Realizability

Lecture 1: Kleene Realizability

Dr Paulo Oliva
Mon, 13 June 2022
4th Nordic Logic Summer School (Bergen)

The Journal of Symbolic Logic
Volume 10, Number 4, Dec. 1945

ON THE INTERPRETATION OF INTUITIONISTIC NUMBER THEORY

S. C. KLEENE

The purpose of this article is to introduce the notion of "recursive realizability."[1]

First we define by a metamathematical recursion a formula e Ⓡ A depending on A, which represents the realization predicate "*e* realizes A."

1. e Ⓡ F is $e=0$ & F.
2. e Ⓡ A & B is $\exists a \exists b[e=2^a \cdot 3^b$ & a Ⓡ A & b Ⓡ B].
3. e Ⓡ A ∨ B is $\exists a[e=2^0 \cdot 3^a$ & a Ⓡ A] ∨ $\exists b[e=2^1 \cdot 3^b$ & b Ⓡ B].
4. e Ⓡ A ⊃ B is $\forall a$[a Ⓡ A ⊃ $\exists y[T_1(e,a,y)$ & U(y) Ⓡ B]].
5. e Ⓡ ¬A is e Ⓡ A ⊃ $1=0$.
6. e Ⓡ $\exists x$A(x) is $\exists x \exists a[e=2^x \cdot 3^a$ & a Ⓡ A(x)].
7. e Ⓡ $\forall x$A(x) is $\forall x \exists y[T_1(e,x,y)$ & U(y) Ⓡ A(x)].

THE JOURNAL OF SYMBOLIC LOGIC
Volume 10, Number 4, Dec. 1945

ON THE INTERPRETATION OF INTUITIONISTIC NUMBER THEORY

S. C. KLEENE

The purpose of this article is to introduce the notion of "recursive realizability."[1]

First we define by a metamathematical recursion a formula e ® A depending on A, which represents the realization predicate "*e* realizes A."

1. e ® F is $e=0$ & F.
2. e ® A & B is $\exists a \exists b[e=2^a \cdot 3^b$ & a ® A & b ® B].
3. e ® A ∨ B is $\exists a[e=2^0 \cdot 3^a$ & a ® A] ∨ $\exists b[e=2^1 \cdot 3^b$ & b ® B].
4. e ® A ⊃ B is $\forall a$[a ® A ⊃ $\exists y[T_1(e,a,y)$ & U(y) ® B]].
5. e ® ¬A is e ® A ⊃ $1=0$.
6. e ® $\exists x$A(x) is $\exists x \exists a[e=2^x \cdot 3^a$ & a ® A(x)].
7. e ® $\forall x$A(x) is $\forall x \exists y[T_1(e,x,y)$ & U(y) ® A(x)].

Plan

- **Lecture 1: Kleene (number) realizability**
- Lecture 2: Kreisel modified realizability
- Lecture 3: General structure of realizability
- Lecture 4: Realizability of arithmetic and analysis

Lecture 1: Kleene Realizability

- Intuitionistic logic and Heyting arithmetic
- BHK Interpretation
- Slash Translation
- Kleene Realizability
- Soundness and Characterisation
- Applications

Intuitionistic Logic

Intuitionistic Logic

$\neg A \equiv A \to \bot$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\wedge\mathrm{I}) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ (\to\mathrm{I}) \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\ (\vee\mathrm{I})$$

$$\frac{\Gamma \vdash A_1 \wedge A_2}{\Gamma \vdash A_i}\ (\wedge\mathrm{E}) \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash A \to B}{\Gamma \vdash B}\ (\to\mathrm{E}) \qquad \frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}\ (\vee\mathrm{E})$$

$$\frac{}{\Gamma, A \vdash A}\ (\mathrm{Ax}) \qquad \frac{\Gamma \vdash A(x)}{\Gamma \vdash \forall x A(x)}\ (\forall\mathrm{I}) \qquad \frac{\Gamma \vdash A(t)}{\Gamma \vdash \exists x A(x)}\ (\exists\mathrm{I})$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}\ (\mathrm{EFQ}) \qquad \frac{\Gamma \vdash \forall x A(x)}{\Gamma \vdash A(t)}\ (\forall\mathrm{E}) \qquad \frac{\Gamma \vdash \exists x A(x) \quad \Gamma, A(x) \vdash B}{\Gamma \vdash B}\ (\exists\mathrm{E})$$

