Proof Mining in L_1-approximation

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(Extended Abstract)

1 Introduction

We report on the extraction (presented in [KO01]) of the first effective uniform modulus of uniqueness (this notion is defined in section 2) for best polynomial L_1-approximation of continuous functions from the ineffective proof of uniqueness due to Cheney. The extraction of the modulus is based on the technique of monotone functional interpretation (developed in [Koh96]) and is an instance of the following meta-theorem,[1]

Theorem 1.1 ([Koh93a], theorem 4.1) *Let X, K be $\boldsymbol{PA}^\omega$-definable Polish spaces, K compact and consider a sentence which can be written (when formalized in the language of $\boldsymbol{PA}^\omega$) in the form*

$$A := \forall n \in \mathbb{N}; x \in X; y \in K\ \exists k \in \mathbb{N}\, A_1(n, x, y, k),$$

where A_1 is a purely existential. Then the following rule holds:

$$\begin{cases} \boldsymbol{PA}^\omega + AC_{qf}^{1,0} + WKL \vdash \forall n \in \mathbb{N}; x \in X; y \in K\ \exists k \in \mathbb{N}\, A_1(n, x, y, k) \\ \textit{then one can extract a primitive recursive (in the sense of [Göd58]) term } \Phi \textit{ s.t.} \\ \boldsymbol{HA}^\omega \vdash \forall n \in \mathbb{N}; x \in X; y \in K \exists k \leq \Phi(n, x)\, A_1(n, x, y, k). \end{cases}$$

A crucial feature of the functional Φ above is that it does not depend on the element $y \in K$. It should also be noted that Φ depends *on the representation* of x as an element of X. In the present case X is the space of continuous functions on the unit interval (denoted by $C[0,1]$), and according to the representation of $C[0,1]$ [2] the elements $f \in C[0,1]$ are given together with a modulus of uniform continuity, ω_f,

$$\forall x, y \in [0,1] \forall \varepsilon \in \mathbb{Q}^*_+ (|x - y| < \omega_f(\varepsilon) \to |f(x) - f(y)| < \varepsilon),$$

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[*]Basic Research in Computer Science, funded by the Danish National Research Foundation.

[1]$\mathbf{PA}^\omega$ denotes Peano Arithmetic (with extensionality) in all finite types and $\mathbf{HA}^\omega$ its intuitionistic variant.

[2]We have to represent $C[0,1]$ w.r.t. the uniform norm $\|\cdot\|_\infty$ (although we below consider the L_1-norm) since $C[0,1]$ is a Polish space only with respect to the former.

which means that the extracted uniform modulus of uniqueness will depend a priori on such a modulus of uniform continuity of f.

In the theorem above, WKL denotes the non-computational principle "binary ('weak') König's lemma" and $AC^{1,0}_{qf}$ the axiom of choice for quantifier free formulas,

$$AC^{1,0}_{qf}\,:\,\forall f^1 \exists x^0 A_{qf}(f,x) \rightarrow \exists F^2 \forall f^1 A_{qf}(f,F(f)).$$

We show below that the uniqueness theorem for L_1-approximation (expressed as a sentence B) can be written in the form A (of meta-theorem 1.1), and the functional Φ guaranteed by the meta-theorem is exactly the uniform modulus of uniqueness. In order to apply theorem 1.1 we just have to show that B can be proved in the system $\mathcal{A} := \mathbf{PA}^\omega + AC^{1,0}_{qf} + WKL$ which can be achieved by showing that Cheney's proof of B can be formalized in $\mathcal{A}$. The latter fact was shown in [Koh90].

Note that our case study is twofold: we test a technique of proof mining (monotone functional interpretation in the present case) with respect to its applicability and feasibility, and at the same time we obtain new results in analysis.

In the next section we introduce some notions from L_1-approximation theory and present the uniform modulus of uniqueness Φ. In section 3 we show how Φ can be used to compute the best L_1-approximation of a given function $f \in C[0,1]$ by polynomials of degree $\leq n$.

