Realizability

Lecture 3: General Structure of Realizability

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4th Nordic Logic Summer School (Bergen)

Lecture 2: Homework

Find IL derivation of

$$(\neg\neg A \to \neg\neg B) \to \neg\neg(A \to B)$$

Lecture 2: Homework

MP (Markov Principle). $\forall m(\neg \forall n A_{\mathrm{qf}}(m,n) \rightarrow \exists n \neg A_{\mathrm{qf}}(m,n))$

Show, using **mr**, that

$\mathbf{HA}^{\omega} \nvdash \mathbf{MP}$

Plan

- Lecture 1: Kleene (number) realizability
- Lecture 2: Kreisel modified realizability
- **Lecture 3: General structure of realizability**
- Lecture 4: Realizability of arithmetic and analysis

Lecture 3: General Structure of Realizability

- Pointwise Realizability
- Linear Logic (LL) / Girard Translations
- Realizability Interpretation of LL
- Recovering Interpretation of IL
- Unifying Functional Interpretations

Pointwise Realizability

Modified Realizability

$$\langle\rangle\ \mathbf{mr}\ s = t \ \equiv\ s = t$$

$$\mathbf{x}, \mathbf{y}\ \mathbf{mr}\ A \wedge B \ \equiv\ (\mathbf{x}\ \mathbf{mr}\ A) \wedge (\mathbf{y}\ \mathbf{mr}\ B)$$

$$\mathbf{f}\ \mathbf{mr}\ A \to B \ \equiv\ \forall \mathbf{x}(\mathbf{x}\ \mathbf{mr}\ A \to \mathbf{fx}\ \mathbf{mr}\ B)$$

$$\mathbf{f}\ \mathbf{mr}\ \forall z^{\tau} A \ \equiv\ \forall z^{\tau}(\mathbf{f}z\ \mathbf{mr}\ A)$$

$$a, \mathbf{x}\ \mathbf{mr}\ \exists z^{\tau} A \ \equiv\ \mathbf{x}\ \mathbf{mr}\ A[a/z]$$

Pointwise Modified Realizability

$$\langle\rangle \ \mathbf{pmr}_{\langle\rangle} \ s = t \ \equiv \ s = t$$

$$\mathbf{x}, \mathbf{v} \ \mathbf{pmr}_{\mathbf{y},\mathbf{w}} \ A \wedge B \ \equiv \ (\mathbf{x} \ \mathbf{pmr}_{\mathbf{y}} \ A) \wedge (\mathbf{v} \ \mathbf{pmr}_{\mathbf{w}} \ B)$$

$$\mathbf{f} \ \mathbf{pmr}_{\mathbf{x},\mathbf{w}} \ A \to B \ \equiv \ \forall \mathbf{y}(\mathbf{x} \ \mathbf{pmr}_{\mathbf{y}} \ A) \to \mathbf{fx} \ \mathbf{pmr}_{\mathbf{w}} \ B$$

$$\mathbf{f} \ \mathbf{pmr}_{a,\mathbf{y}} \ \forall z^{\tau} A \ \equiv \ \mathbf{f}a \ \mathbf{pmr}_{\mathbf{y}} \ A[a/z]$$

$$a, \mathbf{x} \ \mathbf{pmr}_{\mathbf{y}} \ \exists z^{\tau} A \ \equiv \ \mathbf{x} \ \mathbf{pmr}_{\mathbf{y}} \ A[a/z]$$