Homework

Find IL derivation of

$$\neg\neg A \to \neg\neg B \to \neg\neg(A \land B)$$

The Brouwer-Heyting-Kolmogorov Interpretation (BHK Interpretation)

BHK Interpretation

Aim is to "explain" the meaning of "intuitionistic" proof

A proof of $A \wedge B$ consists of a proof of A and a proof of B

A proof of $A \vee B$ consists of either a proof of A or a proof of B

A proof of $A \rightarrow B$ is a "construction" which allows us to transform a proof of A into a proof of B

A proof of $\forall x^D A$ is a "construction" which allows us to transform a proof of $d \in D$ (intended range of x) into a proof of $A[d/x]$

A proof of $\exists x^D A$ consists of giving $d \in D$ and a proof of $A[d/x]$

https://plato.stanford.edu/entries/intuitionistic-logic-development/

The Slash Translation

Slash Translation

$$
\begin{aligned}
\Gamma | A_{\mathrm{at}} &\equiv \Gamma \vdash A_{\mathrm{at}} \\
\Gamma | A \wedge B &\equiv \Gamma | A \text{ and } \Gamma | B \\
\Gamma | A \vee B &\equiv \Gamma | A \text{ or } \Gamma | B \\
\Gamma | A \to B &\equiv \text{if } \Gamma | A \text{ then } \Gamma | B \\
\Gamma | \forall x^{D} A &\equiv \Gamma | A[d/x] \text{ for all } d \in D \\
\Gamma | \exists x^{D} A &\equiv \Gamma | A[d/x] \text{ for some } d \in D
\end{aligned}
$$

$$
\begin{aligned}
\Gamma \,|\, A_{\text{at}} &\equiv \Gamma \vdash A_{\text{at}} \\
\Gamma \,|\, A \wedge B &\equiv \Gamma \,|\, A \text{ and } \Gamma \,|\, B \\
\Gamma \,|\, A \vee B &\equiv \Gamma \,|\, A \text{ or } \Gamma \,|\, B \\
\Gamma \,|\, A \to B &\equiv \text{if } \Gamma \,|\, A \text{ then } \Gamma \,|\, B \\
\Gamma \,|\, \forall x^D A &\equiv \Gamma \,|\, A[d/x] \text{ for all } d \in D \\
\Gamma \,|\, \exists x^D A &\equiv \Gamma \,|\, A[d/x] \text{ for some } d \in D
\end{aligned}
$$

Soundness Theorem. If $\Gamma \,|\, \Gamma$ and $\Gamma \vdash A$ then $\Gamma \,|\, A$

Heyting Arithmetic

Intuitionistic Logic (IL)

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\wedge\mathrm{I})$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ (\to\mathrm{I})$$

$$\frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\ (\vee\mathrm{E})$$

$$\frac{\Gamma \vdash A_1 \wedge A_2}{\Gamma \vdash A_i}\ (\wedge\mathrm{E})$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash A \to B}{\Gamma \vdash B}\ (\to\mathrm{E})$$

$$\frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}\ (\vee\mathrm{E})$$

$$\frac{}{\Gamma, A \vdash A}\ (\mathrm{Ax})$$

$$\frac{\Gamma \vdash A(x)}{\Gamma \vdash \forall x A(x)}\ (\forall\mathrm{I})$$

$$\frac{\Gamma \vdash A(t)}{\Gamma \vdash \exists x A(x)}\ (\exists\mathrm{I})$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}\ (\mathrm{EFQ})$$

$$\frac{\Gamma \vdash \forall x A(x)}{\Gamma \vdash A(t)}\ (\forall\mathrm{E})$$

$$\frac{\Gamma \vdash \exists x A(x) \quad \Gamma, A(x) \vdash B}{\Gamma \vdash B}\ (\exists\mathrm{E})$$

