

# The Logic of the Unit Interval $[0, 1]$

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*There are no whole truths; all truths are half-truths.*

*It is trying to treat them as whole truths that plays the devil.*

- Alfred North Whitehead

# Outline

## Łukasiewicz Logic

Background

Ulam Game

McNaughton Functions

## Intuitionistic Łukasiewicz Logic

Hoops

Prover9 and Mace4

De Morgan Properties

Double Negation a Homomorphism

## Double Negation Translations

Affine and Łukasiewicz Logic

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**Q.** Are the usual rules of logic consistent with this view?

**A.** Yes! (almost)

## Contraction axiom not valid

The **contraction** axiom says

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However, throwing away the contraction axiom is too much

For instance, the formulas

$$(A \Rightarrow B) \Rightarrow (A \Rightarrow (B \wedge (B \Rightarrow A)))$$

are

- ▶ valid under our interpretation, but
- ▶ not derivable in linear logic

## Łukasiewicz Axiomatisation

The following axioms are **sound** and **complete** for  $[0, 1]$

$$(A1) \quad A \Rightarrow (B \Rightarrow A)$$

$$(A2) \quad (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$$

$$(A3) \quad ((A \Rightarrow B) \Rightarrow B) \Rightarrow ((B \Rightarrow A) \Rightarrow A)$$

$$(A4) \quad (\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)$$

with the **cut rule**, i.e. from  $A$  and  $A \Rightarrow B$  derive  $B$

Conjectured by Łukasiewicz (1920's)

Proven by Wajsberg (1935) and Chang (1959)

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Contrast this with the (type of the)  $S$  and  $K$  combinators

$$(K) \quad A \Rightarrow (B \Rightarrow A)$$

$$(S) \quad (A \Rightarrow B) \Rightarrow (A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow C)$$

# The Ulam Game

The Ulam Game is a twist on the classical 20-question game:

- ▶ Player B thinks of a number between 1 and  $10^6$
- ▶ Player A is allowed to ask up to 20 questions
- ▶ Player B is supposed to answer only yes or no
- ▶ Suppose Player B were allowed to lie once (or  $n$  times)

*How many questions would A need to get the right answer?*

# The Ulam Game

Classical reasoning no longer works

- ▶ Conjunction of two **equal** answers to the same question no longer equivalent to a single answer
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*Player A can record current knowledge by taking the Łukasiewicz conjunction of information contained in answers*

# McNaughton Functions

A function  $f : [0, 1]^n \rightarrow [0, 1]$  is **McNaughton** if it is

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## **McNaughton theorem (1951)**

A function  $f : [0, 1]^n \rightarrow [0, 1]$  is a “truth table” of a Łukasiewicz formula iff it is a McNaughton function

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# The Logic

In a sub-structural setting (no contraction) we use:

- ▶  $A \otimes B$  for “ $A$  and  $B$ ”
- ▶  $A \multimap B$  for “ $A$  implies  $B$ ”
- ▶ Falsehood is denoted by  $1$
- ▶ Negation is defined as  $A^\perp = A \multimap 1$

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**Ex falso quodlibet (EFQ)**

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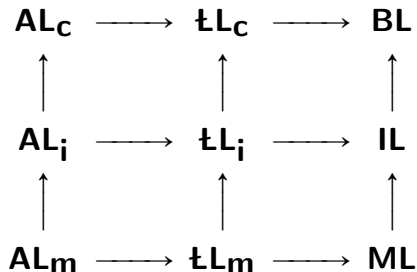
**Ex falso quodlibet (EFQ)**

