On the computational complexity of L_1-approximation

Paulo Oliva

BRICS
University of Århus
Denmark

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Plan of the talk

- Numerical Analysis
  - The problem of L_1-approximation
  - Modulus of Uniqueness
- Computing p_n
- Complexity Analysis of p_n

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Mathematical background

Notation

- $C[0,1]$ = real valued continuous functions on the interval $[0,1]$,
- P_n = algebraic polynomials of degree $\leq n$,
- $\|\cdot\|_1$ = L_1-norm, i.e. $\|f\|_1 = \int_0^1 |f(x)|\, dx$
- $dist_1(f, P_n) = \inf_{p \in P_n} \|f - p\|_1$

Jackson's thm: Let $f \in C[0,1]$ and $n \in \mathbb{N}$. There exists a unique polynomial $p \in P_n$ such that $\|f - p\|_1 = dist_1(f, P_n)$.

- The existence of such polynomial follows from a general theorem on the existence of best approximations in metric spaces.
- We call the polynomial above p_n.

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Computing p_n

Inputs:

- continuous function $f \in C[0,1]$,
- natural numbers $n, k \in \mathbb{N}$.

Output: $(n+1)$-tuple $(d_0, \ldots, d_n)$ of dyadic numbers such that $|q_i - p_i| \le 2^{-k}$, for $0 \le i < n$, where $\sum_{i=0}^{n} q_i x^i$ is the best L_1-approximation of f from P_n.

The modulus of uniqueness (I)

The computation of p_n relies on the modulus of uniqueness for L_1-approximation.

Thm[KO'01] Let $\Phi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined as,

$$\Phi(\omega, n, k) := 2k + C_1 + \omega_f(k + C_2),$$

where $C_1 := (4n^2 + 10n + 18)\log(n+2)$ and $C_2 := (2n^2 + 5n + 6)\log(n+2)$.

For any $f \in C[0,1]$ with modulus of continuity ω,

$$\forall n \in \mathbb{N}; p_1, p_2 \in P_n; k \in \mathbb{N}$$

$$\left(\bigwedge_{i=1}^{2}(\|f - p_i\|_1 - dist_1(f, P_n) \leq 2^{-\Phi(\omega,n,k)}) \to \|p_1 - p_2\|_1 \leq 2^{-k}\right).$$

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The modulus of uniqueness (II)

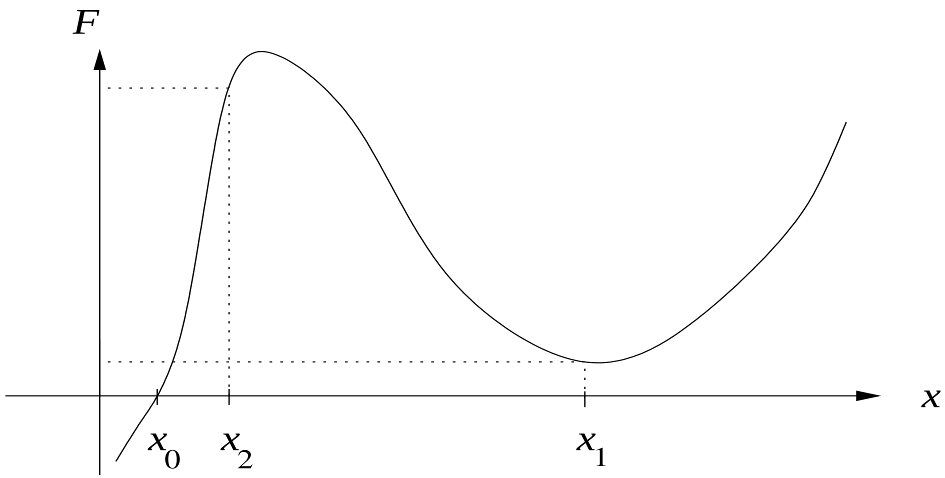
- The first such Φ is presented in [KO'01].
- Φ has optimal k dependency as shown in Kroo'79.
- The modulus Φ was extracted from Cheney's non-effective proof of uniqueness (1965) for L_1-approximation.
- Φ is polynomial in n and k (given that f is polynomial time computable, i.e. ω_f a polynomial).

How the modulus of uniqueness helps

Given that a continuous $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ has a unique root (let x_0 be the root and $k \in \mathbb{N}$):

(i) Find k-roots, i.e. x_1 such that $|\mathcal{F}(x_1)| < 2^{-k}$.

(ii) Find x_2 k-close to x_0, i.e. $|x_2 - x_0| < 2^{-k}$.