Heyting Arithmetic = IL+eq+ numbers+induction

Terms: Constants 0, succ (+1), and symbols for all primitive recursive functions

Atomic formulas: Equality between numbers: $s = t$

Axioms for equality relation

Axioms for 0, succ (+1), and each primitive recursive function

Axiom schema for induction:

$$A(0) \wedge \forall n(A(n) \to A(n+1)) \to \forall n A(n) \quad \text{(IND)}$$

Partial computable functions in HA

Definable in **HA**:

- Kleene's (primitive recursive) predicate $T(x, y, z)$, z is the code of the computation of the Turing machine x on input y
- $U(z)$, calculating the output of the computation z

We can express that Turing machine x terminates on input y as $\exists z T(x, y, z)$

- This is often written as $\{x\}(y) \downarrow$

We can talk about the output of x on y as $\exists z(T(x, y, z) \wedge A(U(z))$

- It's easier to assume that $\{x\}$ is the partial computable function with code x, so we can write $\{x\}(y) \downarrow \wedge A(\{x\}(y))$

Kleene Realizability

Kleene Realizability

$$
\begin{aligned}
n \ \mathbf{r} \ s = t &\equiv (n = 0) \wedge (s = t) \\
n \ \mathbf{r} \ A \wedge B &\equiv (n_0 \ \mathbf{r} \ A) \wedge (n_1 \ \mathbf{r} \ B) \\
n \ \mathbf{r} \ A \vee B &\equiv (n_0 = 0 \rightarrow n_1 \ \mathbf{r} \ A) \wedge (n_0 = 1 \rightarrow n_1 \ \mathbf{r} \ B) \\
n \ \mathbf{r} \ A \rightarrow B &\equiv \forall m (m \ \mathbf{r} \ A \rightarrow \exists z (T(n, m, z) \wedge U(z) \ \mathbf{r} \ B)) \\
n \ \mathbf{r} \ \forall x A &\equiv \forall m \exists z (T(n, m, z) \wedge U(z) \ \mathbf{r} \ A[m/x]) \\
n \ \mathbf{r} \ \exists x A &\equiv n_1 \ \mathbf{r} \ A[n_0/x]
\end{aligned}
$$

Kleene, JSL, 1945

Kleene Realizability

$$
\begin{aligned}
n \ \mathbf{r}\ s = t &\equiv s = t \\
n \ \mathbf{r}\ A \wedge B &\equiv (n_0 \ \mathbf{r}\ A) \wedge (n_1 \ \mathbf{r}\ B) \\
n \ \mathbf{r}\ A \vee B &\equiv (n_0 = 0 \to n_1 \ \mathbf{r}\ A) \wedge (n_0 \neq 0 \to n_1 \ \mathbf{r}\ B) \\
n \ \mathbf{r}\ A \to B &\equiv \forall m (m \ \mathbf{r}\ A \to \{n\}(m) \downarrow \wedge \{n\}(m) \ \mathbf{r}\ B) \\
n \ \mathbf{r}\ \forall x A &\equiv \forall m (\{n\}(m) \downarrow \wedge \{n\}(m) \ \mathbf{r}\ A[m/x]) \\
n \ \mathbf{r}\ \exists x A &\equiv n_1 \ \mathbf{r}\ A[n_0/x]
\end{aligned}
$$

Kleene, JSL, 1945

Kleene Realizability: Characterisation

CT (Church-Turing thesis).
$\forall m \exists n A(m, n) \to \exists e \forall m(\{e\}(m) \downarrow \wedge A(m, \{e\}(m)))$