$$1 \multimap A$$

**Double negation elimination (DNE)**

$$A^{\perp\perp} \multimap A$$

# Affine, Łukasiewicz and Boolean Logic



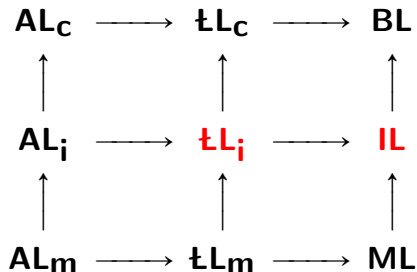
*minimal:* only weakening rule

*intuitionistic:* minimal plus EFQ

*classical:* intuitionistic plus DNE



# Affine, Łukasiewicz and Boolean Logic



*minimal:* only weakening rule

*intuitionistic:* minimal plus EFQ

*classical:* intuitionistic plus DNE

## Some Theorems of **IL**

The following are provable in **IL**

$$\neg\neg(\neg\neg A \Rightarrow A)$$

$$\neg(A \Rightarrow B) \simeq \neg\neg A \wedge \neg B$$

$$\neg(A \wedge B) \simeq A \Rightarrow \neg B$$

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How many of these are valid in **IL<sub>i</sub>**?

For instance:  $\neg\neg(\neg\neg A \Rightarrow A)$

Short derivation in **intuitionistic logic**

$$\frac{\frac{\frac{[A]_\alpha}{\neg\neg A \Rightarrow A} \quad [\neg(\neg\neg A \Rightarrow A)]_\delta}{\perp} \quad \frac{\frac{\perp}{\neg A} \alpha \quad [\neg\neg A]_\beta}{\perp}}{\frac{\frac{\perp}{A} \quad [\neg(\neg\neg A \Rightarrow A)]_\delta}{\neg\neg A \Rightarrow A} \beta} \delta$$

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How about **intuitionistic Łukasiewicz logic**?

For instance:  $\neg(A \Rightarrow B) \Rightarrow (\neg\neg A \wedge \neg B)$

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How about **intuitionistic Łukasiewicz logic**?

# The Algebras of $\mathbf{tL}_m$ and $\mathbf{tL}_i$ : Hoops

- A **pocrim**  $(+, 0, \rightarrow)$  is a commutative monoid  $(+, 0)$  which is
- ▶ partially ordered (with  $x \geq y$  defined as  $x \rightarrow y = 0$ )
  - ▶ residuated ( $x + y \geq z$  iff  $x \geq y \rightarrow z$ )
  - ▶ integral ( $x \geq 0$ )

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A **hoop** is a pocrim that satisfies the *divisibility axiom*:

$$x + (x \rightarrow y) = y + (y \rightarrow x)$$

## The Algebras of $\mathbf{tLL}_m$ and $\mathbf{tL}_i$ : Hoops

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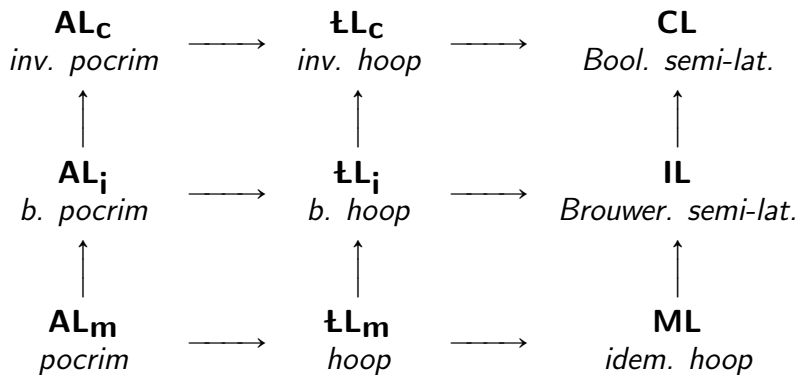
A **hoop** is a pocrim that satisfies the *divisibility axiom*:

$$x + (x \rightarrow y) = y + (y \rightarrow x)$$

**Thm.**  $A$  is provable  $\mathbf{tLL}_m$  iff  $[A]_{\mathcal{H}} = 0$  in all hoops  $\mathcal{H}$

**Thm.**  $A$  is provable  $\mathbf{tL}_i$  iff  $[A]_{\mathcal{H}} = 0$  in all bounded hoops  $\mathcal{H}$

# Logics and Algebras



*b.* = bounded

*inv.* = involutive

*idem.* = idempotent

# Hoops

The class of (bounded) hoops is a variety

One possible equational axiomatisation is

$$(x + y) + z = x + (y + z)$$
$$x + y = y + x \quad (\textit{commutative monoid})$$

$$x + 0 = x$$

$$x \rightarrow 0 = 0 \quad (\textit{integral})$$

$$x \rightarrow x = 0 \quad (\textit{poset})$$

$$x + y \rightarrow z = x \rightarrow (y \rightarrow z) \quad (\textit{residuation})$$

$$x + (x \rightarrow y) = y + (y \rightarrow x) \quad (\textit{divisibility})$$

$$x + 1 = 1 \quad (\textit{bounded})$$

DEMO!