$$
\begin{aligned}
\langle\rangle \ \mathbf{pmr}\ s = t \ &\equiv\ s = t \\
\mathbf{x}, \mathbf{v}\ \mathbf{pmr}_{\mathbf{y},\mathbf{w}}\ A \wedge B \ &\equiv\ (\mathbf{x}\ \mathbf{pmr}_{\mathbf{y}}\ A) \wedge (\mathbf{v}\ \mathbf{pmr}_{\mathbf{w}}\ B) \\
\mathbf{f}\ \mathbf{pmr}_{\mathbf{x},\mathbf{w}}\ A \to B \ &\equiv\ \forall \mathbf{y}(\mathbf{x}\ \mathbf{pmr}_{\mathbf{y}}\ A) \to \mathbf{fx}\ \mathbf{pmr}_{\mathbf{w}}\ B \\
\mathbf{f}\ \mathbf{pmr}_{a,\mathbf{y}}\ \forall z^{\tau} A \ &\equiv\ \mathbf{f}a\ \mathbf{pmr}_{\mathbf{y}}\ A[a/z] \\
a, \mathbf{x}\ \mathbf{pmr}_{\mathbf{y}}\ \exists z^{\tau} A \ &\equiv\ \mathbf{x}\ \mathbf{pmr}_{\mathbf{y}}\ A[a/z]
\end{aligned}
$$

$$
\mathbf{x}\ \mathbf{mr}\ A \Leftrightarrow \forall \mathbf{y}\boxed{(\mathbf{x}\ \mathbf{pmr}_{\mathbf{y}}\ A)}
$$

More primitive notion/concept!

Linear Logic

Linear (Affine) Logic (LL)

Additive conjunction