2 The uniform modulus of uniqueness

For any given continuous function f on the interval $[0,1]$ (we write $f \in C[0,1]$) the L_1-*norm of* f is defined as,

$$||f||_1 := \int_0^1 |f(x)|\,dx.$$

Given a function $f \in C[0,1]$ we define the L_1-*distance of* f from some subspace $H \subset C[0,1]$ by,

$$dist_1(f,H) := \inf_{p\in H} \|f-p\|_1.$$

By 'the problem of L_1-approximation from H' we mean the problem of finding, for a given $f \in C[0,1]$, an element $p \in H$ such that $\|f-p\|_1 = dist_1(f,H)$. L_1-approximation (or 'approximation in the mean') has been extensively studied in numerical mathematics since 1859 when was first considered by Chebycheff (see [Pin89] for a comprehensive survey). The space of polynomials up to degree n (including n) is denoted by P_n. In 1921, Jackson [Jac21] proved that the problem of L_1-approximation from P_n has a unique solution, i.e. for any given function $f \in C[0,1]$ and for a fixed $n \in \mathbb{N}$ there exists a unique polynomial $p_n \in P_n$ such that

$$\|f-p_n\|_1 = dist_1(f,P_n).$$

In 1965, Cheney (cf. [Che65], [Che66]) simplified (in the sense that he eliminated the use of measure theory) and generalized Jackson's uniqueness proof to arbitrary Haar subspaces[3] of $C[0,1]$. This is the proof we analyze in [KO01].

[3] H is a n-dimensional Haar subspace of $C[0,1]$ if 0 is the only element with n roots. Note that P_n is a Haar subspace of $C[0,1]$ of dimension $n+1$.

The uniqueness of the best L_1-approximation from the space P_n can be written as,

$$\forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in P_n(\bigwedge_{i=1}^{2}(\|f-p_i\|_1 =_{\mathbb{R}} dist_1(f,P_n)) \to p_1 = p_2).$$

which is equivalent in the system $\mathcal{A}$ to,[4]

$$(*)\ \forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in P_n; k \in \mathbb{N} \exists d \in \mathbb{N}$$
$$(\bigwedge_{i=1}^{2}(\|f-p_i\|_1 - dist_1(f,P_n) \leq 2^{-d}) \to \|p_1 - p_2\|_1 \leq 2^{-k}).$$

It is important to note that the best L_1-approximation of any given continuous function $f \in C[0,1]$ from P_n lives in a compact subspace of $C[0,1]$, for instance $K_{f,n} := \{p \in P_n : \|p\|_1 \leq 2\|f\|_1\}$, i.e. for a fixed $f \in C[0,1]$ its best L_1-approximation p_n must belong to $K_{f,n}$ for if not we would have $\|p_n\|_1 > 2\|f\|_1$ which, by the triangle inequality for the L_1-norm, implies $\|f - p_n\|_1 > \|f\|_1$, and 0 would be better approximant than p_n, a contradiction. Therefore, $(*)$ can be written as,

$$\forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n}; k \in \mathbb{N} \exists d \in \mathbb{N}$$
$$(\bigwedge_{i=1}^{2}(\|f-p_i\|_1 - dist_1(f,P_n) \leq 2^{-d}) \to \|p_1 - p_2\|_1 < 2^{-k}),$$

which has the exact form of the formula A from theorem 1.1. Now, since Cheney's uniqueness proof can be formalized in $\mathcal{A}$, metatheorem 1.1 guarantees the existence of a functional Φ realizing $\exists d$ depending only on f, n and k, i.e. independent of p_1 and p_2,

$$(**)\ \forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n}; k \in \mathbb{N}$$
$$(\bigwedge_{i=1}^{2}(\|f-p_i\|_1 - dist_1(f,P_n) \leq 2^{-\Phi(f,n,k)}) \to \|p_1 - p_2\|_1 < 2^{-k}).$$

In fact, Φ can be easily extended to a modulus on the whole space P_n. In [Koh93a] and [Koh93b], such functionals were called *uniform modulus of uniqueness*.[5]

