Closure of System T under the Bar Recursion Rule

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Friday, 24 February 2017
Outline

1. Spector’s bar recursion
2. Schwichtenberg’s proof
3. A new (more direct) proof
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3. A new (more direct) proof
Spector’s Bar Recursion

(1958) Gödel’s Dialectica interpretation of arithmetic (system T)
(1962) Spector extends interpretation to analysis (T + BR)
(1968) Howard interpretation of bar induction (T + BR)
(1971) Scarpellini shows \( \mathcal{C} \) is a model of BR
(1979) Schwichtenberg closure theorem (low types)
(1981) Howard’s ordinal analysis of BR (low types)
(1985) Bezem shows \( \mathcal{M} \) is a model of BR
Spector’s Bar Recursion (Rule)

Given \( s : \tau^* \) let \( \hat{s} : \tau^\mathbb{N} \) be the extension of \( s \) with \( 0 \)’s

For each pair of types \( \tau, \sigma \), and given \( G, H \) and \( Y \)

\[
\text{BR}^{\tau,\sigma}(s) \equiv \begin{cases} 
G(s) & \text{if } Y(\hat{s}) < |s| \\
H(s)(\lambda x.\text{BR}(s * x)) & \text{otherwise}
\end{cases}
\]

where

\[
G : \tau^* \to \sigma \\
Y : \tau^\mathbb{N} \to \mathbb{N} \\
H : \tau^* \to (\tau \to \sigma) \to \sigma
\]
Schwichtenberg’s Closure Theorem

Theorem

System $T$ is closed under the bar recursion rule when $\tau$’s type level is either 0 or 1

That is, given $G, H$ and $Y$ terms in $T$, the functional

$$BR^{\tau,\sigma}(s) \overset{\sigma}{=} \begin{cases} G(s) & \text{if } Y(\hat{s}) < |s| \\ H(s)(\lambda x^{\tau}.BR(s \ast x)) & \text{otherwise} \end{cases}$$

is also $T$ definable
Counter-example for $\tau > 1$

Howard (1968) showed that bar recursion of type $\rho$ can be defined using the bar recursion rule of type $(\mathbb{N} \rightarrow \rho) \rightarrow \rho$
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Since bar recursion, even of type $\rho = \mathbb{N}$, is not T definable.
Closure of System T under the Bar Recursion Rule

Spector’s bar recursion

Counter-example for \( \tau > 1 \)

Howard (1968) showed that bar recursion of type \( \rho \) can be defined using the bar recursion rule of type \( (\mathbb{N} \to \rho) \to \rho \)

Since bar recursion, even of type \( \rho = \mathbb{N} \), is not T definable

it follows that T is not closed under the bar recursion rule for \( \tau = (\mathbb{N} \to \mathbb{N}) \to \mathbb{N} \)
Outline

1. Spector’s bar recursion
2. Schwichtenberg’s proof
3. A new (more direct) proof
Schwichtenberg’s Proof

Published in The Journal of Symbolic Logic (1971)

“On bar recursion of type 0 and 1”

5 pages long (actual proof only two pages long)
Schwichtenberg’s Proof

1. Translate terms $G, H, Y$ into infinitary terms (get rid of recursor)
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2. Define a bar $S_Y(s) = \text{“sequence } s \text{ is secure for term } Y\text{”}$
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2. Define a bar $S_Y(s) = \text{“sequence } s \text{ is secure for term } Y\text{”}$

3. Complement of $S_Y(s)$ is a tree
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4. See BR as a recursion on this tree
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5. Define order-preserving embedding of tree into $\varepsilon_0$-ordinals
Schwichtenberg’s Proof

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4. See BR as a recursion on this tree

5. Define order-preserving embedding of tree into $\varepsilon_0$-ordinals

6. Hence, BR can be mimicked by $\varepsilon_0$-ordinal recursion
Schwichtenberg’s Proof

1. Translate terms $G, H, Y$ into infinitary terms (get rid of recursor)
2. Define a bar $S_Y(s) =$ “sequence $s$ is secure for term $Y$”
3. Complement of $S_Y(s)$ is a tree
4. See BR as a recursion on this tree
5. Define order-preserving embedding of tree into $\varepsilon_0$-ordinals
6. Hence, BR can be mimicked by $\varepsilon_0$-ordinal recursion
7. By Tait, we can find equivalent T definition of $BR(s)$
Outline

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3. A new (more direct) proof
Base case: \( Y(\alpha) \) is constant

When \( Y(\alpha) \) is constant \( n \), \( \text{BR} \) becomes

\[
\text{BR}^{\tau,\sigma}(s) \overset{\sigma}{=} \begin{cases} 
G(s) & \text{if } |s| > n \\
H(s)(\lambda x^\tau.\text{BR}(s \ast x)) & \text{if } |s| \leq n
\end{cases}
\]
Base case: $Y(\alpha)$ is constant

