Bar Recursion: A Survey

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Classical logic = Continuation/Backtracking

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$$A \vee \neg A$$

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$$\forall N \exists s^{\mathbb{B}^N} \forall n < N \, (s_n \Leftrightarrow A(n))$$

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Analysis (comprehension)

$$\exists \alpha^{\mathbb{B}^{\mathbb{N}}} \forall n (\alpha(n) \Leftrightarrow A(n))$$

Outline

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(1958) Gödel publishes Dialectica interpretation of **arithmetic**

(1962) Spector extends Gödel's interpretation to **analysis**
Introduces **bar recursion** as an extension of system T
Essentially recursion on well-founded trees

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(1971) Scarpellini model $\mathcal{C}$ of **continuous functionals**
$\forall \varphi^{X^{\mathbb{N}} \to \mathbb{N}}, \alpha^{X^{\mathbb{N}}} \exists n^{\mathbb{N}} \forall \beta (\alpha[n] = \beta[n] \to \varphi\alpha = \varphi\beta)$

(1985) Bezem model $\mathcal{M}$ of **majorizable functionals**
$\forall \varphi^{X^{\mathbb{N}} \to \mathbb{N}}, \alpha^{X^{\mathbb{N}}} \exists n^{\mathbb{N}} \forall \beta (\alpha[n] = \beta[n] \to \varphi\beta \leqslant n)$

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Novel bar recursion and **realizability** interp. of analysis

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Defined **modified bar recursion**

Use of standard (modified) realizability

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(2006) Escardó rediscovers (variant of) modified bar recursion
Defining searchable sets and their countable products
Computational counterpart of compactness

Outline

Strong Monad

Let TX be a **type constructor** *(working in system T)*

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$$\begin{aligned} \eta_X &: X \to TX \\ (\cdot)^\dagger &: (X \to TY) \to (TX \to TY) \end{aligned}$$

we have ($f\colon X \to TY$ and $g\colon Y \to TZ$)

(i) $(\eta_X)^\dagger = \mathrm{id}_{TX}$

(ii) $f^\dagger \circ \eta_X = f$

(iii) $(g^\dagger \circ f)^\dagger = g^\dagger \circ f^\dagger$

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E.g. $TX = X$ *(identity monad)*

$TX = (X \to R) \to R$ *(continuation monad, KX)*

$TX = (X \to R) \to X$ *(selection monad, JX)*

Binary Product

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For $TX = KX$ and $\phi \colon KX$ and $\psi \colon X \to KY$

$$(\phi \otimes \psi)(q) = \phi(\lambda x.\psi(x)(\lambda y.q(x, y))) \qquad (q \colon X \times Y \to R)$$

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$$(\phi \otimes \psi)(q) = \phi(\lambda x.\psi(x)(\lambda y.q(x,y))) \qquad (q : X \times Y \to R)$$

For $TX = JX$ and $\varepsilon : JX$ and $\delta : X \to JY$ we have

$$(\varepsilon \otimes \delta)(q) = (a, f(a))$$

where $f(x) = \delta(x)(\lambda y.q(x,y))$ and $a = \varepsilon(\lambda x.q(x, f(x)))$

Finite Product

Given

$$f \quad : \quad X^* \to TX$$

Define (for $|s| \leqslant n$)

$$\bigotimes_s^n f = f(s) \otimes (\lambda x. \bigotimes_{s*x}^n f)$$

with $\bigotimes_s^{|s|} f = \eta(1)$

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Theorem (Escardó/Powell/O.'2011).
For all three monads $TX = X$, $TX = KX$ and $TX = JX$ the finite product is equivalent to Gödel primitive recursion

Unbounded Product (Implicitly Controlled)

Given

$$\phi_s \quad : \quad (X \to R) \to R \qquad \text{(quantifiers)}$$

$$\varepsilon_s \quad : \quad (X \to R) \to X \qquad \text{(selection functions)}$$

Define

$$\mathsf{IPQ}_s \quad : \quad (X^* \to KX)^{\mathbb{N}} \to K(X^{\mathbb{N}})$$

$$\mathsf{IPQ}_s \quad = \quad \phi_s \otimes (\lambda x.\mathsf{IPQ}_{s*x})$$

$$\mathsf{IPS}_s \quad : \quad (X^* \to JX)^{\mathbb{N}} \to J(X^{\mathbb{N}})$$

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IPQ is inconsistent

For discrete R, IPS exists in $\mathcal{C}$ and (not uniquely) in $\mathcal{M}$

Unbounded Product (Explicitly Controlled)

