Calculating Games with Higher-Order Functions

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(based on joint work with M. Escardó)

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Outline

1. Game Theory
2. Quantifiers and Selection Functions
3. Generalisation
4. Monads
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1. Game Theory
2. Quantifiers and Selection Functions
3. Generalisation
4. Monads
Game Theory

- Early development in the 19th century
- Formal approach with von Neumann (1930’s)

John von Neumann
Game Theory

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- \( n \) players
- \( n \) strategy sets \( X_1, \ldots, X_n \)
- payoff function \( q: \bar{X} \rightarrow \mathbb{R}^n \)

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How should players choose their strategies in order to maximise their individual payoffs?

John von Neumann
Game Theory
Penalties

Two players

Strategy sets $X_1 = X_2 = \{L, R\}$

Payoff function

<table>
<thead>
<tr>
<th></th>
<th>$L$</th>
<th>$R$</th>
</tr>
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<tbody>
<tr>
<td>$L$</td>
<td>(1, 0)</td>
<td>(0, 1)</td>
</tr>
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Game Theory

- No **winning** strategy!
- What about strategies in **equilibrium**?
Game Theory

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**Definition (Nash)**

Strategy profile $\vec{x}$ is in equilibrium if no player has an incentive to unilaterally change his strategy.
Game Theory

- No winning strategy!
- What about strategies in equilibrium?

Definition (Nash)

Strategy profile $\vec{x}$ is in equilibrium if no player has an incentive to unilaterally change his strategy.

The “penalty” example shows that strategy profiles in equilibrium not necessarily exist either.
Game Theory

- What if players choose “mixed” strategies
  i.e. player chooses probability distribution on strategies
Game Theory

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Theorem (Nash)

*Mixed strategies in equilibrium always exist*
Game Theory

- What if players choose “mixed” strategies
  i.e. player chooses probability distribution on strategies

Theorem (Nash)

Mixed strategies in equilibrium always exist

The “penalty” example is again an illustration of this:
Players randomly choosing left or right is best they can do
Simultaneous versus Sequential Games

- That’s all in the case of **simultaneous** games

- With **sequential** games things are simpler and nicer

- Strategies: mappings from previous moves to current move

- Similar definition of Nash equilibrium
Simultaneous versus Sequential Games

- That’s all in the case of **simultaneous** games
- With **sequential** games things are simpler and nicer
- Strategies: mappings from previous moves to current move
- Similar definition of Nash equilibrium

But equilibrium always exists and can be computed by a technique called **backward induction**
Backward Induction

\[ q : X \times Y \times Z \to \mathbb{R}^3 \]

- \( q(x_0, y_0, z_0) = (0,1,2) \)
- \( q(x_0, y_0, z_1) = (2,1,1) \)
- \( q(x_0, y_1, z_0) = (3,0,2) \)
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Our Recent Work

1. Generalised notions of sequential game, Nash equilibrium and backward induction
Our Recent Work

1. Generalised notions of sequential game, Nash equilibrium and backward induction

2. Showed how general notions appear in topology, proof theory, and algorithms, amongst others
Outline

1. Game Theory
2. Quantifiers and Selection Functions
3. Generalisation
4. Monads
Single-player Games
Two-player Games

Two **players**: Black and White
Two-player Games

Two players: Black and White

Possible outcomes:
- Black wins
- White wins
- Draw
Two-player Games

Two **players**: Black and White

Possible **outcomes**:
- Black wins
- White wins
- Draw

**Strategy**: Choice of move at round $k$ given previous moves
Another Game

Two players: John and Julia
Another Game

Two players: John and Julia

John splits a cake. Julia chooses one of the two pieces
Another Game

Two players: John and Julia

John splits a cake. Julia chooses one of the two pieces

Possible outcomes:
- John gets $N\%$ of the cake (John’s payoff)
- Julia gets $(100 - N)\%$ of the cake (Julia’s payoff)
Another Game

Two **players**: John and Julia

*John splits a cake. Julia chooses one of the two pieces*

Possible **outcomes**:
- John gets $N\%$ of the cake (John’s payoff)
- Julia gets $(100 - N)\%$ of the cake (Julia’s payoff)

Best strategy for John is to split cake into half

It is not a “winning strategy” but it is an **optimal strategy**

It maximises his payoff
Number of Player vs Number of Rounds

Number of players is not essential

It is important what the “goal” at each round is

Rounds with “same goal” mean played by “same player”
Number of Player vs Number of Rounds

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It is important what the “goal” at each round is

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How to describe the goal at a particular round?
Number of Player vs Number of Rounds

Number of players is not essential

It is important what the “goal” at each round is

Rounds with “same goal” mean played by “same player”

How to describe the goal at a particular round?

You could say: The goal is to win!

But maybe this is not possible (or might not even make sense)

Instead, the goal should be described as:

*a choice of outcome from each set of possible outcomes*
As in...