When $Y(\alpha)$ is constant $n$, BR becomes

$$BR^{\tau,\sigma}(s) \equiv \begin{cases} 
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It is easy to write down a T term (uniformly in $G$ and $H$) computing the same function
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It is easy to write down a T term (uniformly in $G$ and $H$) computing the same function

Needs primitive recursion of type $\tau^* \rightarrow \sigma$
Closure of System $T$ under the Bar Recursion Rule

A new (more direct) proof

Base case: $Y(\alpha)$ is constant

When $Y(\alpha)$ is constant $n$, BR becomes

$$BR^{\tau,\sigma}(s) \equiv \begin{cases} G(s) & \text{if } |s| > n \\ H(s)(\lambda x^\tau.BR(s \ast x)) & \text{if } |s| \leq n \end{cases}$$

It is easy to write down a $T$ term (uniformly in $G$ and $H$) computing the same function

Needs primitive recursion of type $\tau^* \rightarrow \sigma$

Let us refer to this $T$ term as $cBR$
Proof Idea

Part 1: Show that BR is definable in “general BR”
Part 2: Show that T is closed under “general BR”

(first part works for any type, second part requires the type restriction)
General BR

For any \textit{bar} $S$ consider the defining equation

\[
g_{BR}^S(s) \equiv \begin{cases} 
G(s) & \text{if } S(s) \\
H(s)(\lambda x^\tau.g_{BR}^S(s \ast x)) & \text{if } \neg S(s)
\end{cases}
\]
General BR

For any bar $S$ consider the defining equation

$$g_{\text{BR}}^S(s) = \sigma \begin{cases} G(s) & \text{if } S(s) \\ H(s)(\lambda x^\tau.g_{\text{BR}}^S(s \ast x)) & \text{if } \neg S(s) \end{cases}$$

Definition

We say that a bar $S$ secures $Y : \tau^\mathbb{N} \rightarrow \mathbb{N}$ if for all $s^{\tau^*}$

$$S(s) \Rightarrow \lambda \beta.Y(s \ast \beta) \text{ is constant}$$
Part 1: BR definable in general BR

**Theorem**

Fix $Y : \tau^\mathbb{N} \rightarrow \mathbb{N}$. The functional

$$\lambda G, H, s. \text{BR}^{\tau,\sigma}(G, H, Y)(s)$$

is $T$-definable in $\text{gBR}^S$, for any bar $S$ securing $Y$
Part 1: BR definable in general BR

**Theorem**

*Fix* $Y : \tau^\mathbb{N} \rightarrow \mathbb{N}$. The functional

$$\lambda G, H, s.\text{BR}^{\tau, \sigma}(G, H, Y)(s)$$

*is T-definable in* $\text{gBR}^S$, *for any bar* $S$ *securing* $Y$.

**Proof.**

Use the bar $S$ to spot when $Y$ becomes constant, then apply the T construction for the case when $Y$ is constant.
Part 2: Closure of T under gBR rule

**Theorem**

Fix a $T$-term $Y : \tau^\mathbb{N} \rightarrow \mathbb{N}$. For some $S$ securing $Y$ the functional $gBR^S$ is $T$ definable.
Part 2: Closure of T under gBR rule

**Theorem**

Fix a T-term $Y : \tau^\mathbb{N} \rightarrow \mathbb{N}$. For some $S$ securing $Y$ the functional $gBR^S$ is $T$ definable.

**Proof.**

*(Construction)* By induction on $Y$.

*(Correctness proof)* Use a logical relation to show that the constructed term is indeed equivalent to $gBR^S$. 

□
The Construction (case $\tau = \mathbb{N}$)

Let $\mathbb{N}^\circ \equiv$ the type of $g\text{BR}$. We will map $\mathbb{N}$ to $\mathbb{N}^\circ$.

Let $\alpha$ be a special variable of type $\mathbb{N} \rightarrow \mathbb{N}$ (generic)

\[
0^\circ = \lambda G. G
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(\lambda x^\eta. t)^\circ & = \lambda x^\circ. t^\circ
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The Construction (case $\tau = \mathbb{N}$)

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\[ (uv)^\circ = u^\circ v^\circ \]
The Construction (case $\tau = \mathbb{N}$)

Let $\mathbb{N}^\circ \equiv$ the type of $\text{gBR}$. We will map $\mathbb{N}$ to $\mathbb{N}^\circ$.

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0^\circ &= \lambda G. G \\
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(\lambda x^\eta.t)^\circ &= \lambda x^\circ.t^\circ \\
(\text{uv})^\circ &= u^\circ v^\circ \\
(\text{Rec}^\eta)^\circ &= \ldots
\end{align*}
\]
The Construction (case $\tau = \mathbb{N}$)

Let $\mathbb{N}^\circ \equiv$ the type of gBR. We will map $\mathbb{N}$ to $\mathbb{N}^\circ$.