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B} \;(\&\text{I})$$

$$\frac{\Gamma \vdash A_1 \,\&\, A_2}{\Gamma \vdash A_i} \;(\&\text{E})$$

$$\frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \oplus A_2} \;(\oplus\text{I})$$

$$\frac{\Gamma \vdash A \oplus B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C} \;(\oplus\text{E})$$

Additive disjunction

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \;(\otimes\text{I})$$

$$\frac{\Gamma \vdash A \otimes B \quad \Delta, A, B \vdash C}{\Gamma, \Delta \vdash C} \;(\otimes\text{E})$$

Multiplicative conjunction

Exponentials

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B} \text{ (weakening)} \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B} \text{ (contraction)}$$

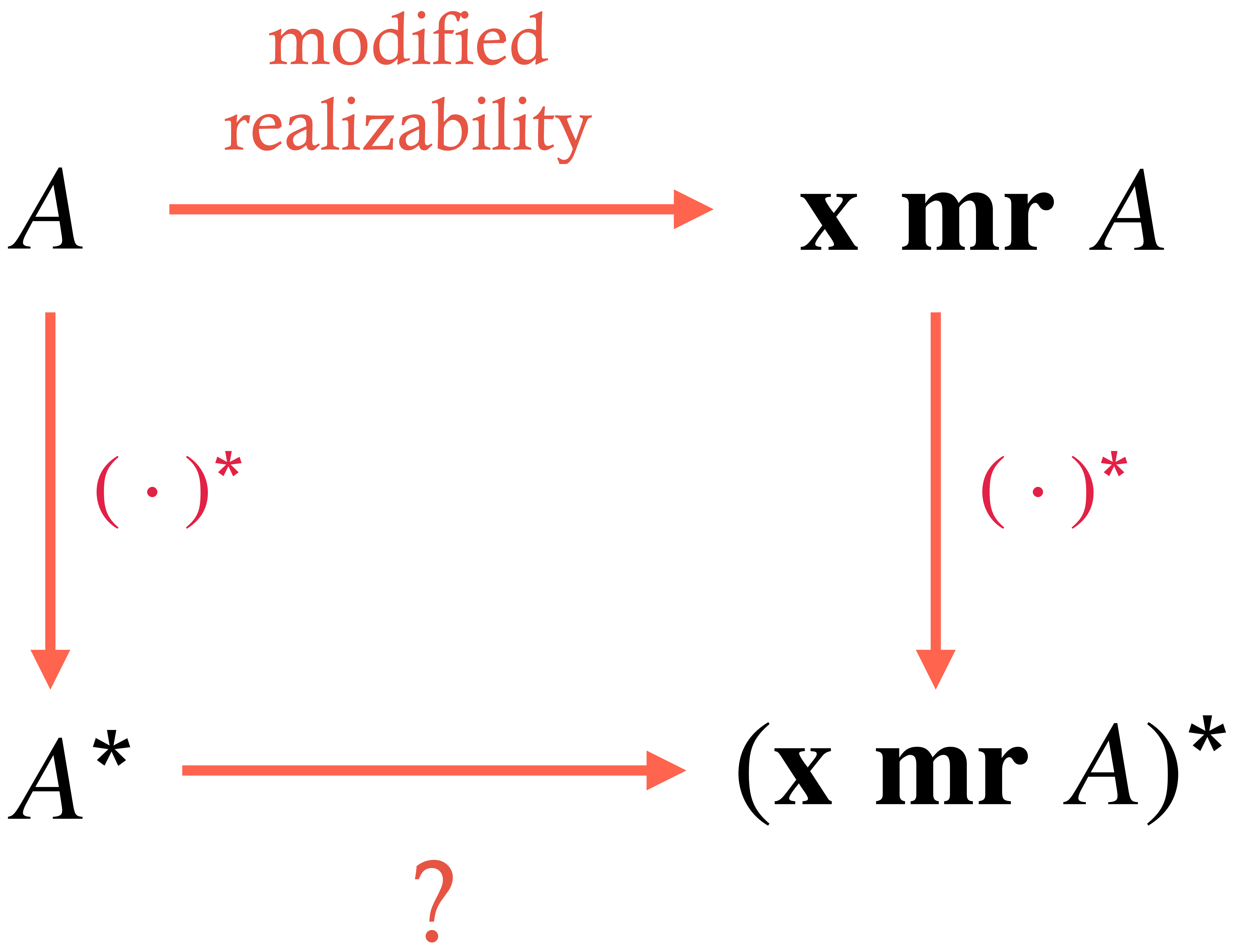
$$\frac{\Gamma \vdash !A}{\Gamma \vdash A} \text{ (dereliction)} \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A} \text{ (promotion)}$$