Q: How much would you like to pay for your flight?
As in...

Q: How much would you like to pay for your flight?
A: As little as possible!
Quantifiers

\[ R = \text{set of outcomes} \]
\[ X = \text{set of possible moves} \]

\[ \phi \in (X \to R) \to R \]

Describes the desired outcome \( \phi p \in R \) given \( p \in X \to R \)
Quantifiers

\( R = \) set of outcomes
\( X = \) set of possible moves

\[ \phi \in (X \rightarrow R) \rightarrow R \]

describes the desired outcome \( \phi p \in R \) given \( p \in X \rightarrow R \)

In the example:

\( R = \) prices (real numbers)
\( X = \) possible flights
\( X \rightarrow R = \) price of each flight
\( \phi = \) minimal value functional
Quantifiers

\[ \phi : (X \to R) \to R \]
## Quantifiers

$$\phi : (X \to R) \to R$$

## Other Examples

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### Quantifiers

\[ \phi : (X \to R) \to R \quad (\equiv K_RX) \]

### Other Examples

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Theorem (Maximum Value Theorem)

For any $p \in C[0, 1]$ there is a point $a \in [0, 1]$ such that

$$\sup p = p(a)$$
Theorem (Maximum Value Theorem)

For any $p \in C[0, 1]$ there is a point $a \in [0, 1]$ such that

$$\sup p = p(a)$$

Theorem (Mean Value Theorem)

For any $p \in C[0, 1]$ there is a point $a \in [0, 1]$ such that

$$\int_0^1 p = p(a)$$
Theorem (Witness Theorem)

For any \( p : X \rightarrow \mathbb{B} \) there is a point \( a \in X \) such that

\[
\exists x \in X \ p(x) \iff p(a)
\]

(similar to Hilbert's \( \varepsilon \)-term).
Theorem (Witness Theorem)

For any $p: X \rightarrow \mathbb{B}$ there is a point $a \in X$ such that

$$\exists x^X p(x) \iff p(a)$$

(similar to Hilbert's $\varepsilon$-term).

Theorem (Counter-example Theorem)

For any $p: X \rightarrow \mathbb{B}$ there is a point $a \in X$ such that

$$\forall x^X p(x) \iff p(a)$$

($a$ is counter-example to $p$ if one exists).
Let $\mathcal{J}_R X \equiv (X \to R) \to X$
Let $J^X_R \equiv (X \to R) \to X$

**Definition (Selection Functions)**

$\varepsilon : J^X_R$ is called a **selection function** for $\phi : K^X_R$ if

$$\phi(p) = p(\varepsilon p)$$

holds for all $p : X \to R$
Let $J_R X \equiv (X \to R) \to X$

**Definition (Selection Functions)**

$\varepsilon : J_R X$ is called a **selection function** for $\phi : K_R X$ if

$$\phi(p) = p(\varepsilon p)$$

holds for all $p : X \to R$

**Definition (Attainable Quantifiers)**

A quantifier $\phi : K_R X$ is called **attainable** if it has a selection function $\varepsilon : J_R X$
For Instance

- \( \sup : K_\mathbb{R}[0, 1] \) is an attainable quantifier

\[
\sup(p) = p(\text{argsup}(p))
\]

where \( \text{argsup} : J_\mathbb{R}[0, 1] \)
For Instance

- \( \text{sup}: K_{\mathbb{R}}[0, 1] \) is an attainable quantifier
  \[
  \text{sup}(p) = p(\text{argsup}(p))
  \]
  where \( \text{argsup}: J_{\mathbb{R}}[0, 1] \)

- \( \text{fix}: K_X X \) is an attainable quantifier
  \[
  \text{fix}(p) = p(\text{fix}(p))
  \]
  where \( \text{fix}: J_X X \ (= K_X X) \)
Selection Functions and Quantifiers

Every selection function \( \varepsilon : J_X \) defines a quantifier \( \overline{\varepsilon} : K_X \)

\[
\overline{\varepsilon}(p) = p(\varepsilon(p))
\]
Selection Functions and Quantifiers

Not all quantifiers are attainable, e.g. \( R = \{0, 1\} \)

\[
\phi(p) = 0
\]
Selection Functions and Quantifiers

Different $\varepsilon$ might define same $\phi$, e.g. $X = [0, 1]$ and $R = \mathbb{R}$

$$\varepsilon_0(p) = \mu x. \sup p = p(x)$$

$$\varepsilon_1(p) = \nu x. \sup p = p(x)$$
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Finite Sequential Games \((n \text{ rounds})\)

**Definition (A tuple \((R, (X_i)_{i<n}, (\phi_i)_{i<n}, q)\) where)**

- \(R\) is the set of **possible outcomes**
- \(X_i\) is the set of **available moves** at round \(i\)
- \(\phi_i: K_R X_i\) is the **goal quantifier** for round \(i\)
- \(q: \prod_{i=0}^{n-1} X_i \rightarrow R\) is the **outcome function**
Finite Sequential Games ($n$ rounds)