Given

$$\begin{array}{lll} \phi_s & : & (X \to R) \to R \\ \varepsilon_s & : & (X \to R) \to X \\ \varphi & : & X^{\mathbb{N}} \to \mathbb{N} \end{array}$$

(quantifiers)

(selection functions)

(control function)

Define ($\hat{s}$ = *infinite extension of finite sequence* s)

$$\mathsf{EPQ}_s = \begin{cases} \mathbf{0} & \text{if } \varphi(\hat{s}) < |s| \\ \phi_s \otimes (\lambda x.\mathsf{EPQ}_{s*x}) & \text{otherwise} \end{cases}$$

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$$\varphi : X^{\mathbb{N}} \to \mathbb{N} \qquad \text{(control function)}$$

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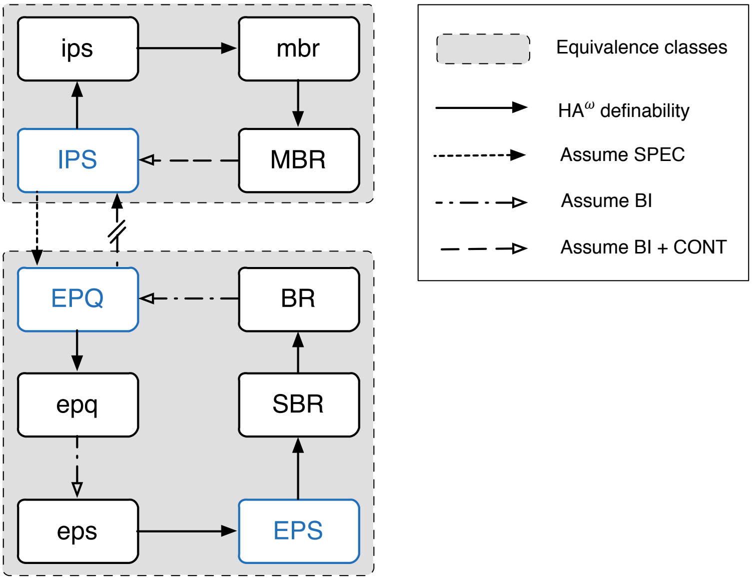
Both exist (uniquely) in $\mathcal{C}$ and $\mathcal{M}$, for arbitrary R

Interdefinability (Escardó/O.'2014)



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Game Contexts and Players

X = set of available moves

R = set of possible outcomes

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Maps $p\colon X \to R$ can be thought of as **game contexts**

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Selection functions describe **players**

$$(X \to R) \to X$$

by determining the **optimal move** for each game context

Product of Selection Functions

Product of selection functions = way of **combining** players

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The selection function $\varepsilon \otimes \delta$ will

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Theorem (Escardó/O.'2010).
Given n players, finite product $\otimes$ calculates optimal play

When selection functions are maximisation functions
finite product implements **backward induction**
(sub-game perfect equilibrium)

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Bar Recursion and Games

(2012) Infinite product IPS extends backward induction to **unbounded** games (Escardó/O.)

(2013) Selection function generalisation of Nash's theorem on the existence of **mixed equilibrium** (Hedges)

(2014) Application of selection functions and quantifiers to **"classical" game theory** (Hedges/O./Meinheim)
Novel equilibrium (multi-valued selection functions)

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Novel equilibrium (multi-valued selection functions)

(??) Selection function/bar recursion for mixed strategies

(??) Consider repeated games and approximate equilibria

Bar Recursion and Monads

(2013) Selection + State monad for DPLL (Hedges)

(2014) Multi-valued selection functions and the **Herbrand interpretation** of DNS (Escardó/O.)

Equivalence of bar recursion and "monadic" bar recursion

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Equivalence of bar recursion and "monadic" bar recursion

(??) Combination of selection functions and probability monad

Implication to games and mixed equilibrium

(??) Combination of selection functions with searchable set monad (Hedges)

Novel variant of functional interpretation

Applied Bar Recursion

(2013) Selection function (game-theoretic) interpretation of **Bolzano-Weierstrass and Ramsey thms** (Powell/O.)

(2014) Optimised variant of Spector bar recursion (Powell/O.) Better use of the control function

(2014) Bar recursive interpretation of **"termination" theorem** Based on analysis of transitive Ramsey theorem for pairs (Berardi/Steila/O.)

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(??) Game-theoretic interpretation of Analysis
E.g. Fixed point theory, Approximation theory,
Diophantine approximation, Ergodic theory

THE END