- The modulus of uniqueness relates those two problems, for all $x_0, x_1 \in K$,

$$\forall k \in \mathbb{N} \left(|\mathcal{F}(x_0) - \mathcal{F}(x_1)| \leq 2^{-\Phi(k)} \to |x_0 - x_1| < 2^{-k} \right)$$

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Computing the best L_1-approximation (1)

Our goal is to produce an $(n+1)$-tuple $(d_0, \ldots, d_n)$ of dyadic numbers such that $|q_i - d_i| \leq 2^k$, for any given k, where $p_n = \sum q_i x^i$.

From the Φ above we can easily obtain a Φ' such that,

$$\forall n \in \mathbb{N}; p_1, p_2 \in P_n; k \in \mathbb{N}$$

$$\left(\bigwedge_{i=1}^{2} (\|f - p_i\|_1 - dist_1(f, P_n) \leq 2^{-\Phi'(\omega,n,k)}) \to \|p_1 - p_2\|_{\max} \leq 2^{-k} \right),$$

where $\|(a_0, \ldots, a_{n-1})\|_{\max} := \max\{|a_i|\}$.

Now we replace p_2 with p_n above, (and rename p_1)

$$\forall n \in \mathbb{N}; p \in P_n; k \in \mathbb{N}$$

$$\left(\|f - p\|_1 - dist_1(f, P_n) \leq 2^{-\Phi'(\omega,n,k)} \to \|p - p_n\|_{\max} \leq 2^{-k} \right),$$

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Computing the best L_1-approximation (2)

Our problem is now reduce to finding a p such that,

$$(*) \quad \|f - p\|_1 - dist_1(f, P_n) \leq 2^{-\Phi'(\omega,n,k)}.$$

Here we make strong use of the fact that, for a fixed n, the best L_1-approximation of f from P_n, p_n, lives in the compact set

$$K_{f,n} := \{p \in P_n : \|p\|_1 \leq 2\,\|f\|_1\}.$$

The next step is then to search through a (sufficiently fine) net (which 'covers' $K_{f,n}$) for a p satisfying $(*)$.

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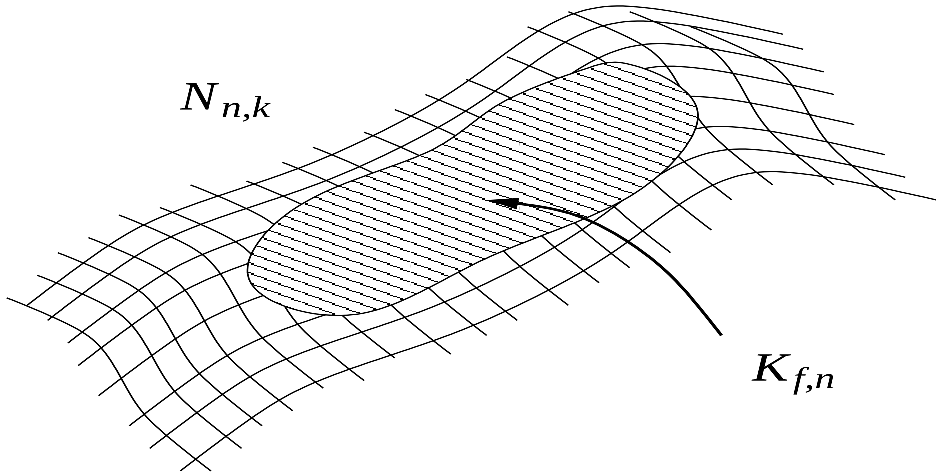
The net



The properties we want from the net $N_{n,k}$ are,

- $N_{n,k}$ is a finite set,
- elements of $N_{n,k}$ are finite (tuples of dyadic numbers),
- $\forall \tilde{p} \in K_{f,n} \exists p \in N_{n,k} \|p - \tilde{p}\|_1$.

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Complexity Analysis (I/1)

We first give a complexity analysis using the two following oracles,

- oracle A_f deciding membership for the set $\{\langle n, d\rangle : d \in L_n\}$, where $n \in S_1$, $d \in S_2$ and L_n is the left cut of d_n.
- oracle B_f satisfying the following,

$$\begin{aligned}
\langle k, p, d\rangle \in B_f \quad &\rightarrow \quad |\, \|f - p\|_1 - d \,| \leq 3 \cdot 2^{-k-2} \\
\langle k, p, d\rangle \notin B_f \quad &\rightarrow \quad |\, \|f - p\|_1 - d \,| > 2 \cdot 2^{-k-2},
\end{aligned}$$

  where $k \in S_1$ and $a_0, \ldots, a_n, d \in S_2$.

Obs.: For the case of Chebyshev approximation the oracle A_f is NP – using de la Vallée Poussin theorem, [Ko'86]. For L_1-approximation there is no algorithm for computing d_n without first computing p_n.