Notice that, although the first proof of uniqueness of best L_1-approximation was given already in 1921, no effective uniform modulus of uniqueness had ever been presented before. In [Bjö75] (see also [Bjö79]) Björnestal proved that a uniform modulus of uniqueness having the form $(k+c)+\omega_f(k+c)$ exists, for some constant c depending at most on f and n. Kroo ([Kro78] and [Kro81]) improved this result by showing that c needed only to depend on the modulus of uniform continuity of f (but not on any particular value of f). Moreover, he showed that the k-dependency, i.e. $k + \omega_f(k)$, is optimal. Note, however, that neither Björnestal nor Kroo presented the constant c, which means that only the ε-dependency of the uniform modulus of uniqueness was known before.

By applying the techniques of monotone functional interpretation to Cheney's ineffective uniqueness proof we obtained the following explicit uniform modulus of uniqueness (with $\varepsilon \in \mathbb{Q}_+^*$ instead of 2^{-k}),

[4] It is worth noticing that, according to the representation of real numbers by Cauchy sequences with fixed rate of convergence, in this way, equality between real numbers is expressed by a $\forall$-statement.

[5] Special cases of such moduli have been studied extensively in approximation theory under the heading of "strong uniqueness" (see [BL95]).

Theorem 2.1 *Let*

$$\Phi(\omega, n, \varepsilon) := \min\{\frac{c_n\varepsilon}{3^{n+2}(n+1)^{n+1}}, \frac{c_n\varepsilon}{2}\omega_n(\frac{c_n\varepsilon}{2})\}, \text{ where}$$
$$c_n := \frac{\lfloor n/2\rfloor!\lceil n/2\rceil!}{2^{n+3}3^{n^2+2n}(n+1)^{n^2+2n+1}} \text{ and}$$
$$\omega_n(\varepsilon) := \min\{\omega(\frac{\varepsilon}{4}), \frac{\varepsilon}{40(n+1)^4\lceil\frac{1}{\omega(1)}\rceil}\}.$$

The functional Φ is a uniform modulus of uniqueness for the best L_1-approximation of any function f in $C[0,1]$ (having modulus of uniform continuity ω) from P_n, i.e.

$$\forall n \in \mathbb{N}; p_1, p_2 \in P_n; \varepsilon \in \mathbb{Q}_+^*(\bigwedge_{i=1}^{2}(\|f - p_i\|_1 - dist_1(f, P_n) < \Phi(\omega, n, \varepsilon)) \to \|p_1 - p_2\|_1 \leq \varepsilon).$$

Let $\mathcal{P}(f, n)$ denote the projection operator which assigns to any given function $f \in C[0,1]$ and any $n \in \mathbb{N}$ the best L_1-approximation of $f \in C[0,1]$ from P_n, i.e,

$$\mathcal{P}(f, n) := p_n, \text{ such that } \|f - p_n\|_1 = dist_1(f, P_n).$$

As a corollary of proposition 5.4 from [Koh93a] and theorem 2.1 above we get,

Theorem 2.2 *Let $\Phi_P(\omega, n, k) := \frac{\Phi(\omega,n,k)}{2}$, Φ as defined in Theorem 2.1. Then, Φ_P is a modulus of pointwise continuity for the operator $\mathcal{P}(f, n)$ for all $f \in C[0,1]$ with modulus of uniform continuity ω, i.e.,*

$$\forall \tilde{f} \in C[0,1]; n \in \mathbb{N}; q \in \mathbb{Q}_+^*(\|f - \tilde{f}\|_1 < \Phi_P(\omega, n, \varepsilon) \to \|\mathcal{P}(f, n) - \mathcal{P}(\tilde{f}, n)\|_1 \leq \varepsilon).$$

3 Computing the best L_1-approximation

Now we present one of the applications of the uniform modulus of uniqueness. First we define another norm. Let $a := (a_0, a_1, \ldots, a_n)$ be an $(n+1)$-dimensional tuple of real numbers. The *max-norm of* a is defined as,

$$\|a\|_{max} := \max\{|a_0|, \ldots, |a_n|\}.$$

Since the polynomials of P_n can be viewed as a $(n+1)$-tuples of real numbers (taking the coefficients as the elements of the tuple) it makes sense to speak about the max-norm of $p \in P_n$. In the same way we treat $(n+1)$-tuples of rational numbers as elements of P_n.

