EFFECTIVE BOUNDS ON STRONG UNICITY IN L_1-APPROXIMATION

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We report the results of another case study [9] in the general project of Proof Mining, which means the logical analysis of prima facie non-effective proofs with the aim of extracting new computationally relevant data (see [5], [6], [7] and [8]). More specifically, we analyze a non-effective proof of the uniqueness of best approximations of continuous functions $f \in C[0,1]$ by polynomials $p \in P_n$ of degree $\leq n$ with respect to the L_1-norm (see [13] for a comprehensive survey on L_1-approximation) due to Cheney [2], [3]. Cheney's proof is a simplification of Jackson's pioneering proof [4] in the sense that it avoids measure theory and only uses the Riemann integral. Because of this simplification Cheney called his proof 'elementary', although, from a logical point of view, it is highly non-constructive relying both on classical logic and non-computational analytical principles which correspond – in logical terminology – to the so-called binary ('weak') König's lemma, WKL, a principle which has received considerable attention in various parts of logic in recent years (see [14]). From Cheney's proof we extract the first effective (in all parameters f, n, ε) uniform rate of strong uniqueness (so-called uniform modulus of uniqueness) for L_1-approximation with optimal ε-dependency ([10]). This work is based on [5] where the first author showed how a quite general class of (non-effective) proofs of uniqueness theorems in analysis can be analyzed such that effective moduli of uniqueness can be extracted. Uniform modulus of uniqueness generalizes the the concept of strong unicity introduced by Newman and Shapiro [11] in 1963 and extensively studied in approximation theory since then (see [1] and the references given there). Using the uniform modulus of uniqueness one can, for instance, compute for any given $f \in C[0,1]$ its best L_1-approximation from P_n, and the computational complexity of this procedure depends on the complexity of the modulus ([12]). In this talk we discuss the logical aspect of the extraction of the mentioned modulus.

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