ECT (Extended Church-Turing thesis). For $B(m)$ almost negative:
$\forall m(B(m) \to \exists n A(m, n)) \to \exists e \forall m(B(m) \to \{e\}(m) \downarrow \wedge A(m, \{e\}(m)))$

Characterisation Theorem. $\mathbf{HA} + \mathbf{ECT} \vdash A \leftrightarrow \exists n(n \ \mathbf{r} \ A)$

$$
\begin{aligned}
n \ \mathbf{r}\ s = t &\equiv s = t \\
n \ \mathbf{r}\ A \wedge B &\equiv (n_0 \ \mathbf{r}\ A) \wedge (n_1 \ \mathbf{r}\ B) \\
n \ \mathbf{r}\ A \vee B &\equiv (n_0 = 0 \rightarrow n_1 \ \mathbf{r}\ A) \wedge (n_0 \neq 0 \rightarrow n_1 \ \mathbf{r}\ B) \\
n \ \mathbf{r}\ A \rightarrow B &\equiv \forall m (m \ \mathbf{r}\ A \rightarrow \{n\}(m) \downarrow \wedge \{n\}(m) \ \mathbf{r}\ B) \\
n \ \mathbf{r}\ \forall x A &\equiv \forall m (\{n\}(m) \downarrow \wedge \{n\}(m) \ \mathbf{r}\ A[m/x]) \\
n \ \mathbf{r}\ \exists x A &\equiv n_1 \ \mathbf{r}\ A[n_0/x]
\end{aligned}
$$

Soundness Theorem. If $\mathbf{HA} + \mathbf{ECT} \vdash A$ (A a closed formula) then $\mathbf{HA} \vdash n \ \mathbf{r}\ A$, for some numeral n

q-realizability and realizability with truth

Slash Translation

$$
\begin{aligned}
\Gamma | A_{\text{at}} &\equiv \Gamma \vdash A_{\text{at}} \\
\Gamma | A \land B &\equiv \Gamma | A \text{ and } \Gamma | B \\
\Gamma | A \lor B &\equiv \Gamma | A \text{ or } \Gamma | B \\
\Gamma | A \to B &\equiv \text{if } \Gamma | A \text{ then } \Gamma | B \\
\Gamma | \forall x^D A &\equiv \Gamma | A[d/x] \text{ for all } d \in D \\
\Gamma | \exists x^D A &\equiv \Gamma | A[d/x] \text{ for some } d \in D
\end{aligned}
$$

Kleene Slash Translation

$$\begin{aligned}
\Gamma \,|\, A_{\text{at}} &\equiv \Gamma \vdash A_{\text{at}} \\
\Gamma \,|\, A \land B &\equiv \Gamma \,|\, A \text{ and } \Gamma \,|\, B \\
\Gamma \,|\, A \lor B &\equiv (\Gamma \,|\, A \text{ and } \Gamma \vdash A) \text{ or } (\Gamma \,|\, B \text{ and } \Gamma \vdash B) \\
\Gamma \,|\, A \to B &\equiv \text{if } (\Gamma \,|\, A \text{ and } \Gamma \vdash A) \text{ then } \Gamma \,|\, B \\
\Gamma \,|\, \forall x A &\equiv \Gamma \,|\, A[n/x] \text{ for all numerals } n \\
\Gamma \,|\, \exists x A &\equiv \Gamma \,|\, A[n/x] \text{ and } \Gamma \vdash A[n/x], \text{ for some numeral } n
\end{aligned}$$

Disjunction property. $\Gamma \,|\, A \lor B$ implies that either $\Gamma \vdash A$ or $\Gamma \vdash B$

Existence property. $\Gamma \,|\, \exists x A$ implies that $\Gamma \vdash A[n/x]$ for some numeral n