Definition 3.1 *An operator $B_f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^{n+1}$ computes the sequence of unique best L_1-approximations, $(p_n)_{n\in\mathbb{N}}$, of a function $f \in C[0,1]$ from P_n if for any given $n, k \in \mathbb{N}$ it generates an $(n+1)$-tuple of rationals $B_f(n,k)$ (which can be viewed as a polynomial with rational coefficients in P_n) such that $\|p_n - B_f(n,k)\|_{max} < 2^{-k}$.*

Let $f \in C[0,1]$ be fixed and assume p_n is the best L_1-approximation of f from P_n. Taking p_2 to be p_n in $(**)$ we have,

$$(1)\ \forall f \in C[0,1]; n \in \mathbb{N}; p \in P_n; k \in \mathbb{N} \\ (\|f-p\|_1 - dist_1(f,P_n) \le 2^{-\Phi(f,n,k)} \to \|p-p_n\|_1 < 2^{-k}).$$

In order to define B_f we need, besides the uniform modulus of uniqueness Φ, a functional $\Psi(f,n,k)$ such that,

$$(2)\ \forall f \in C[0,1]; n,k \in \mathbb{N}(\Psi(f,n,k) \in \mathbb{Q}^{n+1} \wedge \|f-\Psi(f,n,k)\|_1 < dist_1(f,P_n) + 2^{-k}),$$

and a function $\Theta(n,k)$ satisfying,

$$(3)\ \forall n \in \mathbb{N}; p \in P_n; k \in \mathbb{N}(\|p\|_1 < 2^{-\Theta(n,k)} \to \|p\|_{max} < 2^{-k}).$$

It is clear now, by (1), (2) and (3), that the functional defined by

$$B_f(k,n) := \Psi(f,n,\Phi(f,n,\Theta(n,k)))$$

generates the double sequence $(p_{n,k})_{n,k\in\mathbb{N}}$. Note that, if the modulus of uniqueness Φ depended on p_1 and p_2, the functional $B_f(k,n)$ could not be defined, since in order to get (1) we would need to have p_n already.

It is only left to show how the functionals Ψ and Θ can be defined. We sketch here only the definition of $\Psi(f,n,k)$: For $f \in C[0,1]$ and $n,k \in \mathbb{N}$ fixed, we show how to find a polynomial $p \in P_n$ such that $\|f-p\|_1 < dist_1(f,P_n) + 2^{-k}$. As it was pointed out above, such p lives in the compact space $K_{f,n}$. The idea is to build a finite net of elements containing some elements of $K_{f,n}$ (called k-net) such that at least one of them (say p') is $(k+1)$-*close* to p_n, i.e. $\|p_n - p'\|_1 \le 2^{-k-1}$, which by triangle inequality ($\|f-p'\|_1 \le \|f-p_n\|_1 + \|p_n - p'\|_1$) yields $(*)$ $\|f-p'\|_1 \le dist_1(f,P_n) + 2^{-k-1}$. Once we have the k-net we compute $\|f-p\|_1$ for any element p in the net with $(k+1)$-precision and we take a $\tilde{p}$ which gives the minimum value (e.g. the one with smallest code). Since the L_1-norm was computed with $(k+1)$-precision, by $(*)$, we have, $\|f-\tilde{p}\|_1 \le dist_1(f,P_n) + 2^{-k}$.

This important application of moduli of uniqueness for the computation of unique solutions of existence statements was first investigated and used in [Koh93a]. The computational complexity (in the sense of [Ko86]) of generating the best L_1-approximation of a given $f \in C[0,1]$ from P_n will be analyzed in [Oli].

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