call-by-name translation		
$(A \wedge B)^*$	$\equiv$	$A^* \,\&\, B^*$
$(A \vee B)^*$	$\equiv$	$!A^* \oplus !B^*$
$(A \rightarrow B)^*$	$\equiv$	$!A^* \multimap B^*$
$(\forall z\, A)^*$	$\equiv$	$\forall z\, A^*$
$(\exists z\, A)^*$	$\equiv$	$\exists z !A^*$

call-by-value translation		
$(A \wedge B)^\circ$	$\equiv$	$A^\circ \otimes B^\circ$
$(A \vee B)^\circ$	$\equiv$	$A^\circ \oplus B^\circ$
$(A \rightarrow B)^\circ$	$\equiv$	$!(A^\circ \multimap B^\circ)$
$(\forall z\, A)^\circ$	$\equiv$	$!\forall z\, A^\circ$
$(\exists z\, A)^\circ$	$\equiv$	$\exists z\, A^\circ$

$$\Gamma \vdash_{\mathbf{IL}} A \quad \Rightarrow \quad !\Gamma^* \vdash_{\mathbf{LL}} A^*$$

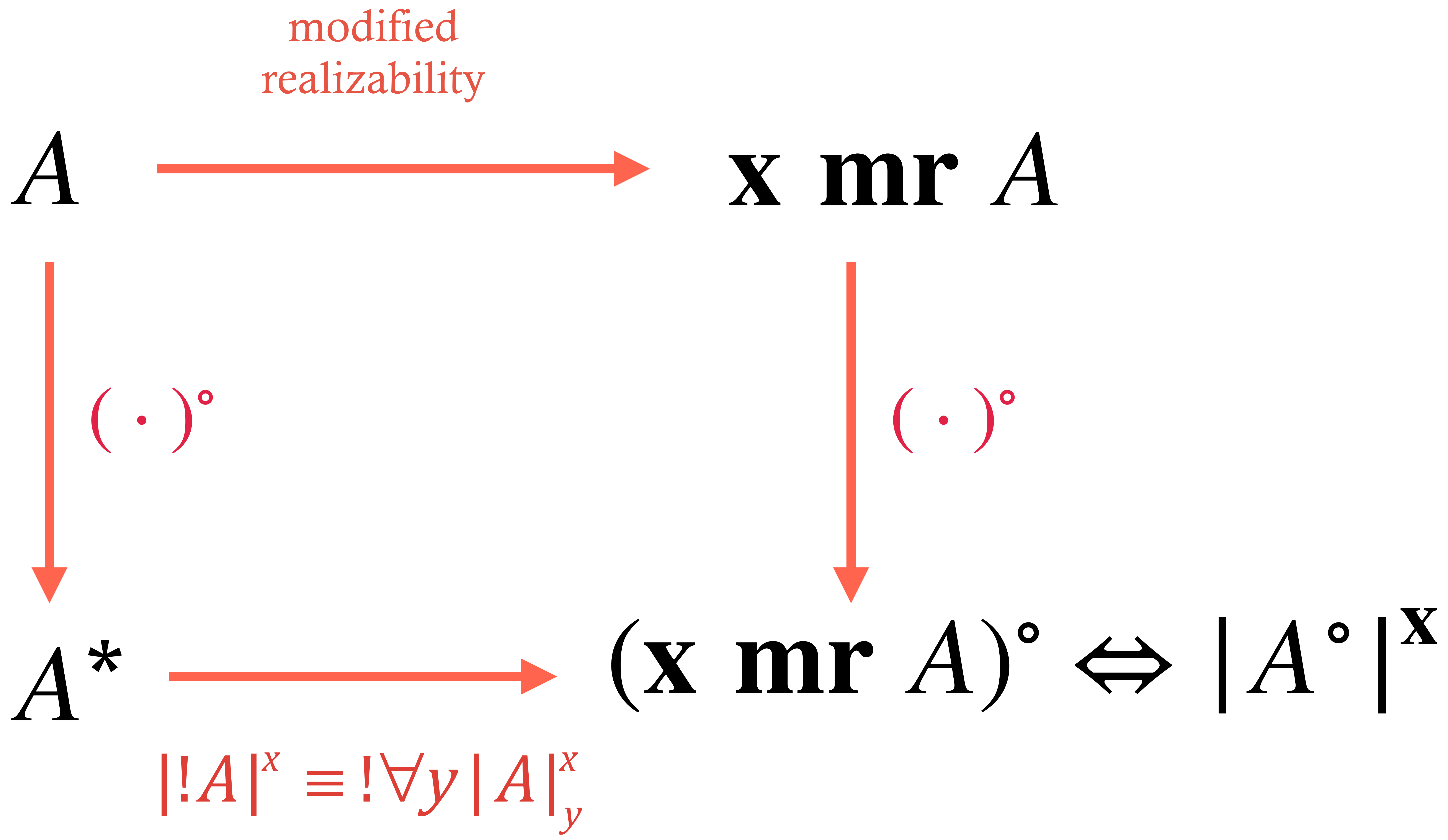
$$\Gamma \vdash_{\mathbf{IL}} A \quad \Rightarrow \quad \Gamma^\circ \vdash_{\mathbf{LL}} A^\circ$$


modified realizability
A
x mr A
(·)*
(·)*
A*
(x mr A)*
?