Aczel Slash Translation

$$
\begin{aligned}
\Gamma | A_{\text{at}} &\equiv \Gamma \vdash A_{\text{at}} \\
\Gamma | A \wedge B &\equiv \Gamma | A \text{ and } \Gamma | B \\
\Gamma | A \vee B &\equiv \Gamma | A \text{ or } \Gamma | B \\
\Gamma | A \to B &\equiv (\text{if } \Gamma | A \text{ then } \Gamma | B) \text{ and } \Gamma \vdash A \to B \\
\Gamma | \forall x A &\equiv (\Gamma | A[n/x] \text{ for all numerals } n) \text{ and } \Gamma \vdash \forall x A \\
\Gamma | \exists x A &\equiv \Gamma | A[n/x], \text{ for some numeral } n
\end{aligned}
$$

Truth property. $\Gamma | A$ implies $\Gamma \vdash A$

q-Realizability

$$
\begin{aligned}
n \;\mathbf{r}\; s = t &\equiv s = t \\
n \;\mathbf{r}\; A \wedge B &\equiv (n_0 \;\mathbf{r}\; A) \wedge (n_1 \;\mathbf{r}\; B) \\
n \;\mathbf{r}\; A \vee B &\equiv (n_0 = 0 \to (n_1 \;\mathbf{r}\; A) \wedge A) \wedge (n_0 \neq 0 \to (n_1 \;\mathbf{r}\; B) \wedge B) \\
n \;\mathbf{r}\; A \to B &\equiv \forall m((m \;\mathbf{r}\; A) \wedge A \to \{n\}(m) \downarrow \wedge \{n\}(m) \;\mathbf{r}\; B) \\
n \;\mathbf{r}\; \forall x A &\equiv \forall m(\{n\}(m) \downarrow \wedge \{n\}(m) \;\mathbf{r}\; A[m/x]) \\
n \;\mathbf{r}\; \exists x A &\equiv (n_1 \;\mathbf{r}\; A[n_0/x]) \wedge A[n_0/x]
\end{aligned}
$$

Realizability with Truth

$$
\begin{aligned}
n \;\mathbf{r}\; s = t &\equiv s = t \\
n \;\mathbf{r}\; A \wedge B &\equiv (n_0 \;\mathbf{r}\; A) \wedge (n_1 \;\mathbf{r}\; B) \\
n \;\mathbf{r}\; A \vee B &\equiv (n_0 = 0 \to n_1 \;\mathbf{r}\; A) \wedge (n_0 \neq 0 \to n_1 \;\mathbf{r}\; B) \\
n \;\mathbf{r}\; A \to B &\equiv \forall m (m \;\mathbf{r}\; A \to \{n\}(m)\downarrow \wedge \{n\}(m) \;\mathbf{r}\; B) \wedge (A \to B) \\
n \;\mathbf{r}\; \forall x A &\equiv \forall m (\{n\}(m)\downarrow \wedge \{n\}(m) \;\mathbf{r}\; A[m/x]) \wedge \forall x A \\
n \;\mathbf{r}\; \exists x A &\equiv n_1 \;\mathbf{r}\; A[n_0/x]
\end{aligned}
$$

Applications

Disjunction Property.
If $\mathbf{HA} \vdash A \vee B$ then either $\mathbf{HA} \vdash A$ or $\mathbf{HA} \vdash B$

Existence Property.
If $\mathbf{HA} \vdash \exists n A(n)$ then $\mathbf{HA} \vdash A(n)$, for some numeral n

Relative Consistency.
If $\mathbf{HA}$ is consistent then so is $\mathbf{HA} + \mathbf{ECT}$

Elimination of ECT. Assuming $\mathbf{HA} \vdash A \leftrightarrow \exists n(n \ \mathbf{r} \ A)$:
If $\mathbf{HA} + \mathbf{ECT} \vdash A$ then $\mathbf{HA} \vdash A$