.\{e_1\}^{\hat{\mathcal{C}}}(p, \overrightarrow{y})]) = k.$$

Hence,

$$x_1(\lambda p.\{e_1\}^{\hat{\mathcal{C}}}(p, \overrightarrow{y})) = k.$$

Note that $\lambda p.\{e_1\}^{\hat{\mathcal{C}}}(p, \overrightarrow{y})$ is total and recursive. Therefore, by assumption (ii),

$$y_1(\lambda p.\{e_1\}^{\hat{\mathcal{C}}}(p, \overrightarrow{y})) = k,$$

and by the definition of S1-S9, $\{e\}^{\hat{\mathcal{C}}}(\overrightarrow{y}) = k$. $\square$

Theorem 8.1 ([9]) `FAN` *is not* S1-S9 *computable in $\mathcal{C}$.*

Proof. Assume $e \in \mathcal{R}ec^{\mathcal{C}}$ is such that $\mathcal{C} \models \mathtt{FAN}(\{e\}^{\mathcal{C}})$. Let O be a total (S1-S9 computable) element of $\hat{\mathcal{C}}$ which is constant zero. Assume $\{e\}^{\mathcal{C}}([O]) = k$. Let F be another type two functional (in $\hat{\mathcal{C}}$) such that $F(f) = 0$ whenever f is total and recursive, but which is $\perp$ for other f. By Lemma 8.2 $\{e\}^{\hat{\mathcal{C}}}(F) = k$. By (+) there must be a compact $G \sqsubseteq F$ (still in $\hat{\mathcal{C}}$) such that $\{e\}^{\hat{\mathcal{C}}}(G) = k$,

G defined on a closed-open set that does not cover all of $\hat{\mathcal{C}}$. We can then extend G to a total G' that is not constant and that k is not a modulus of uniform continuity for G'. Assume $\{e\}^{\mathcal{C}}([G']) = l$. By the *transfer principle* $\{e\}^{\hat{\mathcal{C}}}(G') = l$ and l must equal k, i.e. $\{e\}^{\mathcal{C}}([G']) = k$, a contradiction. $\Box$

Lemma 8.3 `FAN` *is* S1-S9 + MBR *computable in* $\mathcal{C}$*.*

Proof. By Theorem 2.1 there exists a $\Psi \in \mathcal{C}$ such that $\mathcal{C} \models \mathsf{MBR}(\Psi)$. In Theorem 4.2 we have shown that $\mathcal{C} \models \exists e \in \mathcal{R}ec^{\mathcal{C}}.\mathtt{FAN}(\{e\}^{\mathcal{C}}_{\Psi})$. $\Box$

Corollary 8.1 MBR *is not* S1-S9 *computable in* $\mathcal{C}$*.*

Proof. Assume $\mathcal{C} \models \exists e \in \mathcal{R}ec^{\mathcal{C}}.\mathsf{MBR}(\{e\}^{\mathcal{C}})$. By Lemma 8.3 we have that `FAN` is S1-S9 computable in $\mathcal{C}$, contradicting Theorem 8.1. $\Box$

Corollary 8.2 MBR *is not primitive recursively definable in* KBR *nor* SBR*.*

Proof. Follows from the corollary above, Lemma 8.1 and the fact that the set of functionals S1-S9 computable in $\mathcal{C}$ is closed under primitive recursion. $\Box$

Gandy and Hyland also showed that the functional Γ (see Section 5) is not S1-S9 computable in $\mathcal{C}$ even in the fan functional. From Theorem 5.1 we obtain the following corollary.

Corollary 8.3 $\mathsf{MBR}_{\mathbb{N}}$ *is not* S1-S9 *computable in* $\mathcal{C}$*, even in the fan functional.*

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