Realizability Interpretation of LL

$$
\begin{aligned}
|s = t| \quad &\equiv \quad s = t \\
|A \otimes B|^{\mathbf{x},\mathbf{v}}_{\mathbf{y},\mathbf{w}} \quad &\equiv \quad |A|^{\mathbf{x}}_{\mathbf{y}} \otimes |B|^{\mathbf{v}}_{\mathbf{w}} \\
|A \,\&\, B|^{\mathbf{x},\mathbf{v}}_{n,\mathbf{y},\mathbf{w}} \quad &\equiv \quad (n = 0 \,\&\, |A|^{\mathbf{x}}_{\mathbf{y}}) \oplus (n \neq 0 \,\&\, |B|^{\mathbf{v}}_{\mathbf{w}}) \\
|A \oplus B|^{n,\mathbf{x},\mathbf{v}}_{\mathbf{y},\mathbf{w}} \quad &\equiv \quad (n = 0 \,\&\, |A|^{\mathbf{x}}_{\mathbf{y}}) \oplus (n \neq 0 \,\&\, |B|^{\mathbf{v}}_{\mathbf{w}}) \\
|A \multimap B|^{\mathbf{f},\mathbf{g}}_{\mathbf{x},\mathbf{w}} \quad &\equiv \quad |A|^{\mathbf{x}}_{\mathbf{gxw}} \multimap |B|^{\mathbf{fx}}_{\mathbf{w}} \\
|\forall z^{\tau} A|^{\mathbf{f}}_{a,\mathbf{y}} \quad &\equiv \quad |A[a/z]|^{\mathbf{f}a}_{\mathbf{y}} \\
|\exists z^{\tau} A|^{a,\mathbf{x}}_{\mathbf{y}} \quad &\equiv \quad |A[a/z]|^{\mathbf{x}}_{\mathbf{y}}
\end{aligned}
$$

Recovering Interpretation of IL


modified realizability
A
$\mathbf{x}\ \mathbf{mr}\ A$
$(\cdot)^\circ$
$(\cdot)^\circ$
A^*
$(\mathbf{x}\ \mathbf{mr}\ A)^\circ \Leftrightarrow |A^\circ|^{\mathbf{x}}$
$|!A|^x \equiv !\forall y\, |A|^x_y$

Unifying Functional Interpretations

Modified Realizability

$$\begin{aligned}
\langle\rangle \ \mathbf{mr}\ s = t &\equiv s = t \\
\mathbf{x}, \mathbf{y}\ \mathbf{mr}\ A \wedge B &\equiv (\mathbf{x}\ \mathbf{mr}\ A) \wedge (\mathbf{y}\ \mathbf{mr}\ B) \\
n, \mathbf{x}, \mathbf{y}\ \mathbf{mr}\ A \vee B &\equiv (n = 0 \to \mathbf{x}\ \mathbf{mr}\ A) \wedge (n \neq 0 \to \mathbf{y}\ \mathbf{mr}\ B) \\
\mathbf{f}\ \mathbf{mr}\ A \to B &\equiv \forall \mathbf{x}(\mathbf{x}\ \mathbf{mr}\ A \to \mathbf{f}\mathbf{x}\ \mathbf{mr}\ B) \\
\mathbf{f}\ \mathbf{mr}\ \forall z^{\tau} A &\equiv \forall z^{\tau}(\mathbf{f}z\ \mathbf{mr}\ A) \\
a, \mathbf{x}\ \mathbf{mr}\ \exists z^{\tau} A &\equiv \mathbf{x}\ \mathbf{mr}\ A[a/z]
\end{aligned}$$

Modified Realizability with Truth

$$
\begin{aligned}
\langle\rangle \ \mathbf{mrt}\ s = t &\equiv s = t \\
\mathbf{x}, \mathbf{y}\ \mathbf{mrt}\ A \wedge B &\equiv (\mathbf{x}\ \mathbf{mrt}\ A) \wedge (\mathbf{y}\ \mathbf{mrt}\ B) \\
n, \mathbf{x}, \mathbf{y}\ \mathbf{mrt}\ A \vee B &\equiv (n = 0 \to \mathbf{x}\ \mathbf{mrt}\ A) \wedge (n \neq 0 \to \mathbf{y}\ \mathbf{mrt}\ B) \\
\mathbf{f}\ \mathbf{mrt}\ A \to B &\equiv \forall \mathbf{x}(\mathbf{x}\ \mathbf{mrt}\ A \to \mathbf{f}\mathbf{x}\ \mathbf{mrt}\ B) \wedge (A \to B) \\
\mathbf{f}\ \mathbf{mrt}\ \forall z^\tau A &\equiv \forall z^\tau(\mathbf{f}z\ \mathbf{mrt}\ A) \wedge \forall z^\tau A \\
a, \mathbf{x}\ \mathbf{mrt}\ \exists z^\tau A &\equiv \mathbf{x}\ \mathbf{mrt}\ A[a/z]
\end{aligned}
$$