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$$0^\circ = \lambda G. G$$

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$$\alpha^\circ = \lambda \Phi^{\mathbb{N}^\circ} \lambda G. \Phi(\lambda s'. \text{cBR}(G, Y(\hat{s}'))(s'))$$

$$(\lambda x^\eta. t)^\circ = \lambda x^\circ. t^\circ$$

$$(uv)^\circ = u^\circ v^\circ$$

$$(\text{Rec}^\eta)^\circ = \ldots$$

($H$ can be fixed at outset, but extra work to remember $Y$)
The Construction: Recursor

Suppose \( Y(\alpha) = \text{Rec}(n_\alpha, x_\alpha, f_\alpha) \)
The Construction: Recursor

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first ensure term $n_{\alpha}$ is secure (i.e. constant $n$)
The Construction: Recursor

Suppose \( Y(\alpha) = \text{Rec}(n_{\alpha}, x_{\alpha}, f_{\alpha}) \)

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Suppose \( Y(\alpha) = \text{Rec}(n_{\alpha}, x_{\alpha}, f_{\alpha}) \)

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and \( f_{\alpha}(x_{\alpha}) \) is secure
The Construction: Recursor

Suppose $Y(\alpha) = \text{Rec}(n_\alpha, x_\alpha, f_\alpha)$

first ensure term $n_\alpha$ is secure (i.e. constant $n$)

then ensure $x_\alpha$ is secure

and $f_\alpha(x_\alpha)$ is secure

\ldots
The Construction: Recursor

Suppose $Y(\alpha) = \text{Rec}(n_\alpha, x_\alpha, f_\alpha)$

first ensure term $n_\alpha$ is secure (i.e. constant $n$)

then ensure $x_\alpha$ is secure

and $f_\alpha(x_\alpha)$ is secure

\[ \vdots \]

until $f_n(\alpha)$ is secure
The Construction: Recursor

Suppose $Y(\alpha) = \text{Rec}(n_\alpha, x_\alpha, f_\alpha)$

first ensure term $n_\alpha$ is secure (i.e. constant $n$)

then ensure $x_\alpha$ is secure

and $f_\alpha(x_\alpha)$ is secure

... 

until $f^n_\alpha(x_\alpha)$ is secure

Can be done by induction hypothesis + primitive recursion
The Correctness Proof

Recall $\mathbb{N}^\circ \equiv$ the type of $g\text{BR}$

Fix $H$. Define logical relation between $T$ terms

Base case:

$$f^{\mathbb{N}^\circ} \sim_{\mathbb{N}} g^{\mathbb{N}^N \to \mathbb{N}} \equiv \exists S \text{ securing } g \text{ such that } f = g\text{BR}^S$$
The Correctness Proof

Recall $\mathbb{N}^\circ \equiv$ the type of gBR

Fix $H$. Define logical relation between T terms

Base case:

$$f^{\mathbb{N}^\circ} \sim_{\mathbb{N}} g^{\mathbb{N}^N \to \mathbb{N}} \equiv \exists S \text{ securing } g \text{ such that } f = g^{\mathrm{BR}^S}$$

and, as usual:

$$f^{\rho_0 \to \rho_1^\circ} \sim_{\rho_0 \to \rho_1} g^{\mathbb{N}^N \to (\rho_0 \to \rho_1)}$$

$$\equiv \forall x^{\rho_0^\circ} \forall y^{\mathbb{N}^N} \to \rho_0 (x \sim_{\rho_0} y \to f(x) \sim_{\rho_1} \lambda \alpha. g(\alpha)(y\alpha))$$
Main Result

Theorem

Given a closed $T$ term $Y: \mathbb{N}^\mathbb{N} \to \mathbb{N}$, then $(Y^\alpha)^\circ \sim Y$
Main Result

**Theorem**

*Given a closed T term* \( Y : \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N} \), *then* \((Y_\alpha)^\circ \sim Y\).

**Proof.**

*By structural induction on* \( Y \)
Main Result

Theorem

Given a closed $T$ term $Y : \mathbb{N}^\mathbb{N} \to \mathbb{N}$, then $(Y^\alpha)^\circ \sim Y$

Proof.

By structural induction on $Y$

Corollary

Fix $Y : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ in $T$. Then $\lambda G, H, s. \text{BR}(G, H, Y)(s)$ is $T$ definable
Conclusion

Stronger result:
- Only $Y$ needs to be $T$ definable

More explicit construction:
- Given concrete $Y$, reasonably easy to find $T$ definition of
  $\lambda G, H, s.\text{BR}(G, H, Y)(s)$

Easy to calibrate $T$ fragments:
- If $Y$ is $T_i$ then $\lambda G, H, s.\text{BR}(G, H, Y)(s)$ is in $T_j$, where
  $j = 1 + \max\{1, \text{level}(\sigma)\} + i$. 