Definition (A tuple $(R, (X_i)_{i<n}, (\phi_i)_{i<n}, q)$ where)
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Definition (Strategy)
Family of mappings

\[ next_k : \prod_{i=0}^{k-1} X_i \rightarrow X_k \]
Definition (Strategic Play)

Given strategy $\text{next}_k$ and partial play $\vec{a} = a_0, \ldots, a_{k-1}$, the **strategic extension** of $\vec{a}$ is $b^{\vec{a}} = b_k^{\vec{a}}, \ldots, b_{n-1}^{\vec{a}}$ where

$$b_i^{\vec{a}} = \text{next}_i(\vec{a}, b_k^{\vec{a}}, \ldots, b_{i-1}^{\vec{a}})$$
**Definition (Strategic Play)**

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**Definition (Optimal Strategy)**

Strategy $\text{next}_k$ is **optimal** if for any partial play $\vec{a}$

$$q(\vec{a}, b^{\vec{a}}) = \phi_k(\lambda x_k. q(\vec{a}, x_k, b^{\vec{a}, x_k}))$$
**Definition (Strategic Play)**

Given strategy $\text{next}_k$ and partial play $\vec{a} = a_0, \ldots, a_{k-1}$, the **strategic extension** of $\vec{a}$ is $b^{\vec{a}} = b^{\vec{a}}_k, \ldots, b^{\vec{a}}_{n-1}$ where

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$$q(\vec{a}, b^{\vec{a}}) = \phi_k(\lambda x_k.q(\vec{a}, x_k, b^{\vec{a}, x_k}))$$

A product of selection functions computes optimal strategies.
Standard Game Theory

When quantifiers are \( \max \) or \( \sup \) over finite or compact set

Then \( \text{argsup} \) exists (and hence \( \sup \) is attainable)

- Generalised Game \( \mapsto \) Standard Game
- Optimal strategy \( \mapsto \) Strategy in Nash equilibrium
- Product of \( \text{argsup} \) \( \mapsto \) Backward induction!
Fixed Point Theory

Fixed point operators are their own selection function

Generalised Game $\mapsto$ Operators on product space
Optimal strategy $\mapsto$ Bekič’s Lemma
Product of fix’s $\mapsto$ The proof!
Proof Theory

Proof interpretation

$$\exists i \leq n \forall x X_i \exists r R A_i(x, r) \iff \forall \varepsilon(.) \exists i \leq n \exists p A_i(\varepsilon_i p, p(\varepsilon_i p))$$
Proof Theory

Proof interpretation

$$\exists i \leq n \forall x^{X_i} \exists r^R A_i(x, r) \implies \forall \varepsilon(.) \exists i \leq n \exists p A_i(\varepsilon ip, p(\varepsilon ip))$$

\(\varepsilon\)'s define quantifiers, which partially define a game

Computational interpretation relies on completing the definition of the game so optimal strategy solves problem
Proof Theory

Proof interpretation

\[ \exists i \leq n \forall x \exists r^R A_i(x, r) \implies \forall \varepsilon(.) \exists i \leq n \exists p A_i(\varepsilon p, p(\varepsilon p)) \]

\(\varepsilon\)'s define quantifiers, which partially define a game

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**Existence of optimal strategy actually implies the consistency of mathematics!**
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Monads

- $K_R$ and $J_R$ are strong monads
Monads

- $K_R$ and $J_R$ are **strong monads**
- $J_R \hookrightarrow K_R$ is a **monad morphism**
Monads

- $K_R$ and $J_R$ are **strong monads**
- $J_R \mapsto K_R$ is a **monad morphism**
- **Product** of quantifiers

$$K_R X \times K_R Y \rightarrow K_R (X \times Y)$$

calculates optimal outcome
Monads

- $K_R$ and $J_R$ are **strong monads**
- $J_R \rightarrow K_R$ is a **monad morphism**
- **Product** of quantifiers
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- **Product** of selection functions
  \[ J_R X \times J_R Y \rightarrow J_R(X \times Y) \]
  calculates optimal play
Monads

- $K_R$ and $J_R$ are **strong monads**
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  calculates optimal outcome
- **Product** of selection functions
  \[ J_R X \times J_R Y \rightarrow J_R (X \times Y) \]
  calculates optimal play
- **Infinite product** $\Pi_i J_R X_i \rightarrow J_R \Pi_i X_i$ exists
  (**in some models**)
Summary

- Generalised notion of sequential game
- Generalised notion of optimal strategy (equilibrium)
- Product of sel. fct. computes optimal strategies
- Results from fixed point theory, topology, proof theory, etc, can be viewed as optimal strategies in certain games
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