The Logic of the Unit Interval $[0, 1]$

Paulo Oliva

Queen Mary University of London

Florida Atlantic University

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There are no whole truths; all truths are half-truths.
It is trying to treat them as whole truths that plays the devil.

- Alfred North Whitehead

Outline

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What if ...

... we interpret truth values in the $[0, 1]$ interval

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Let 0 denote truth, and 1 falsehood

Let conjunction $A \wedge B$ mean $A \dot{+} B$

Let negation $\neg A$ mean $1 - A$

Or, in general, $A \Rightarrow B$ mean $B \dot{-} A$

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Q. Are the usual rules of logic consistent with this view?

A. Yes! (almost)

Contraction axiom not valid

The **contraction** axiom says

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But clearly $1 - 1/2 \neq 0$, for instance

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However, throwing away the contraction axiom is too much

For instance, the formulas

$$(A \Rightarrow B) \;\Rightarrow\; (A \Rightarrow (B \wedge (B \Rightarrow A)))$$

are

- valid under our interpretation, but
- not derivable in linear logic

Łukasiewicz Axiomatisation

The following axioms are **sound** and **complete** for $[0, 1]$

(A1) $A \Rightarrow (B \Rightarrow A)$

(A2) $(A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$

(A3) $((A \Rightarrow B) \Rightarrow B) \Rightarrow ((B \Rightarrow A) \Rightarrow A)$

(A4) $(\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)$

with the **cut rule**, i.e. from A and $A \Rightarrow B$ derive B

Conjectured by Łukasiewicz (1920's)

Proven by Wajsberg (1935) and Chang (1959)

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Contrast this with the (type of the) S and K combinators

(K) $A \Rightarrow (B \Rightarrow A)$

(S) $(A \Rightarrow B) \Rightarrow (A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow C)$

The Ulam Game

The Ulam Game is a twist on the classical 20-question game:

- Player B thinks of a number between 1 and 10^6
- Player A is allowed to ask up to 20 questions
- Player B is supposed to answer only yes or no
- Suppose Player B were allowed to lie once (or n times)

How many questions would A need to get the right answer?

The Ulam Game

Classical reasoning no longer works

- Conjunction of two **equal** answers to the same question no longer equivalent to a single answer
- Conjunction of two **opposite** answers to the same question need not lead to a contradiction

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Player A can record current knowledge by taking the <u>Łukasiewicz conjunction</u> of information contained in answers

McNaughton Functions

A function $f : [0,1]^n \to [0,1]$ is **McNaughton** if it is

- continuous
- piecewise linear
- each piece has integer coefficients

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McNaughton theorem (1951)

A function $f : [0,1]^n \to [0,1]$ is a "truth table" of a Łukasiewicz formula iff it is a McNaughton function

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The Logic

In a sub-sctructural setting (no contraction) we use:

- $A \otimes B$ for "A and B"
- $A \multimap B$ for "A implies B"
- Falsehood is denoted by 1
- Negation is defined as $A^{\perp} = A \multimap 1$

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Double negation elimination (DNE)

$$A^{\perp\perp} \multimap A$$

Affine, Łukasiewicz and Boolean Logic

$$\begin{array}{ccccc}
\mathbf{AL_c} & \longrightarrow & \textbf{Ł}\mathbf{L_c} & \longrightarrow & \mathbf{BL} \\
\uparrow & & \uparrow & & \uparrow \\
\mathbf{AL_i} & \longrightarrow & \textbf{Ł}\mathbf{L_i} & \longrightarrow & \mathbf{IL} \\
\uparrow & & \uparrow & & \uparrow \\
\mathbf{AL_m} & \longrightarrow & \textbf{Ł}\mathbf{L_m} & \longrightarrow & \mathbf{ML}
\end{array}$$

minimal:	only weakening rule
intuitionistic:	minimal plus EFQ
classical:	intuitionistic plus DNE

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Some Theorems of **IL**

The following are provable in **IL**

$$\neg\neg(\neg\neg A \Rightarrow A)$$

$$\begin{array}{rcl} \neg(A \Rightarrow B) & \simeq & \neg\neg A \wedge \neg B \\ \neg(A \wedge B) & \simeq & A \Rightarrow \neg B \end{array}$$

$$\begin{array}{rcl} \neg\neg(A \Rightarrow B) & \simeq & \neg\neg A \Rightarrow \neg\neg B \\ \neg\neg(A \wedge B) & \simeq & \neg\neg A \wedge \neg\neg B \end{array}$$

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How many of these are valid in **Ł$\mathbf{L_i}$**?

For instance: $\neg\neg(\neg\neg A \Rightarrow A)$

Short derivation in **intuitionistic logic**

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A]_\alpha}{\neg\neg A \Rightarrow A} \qquad [\neg(\neg\neg A \Rightarrow A)]_\delta}{\bot}}{\neg A}\,\alpha \qquad [\neg\neg A]_\beta}{\bot}}{A}}{\neg\neg A \Rightarrow A}\,\beta \qquad [\neg(\neg\neg A \Rightarrow A)]_\delta}{\bot}}{\neg\neg(\neg\neg A \Rightarrow A)}\,\delta
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How about **intuitionistic Łukasiewicz logic**?