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Complexity Analysis (I/2)

We first compute the set Graph(α_f),

$\{\langle n, k, a_0, \ldots, a_n \rangle : \|p_n - \sum_0^n a_i x^i\|_{\max} \leq 2^k\}$

The algorithm for Graph(α_f),

1. Compute d_n with certain precision; (binary search for $d \in S_2$ such that $\langle n, d \rangle \in A_f$ but $\langle n, d + 2^{-\Phi(\omega_f, n, \Theta(n) - k) - 2} \rangle \notin A_f$)
2. Output *yes* iff $\langle \Phi(\omega_f, n, \Theta(n) + k), a_0, \ldots, a_n, d \rangle \in B_f$;

where,

$\|p\|_1 \leq 2^{-\Theta(n) - k} \to \|p\|_{\max} \leq 2^{-k}$.

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Complexity Analysis (I/3a)

Theorem. Let $f \in C[0,1]$ be a fixed polynomial-time computable continuous function. There exists a multi-valued function $\alpha_f : \mathbb{N} \times \mathbb{N} \to P_n$ such that,

$i)$ coefficients of $\alpha_f(n,k)$ have precision $\leq \Phi(\omega_f, n, \Theta(n) + k)$,

$ii)$ α_f is total,

$iii)$ $\langle a_0, \ldots, a_n \rangle \in \alpha_f(n,k) \to \|p_n - \sum a_i x^i\|_{\max} \leq 2^{-k}$,

$iv)$ Graph(α_f) $\in$ **P**$[A_f, B_f]$.

Proof. Let $s = \Phi(\omega_f, n, \Theta(n) + k)$. We define $\langle a_0, \ldots, a_n \rangle \in \alpha_f(n,k)$ iff,

$\langle n, k, a_0, \ldots, a_n \rangle \in$ Graph$(\alpha_f) \wedge \langle a_0, \ldots, a_n \rangle \in N_{n,s+1}$.

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Complexity Analysis (I/3b)

$ii)$ By the construction of $N_{n,s+1}$ we have that exists a $p \in N_{n,s+1}$ such that, $\|p_n - p\|_1 \leq 2^{-s-2}$.

By triangle inequality we get $\|f - p\|_1 - dist_1(f, P_n) \leq 2^{-s-2}$.

By the computation of d we get $|\ \|f - p\|_1 - d\ | \leq 2 \cdot 2^{-s-2}$ and p is accepted.

$iii)$ Suppose $\langle \Phi(\omega_f, n, \Theta(n) + k), a_0, \ldots, a_n, d \rangle \in B_f$.

This means, $|\ \|f - \sum a_i x^i\|_1 - d\ | \leq 3 \cdot 2^{-\Phi(\omega_f, n, \Theta(n)+k)-2}$

since d_n was computed with precision $2^{-\Phi(\omega_f, n, \Theta(n)+k)-2}$ we have,

$|\ \|f - \sum a_i x^i\|_1 - d_n\ | \leq 2^{-\Phi(\omega_f, n, \Theta(n)+k)}$

and by the definition of modulus of uniqueness Φ,

$\|p_n - \sum a_i x^i\|_1 \leq 2^{-\Theta(n)-k}$

which by def of Θ implies, $\|p_n - \sum a_i x^i\|_{\max} \leq 2^{-k}$.

Complexity Analysis (II/1)

We can avoid to compute d_n beforehand by using a slightly different oracle for the integration,

$$\langle k, p, \tilde{p} \rangle \in B_f \to \|f - p\|_1 \le \|f - \tilde{p}\|_1 + 3 \cdot 2^{-k-2}$$

$$\langle k, p, \tilde{p} \rangle \notin B_f \to \|f - p\|_1 \ge \|f - \tilde{p}\|_1 + 2 \cdot 2^{-k-2}.$$

The set Graph(α_f) is now define as,

$$\{\langle n, k, p \rangle : \forall \tilde{p} \in N_{n,s+1}\big(\langle s, p, \tilde{p} \rangle \in B_f\big)\},$$

where $s = \Phi(\omega_f, n, \Theta(n) + k)$.

In this way one obtains

- Graph(α) $\in$ **coNP**$[B_f]$,
- $p_n \in \Sigma_2^P[B_f]$.

Conclusions

- 1921 [Jackson] Proof of the uniqueness of best L_1-approximation.
- 1965 [Cheney] 'Elementary' proof of Jackson's theorem (highly non-constructive, WKL).
- 2001 [Koh, Oli] Fully explicit modulus of uniqueness.
- 2001 [Oli] First complexity upper bound for p_n.