q-Modified Realizability

$$\langle\rangle \textbf{ qmr } s = t \equiv s = t$$

$$\mathbf{x}, \mathbf{y} \textbf{ qmr } A \wedge B \equiv (\mathbf{x} \textbf{ qmr } A) \wedge (\mathbf{y} \textbf{ qmr } B)$$

$$n, \mathbf{x}, \mathbf{y} \textbf{ qmr } A \vee B \equiv (n = 0 \to (\mathbf{x} \textbf{ qmr } A) \wedge A) \wedge (n \neq 0 \to (\mathbf{y} \textbf{ qmr } B) \wedge B)$$

$$\mathbf{f} \textbf{ qmr } A \to B \equiv \forall \mathbf{x}((\mathbf{x} \textbf{ qmr } A) \wedge A \to \mathbf{fx} \textbf{ qmr } B)$$

$$\mathbf{f} \textbf{ qmr } \forall z^{\tau} A \equiv \forall z^{\tau}(\mathbf{f}z \textbf{ qmr } A)$$

$$a, \mathbf{x} \textbf{ qmr } \exists z^{\tau} A \equiv (\mathbf{x} \textbf{ qmr } A[a/z]) \wedge A[a/z]$$

!A	Trans.	Interpretation
$\vert !A\vert^x \equiv !\forall y \vert A\vert^x_y$	$(\cdot)^\circ$	Kreisel modified realizability
$\vert !A\vert^x \equiv !\forall y \vert A\vert^x_y$	$(\cdot)^*$	Pointwise modified realizability
$\vert !A\vert^x \equiv !\forall y \vert A\vert^x_y \otimes !A$	$(\cdot)^\circ$	modified realizability with truth
$\vert !A\vert^x \equiv !\forall y \vert A\vert^x_y \otimes !A$	$(\cdot)^*$	q-variant of modified realizability

call-by-name translation		
$(A \wedge B)^*$	$\equiv$	$A^* \,\&\, B^*$
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$(A \rightarrow B)^*$	$\equiv$	$!A^* \multimap B^*$
$(\forall z A)^*$	$\equiv$	$\forall z A^*$
$(\exists z A)^*$	$\equiv$	$\exists z !A^*$

call-by-value translation		
$(A \wedge B)^\circ$	$\equiv$	$A^\circ \otimes B^\circ$
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$(A \rightarrow B)^\circ$	$\equiv$	$!(A^\circ \multimap B^\circ)$
$(\forall z A)^\circ$	$\equiv$	$!\forall z A^\circ$
$(\exists z A)^\circ$	$\equiv$	$\exists z A^\circ$

$$!A^* \Leftrightarrow_{\mathbf{LL}} A^\circ$$

!A	Trans.	Interpretation
$\lvert !A \rvert^x \equiv !\forall y \lvert A \rvert^x_y$	$(\cdot)^\circ$	Kreisel modified realizability
$\lvert !A \rvert^x \equiv !\forall y \lvert A \rvert^x_y \otimes !A$	$(\cdot)^\circ$	modified realizability with truth
$\lvert !A \rvert^x \equiv !\forall y \lvert A \rvert^x_y \otimes !A$	$(\cdot)^*$	q-variant of modified realizability
$\lvert !A \rvert^x_a \equiv !\forall y \in a \lvert A \rvert^x_y$	$(\cdot)^*$ or $(\cdot)^\circ$	Diller-Nahm interpretation
$\lvert !A \rvert^x_a \equiv ! \lvert A \rvert^x_a$	$(\cdot)^*$ or $(\cdot)^\circ$	Gödel's Dialectica interpretation
$\lvert !A \rvert^x_a \equiv !\forall y \in a \lvert A \rvert^x_y \otimes !A$	$(\cdot)^\circ$	Diller-Nahm with truth

J. Gaspar and P. Oliva, **Proof interpretations with truth**, MLQ, 56(6):591-610, 2010