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The Algebras of **ŁL$_\mathbf{m}$** and **ŁL$_\mathbf{i}$**: Hoops

A **pocrim** $(+, 0, \to)$ is a commutative monoid $(+, 0)$ which is

- partially ordered (with $x \geqslant y$ defined as $x \to y = 0$)
- residuated ($x + y \geqslant z$ iff $x \geqslant y \to z$)
- integral ($x \geqslant 0$)

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A **hoop** is a pocrim that satisfies the *divisibility axiom*:

$$x + (x \to y) = y + (y \to x)$$

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Thm. A is provable **ŁLm** iff $[A]_{\mathcal{H}} = 0$ in all hoops $\mathcal{H}$

Thm. A is provable **ŁLi** iff $[A]_{\mathcal{H}} = 0$ in all bounded hoops $\mathcal{H}$

Logics and Algebras

$$
\begin{array}{ccccc}
\begin{array}{c}\mathbf{AL_c}\\ \textit{inv. pocrim}\end{array} & \longrightarrow & \begin{array}{c}\mathbf{\text{\L}L_c}\\ \textit{inv. hoop}\end{array} & \longrightarrow & \begin{array}{c}\mathbf{CL}\\ \textit{Bool. semi-lat.}\end{array} \\
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$$

b. = *bounded*   *inv.* = *involutive*   *idem.* = *idempotent*

Hoops

The class of (bounded) hoops is a variety

One possible equational axiomatisation is

$$\begin{array}{rcll}
(x+y)+z &=& x+(y+z) & \\
x+y &=& y+x & \textit{(commutative monoid)} \\
x+0 &=& x & \\
x \to 0 &=& 0 & \textit{(integral)} \\
x \to x &=& 0 & \textit{(poset)} \\
x+y \to z &=& x \to (y \to z) & \textit{(residuation)} \\
x+(x \to y) &=& y+(y \to x) & \textit{(divisibility)} \\
x+1 &=& 1 & \textit{(bounded)}
\end{array}$$

Prover9

DEMO!

Derived Connectives

The **primitive** connectives are $\otimes$ and $\multimap$

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Our investigation also led us to consider the following:

$$A \wedge B \equiv A \otimes (A \multimap B) \qquad \text{(\textit{weak conjunction})}$$

$$A \Rightarrow B \equiv A \multimap (A \otimes B) \qquad \text{(\textit{strong implication})}$$

$$A \vee B \equiv (A \multimap B) \multimap B \qquad \text{(\textit{strong disjunction})}$$

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Proofs made sense when we took these connectives seriously

De Morgan Properties

Thm. The following are valid in **Ł**$\mathbf{L_i}$

$$(A \otimes B)^\perp \simeq A \multimap B^\perp$$
$$(A \multimap B)^\perp \simeq A^{\perp\perp} \otimes B^\perp$$

$$(A \wedge B)^\perp \simeq A \Rightarrow B^\perp$$
$$(A \Rightarrow B)^\perp \simeq A^{\perp\perp} \wedge B^\perp$$
$$(A \vee B)^\perp \simeq A^\perp \wedge B^\perp$$

Proofs found by Prover9 (made human-readable by us)

Double Negation a Homomorphism

Thm. The following are valid in $\mathbf{ŁL_i}$

$$(A \multimap B)^{\perp\perp} \simeq A^{\perp\perp} \multimap B^{\perp\perp}$$
$$(A \otimes B)^{\perp\perp} \simeq A^{\perp\perp} \otimes B^{\perp\perp}$$

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	Theorem	Length	Depth	Time
(1)	$(A^{\perp\perp} \multimap A)^{\perp\perp}$	109 steps	9	1 min

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(2)	$(A^{\perp} \multimap B)^{\perp} \simeq A^{\perp} \otimes B^{\perp}$	140 steps*	10	43 sec

$(*)$ using (3)

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(2)	$(A^{\perp} \multimap B)^{\perp} \simeq A^{\perp} \otimes B^{\perp}$	140 steps*	10	43 sec
(4)	$(A \multimap B)^{\perp} \simeq A^{\perp\perp} \otimes B^{\perp}$	73 steps**	11	94 sec

$(*)$ using (3)

$(**)$ using (1) and (2)

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Double Negations

Double negation elimination is only valid **classically**

$$\neg\neg A \Rightarrow A$$

Its double negation, however, is also valid **intuitionistically**

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Idea. Chuck double negations in to constructivize a proof!

Double Negation Translations

For instance: $A \wedge B \Rightarrow C$

Kolmogorov (1925). Place double negations everywhere

$$\neg\neg(\neg\neg(\neg\neg A \wedge \neg\neg B) \Rightarrow \neg\neg C)$$

Glivenko (1929). Place a single double negation in front

$$\neg\neg(A \wedge B \Rightarrow C)$$

Gentzen (1936). Place double negations on the atoms

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Thm. For these translations $(\cdot)^*$, **CL** $\vdash A$ iff **IL** $\vdash A^*$

Double Negation Translations Substructurally

Thm. Neither Gentzen nor Glivenko "work" for affine logic

Thm. All three translations "work" for Łukasiewicz logic

Final Remarks

Question 1. Analytic system for $\mathbf{ŁL_i}$ (cut-elimination)?

Question 2. $\mathbf{ŁL_i}$ decidable, but no complexity bound

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