## Derived Connectives

The **primitive** connectives are  $\otimes$  and  $\dashv$



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Our investigation also led us to consider the following:

$$A \wedge B \equiv A \otimes (A \multimap B) \quad (\textit{weak conjunction})$$

$$A \Rightarrow B \equiv A \multimap (A \otimes B) \quad (\textit{strong implication})$$

$$A \vee B \equiv (A \multimap B) \multimap B \quad (\textit{strong disjunction})$$

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$$A \vee B \equiv (A \multimap B) \multimap B \quad (\text{strong disjunction})$$

Proofs made sense when we took these connectives seriously

# De Morgan Properties

**Thm.** The following are valid in  $\mathbf{tL}_i$ :

$$(A \otimes B)^\perp \simeq A \multimap B^\perp$$

$$(A \multimap B)^\perp \simeq A^{\perp\perp} \otimes B^\perp$$

$$(A \wedge B)^\perp \simeq A \Rightarrow B^\perp$$

$$(A \Rightarrow B)^\perp \simeq A^{\perp\perp} \wedge B^\perp$$

$$(A \vee B)^\perp \simeq A^\perp \wedge B^\perp$$

*Proofs found by Prover9 (made human-readable by us)*

# Double Negation a Homomorphism

**Thm.** The following are valid in  $\mathbf{LL}_i$ :

$$(A \multimap B)^{\perp\perp} \simeq A^{\perp\perp} \multimap B^{\perp\perp}$$

$$(A \otimes B)^{\perp\perp} \simeq A^{\perp\perp} \otimes B^{\perp\perp}$$

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## Prover9

### Theorem

(1)  $(A^{\perp\perp} \multimap A)^{\perp\perp}$

**Length**

109 steps

**Depth**

9

**Time**

1 min

## Prover9

<b>Theorem</b>	<b>Length</b>	<b>Depth</b>	<b>Time</b>
(1) $(A^{\perp\perp} \multimap A)^{\perp\perp}$	109 steps	9	1 min
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(4) $(A \multimap B)^{\perp} \simeq A^{\perp\perp} \otimes B^{\perp}$	73 steps**	11	94 sec

(\*) using (3)

(\*\*) using (1) and (2)

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$$\neg\neg A \Rightarrow A$$

Its double negation, however, is also valid **intuitionistically**

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**Idea.** Chuck double negations in to constructivize a proof!

## Double Negation Translations

For instance:  $A \wedge B \Rightarrow C$

**Kolmogorov (1925)**. Place double negations everywhere

$$\neg\neg(\neg\neg(\neg\neg A \wedge \neg\neg B) \Rightarrow \neg\neg C)$$

**Glivenko (1929)**. Place a single double negation in front

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**Gentzen (1936)**. Place double negations on the atoms

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**Thm.** For these translations  $(\cdot)^*$ , **CL**  $\vdash A$  iff **IL**  $\vdash A^*$

## Double Negation Translations Substructurally

**Thm.** Neither Gentzen nor Glivenko “work” for affine logic

**Thm.** All three translations “work” for Łukasiewicz logic

## Final Remarks

**Question 1.** Analytic system for  $\mathbf{tL}_i$  (cut-elimination)?

**Question 2.**  $\mathbf{tL}_i$  decidable, but no complexity bound



# References



R. Arthan and P. Oliva

On affine logic and Łukasiewicz logic

*arXiv* (<http://arxiv.org/abs/1404.0570>), 2014



R. Arthan and P. Oliva

On pocrimms and hoops

*arXiv* (<http://arxiv.org/abs/1404.0816>), 2014