Proofs and Games

Paulo Oliva

Queen Mary University of London

Classical Logic and Computation
Warwick, 8 July 2012
<table>
<thead>
<tr>
<th>GAMES</th>
<th>LOGIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Game</td>
<td>Formula</td>
</tr>
<tr>
<td>Players</td>
<td>Proponent/Opponent</td>
</tr>
<tr>
<td>Rules + Adjudication</td>
<td>Formal system</td>
</tr>
<tr>
<td>Play</td>
<td>Branch of proof tree</td>
</tr>
<tr>
<td>Strategy</td>
<td>Claimed proof</td>
</tr>
<tr>
<td>Winning Strategy</td>
<td>Proof</td>
</tr>
<tr>
<td>GAMES</td>
<td>LOGIC</td>
</tr>
<tr>
<td>-------------</td>
<td>-------------</td>
</tr>
<tr>
<td>Game</td>
<td>Formula</td>
</tr>
<tr>
<td>GAMES</td>
<td>LOGIC</td>
</tr>
<tr>
<td>-------------</td>
<td>---------------</td>
</tr>
<tr>
<td>Game</td>
<td>Formula</td>
</tr>
<tr>
<td>Players</td>
<td>Proponent/Opponent</td>
</tr>
<tr>
<td>GAMES</td>
<td>LOGIC</td>
</tr>
<tr>
<td>--------</td>
<td>-------</td>
</tr>
<tr>
<td>Game</td>
<td>Formula</td>
</tr>
<tr>
<td>Players</td>
<td>Proponent/Opponent</td>
</tr>
<tr>
<td>Rules + Adjudication</td>
<td>Formal system</td>
</tr>
<tr>
<td>GAMES</td>
<td>LOGIC</td>
</tr>
<tr>
<td>-----------</td>
<td>---------------------------------</td>
</tr>
<tr>
<td>Game</td>
<td>Formula</td>
</tr>
<tr>
<td>Players</td>
<td>Proponent/Opponent</td>
</tr>
<tr>
<td>Rules + Adjudication</td>
<td>Formal system</td>
</tr>
<tr>
<td>Play</td>
<td>Branch of proof tree</td>
</tr>
<tr>
<td>GAMES</td>
<td>LOGIC</td>
</tr>
<tr>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td>Game</td>
<td>Formula</td>
</tr>
<tr>
<td>Players</td>
<td>Proponent/Opponent</td>
</tr>
<tr>
<td>Rules + Adjudication</td>
<td>Formal system</td>
</tr>
<tr>
<td>Play</td>
<td>Branch of proof tree</td>
</tr>
<tr>
<td>Strategy</td>
<td>Claimed proof</td>
</tr>
<tr>
<td>GAMES</td>
<td>LOGIC</td>
</tr>
<tr>
<td>---------------</td>
<td>------------------------</td>
</tr>
<tr>
<td>Game</td>
<td>Formula</td>
</tr>
<tr>
<td>Players</td>
<td>Proponent/Opponent</td>
</tr>
<tr>
<td>Rules + Adjudication</td>
<td>Formal system</td>
</tr>
<tr>
<td>Play</td>
<td>Branch of proof tree</td>
</tr>
<tr>
<td>Strategy</td>
<td>Claimed proof</td>
</tr>
<tr>
<td>Winning Strategy</td>
<td>Proof</td>
</tr>
</tbody>
</table>
Extensive Form versus Strategic Form

Extensive form
Extensive Form versus Strategic Form

**Extensive Form**

```
T
├── L
│   └── (4,3)
├── R
│   └── (-1,-1)
```

```
B
├── L
│   └── (0,0)
├── R
│   └── (3,4)
```

**Strategic Form**

<table>
<thead>
<tr>
<th></th>
<th>LL</th>
<th>LR</th>
<th>RL</th>
<th>RR</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>(4,3)</td>
<td>(4,3)</td>
<td>(-1,-1)</td>
<td>(-1,-1)</td>
</tr>
<tr>
<td>B</td>
<td>(0,0)</td>
<td>(3,4)</td>
<td>(0,0)</td>
<td>(3,4)</td>
</tr>
</tbody>
</table>

Strategic form
Outline

1. Lorenzen Games
2. Blass Games
3. Strategic-form Games
4. Extensive-form Games (Generalised)
Outline

1. Lorenzen Games
2. Blass Games
3. Strategic-form Games
4. Extensive-form Games (Generalised)
Lorenzen Games

- Lorenzen (1961)

- Two players \{P, O\} debating about the truth of a formula
- Players take turns attacking or responding
- A player wins if the other can’t attack or respond

Motivation: alternative semantics for IL

If formula is provable in IL then \(P\) has winning strategy

Felscher (1985) found conditions for completeness

Formula is provable in IL iff \(P\) has winning strategy
Lorenzen Games

- Lorenzen (1961)

- Two players \{P, O\} debating about the truth of a formula
- Players take turns attacking or responding
- A player wins if the other can’t attack or respond

Motivation: alternative semantics for IL

If formula is provable in IL then P has winning strategy
Lorenzen Games

- Lorenzen (1961)
- Two players \( \{ P, O \} \) debating about the truth of a formula
- Players take turns attacking or responding
- A player wins if the other can’t attack or respond

Motivation: alternative semantics for IL
- If formula is provable in IL then \( P \) has winning strategy
- Felscher (1985) found conditions for completeness
  Formula is provable in IL iff \( P \) has winning strategy
Lorenzen Games – E.g. $P \land Q \rightarrow Q \land P$

Possible play in this game:

(0) $P$ starts by asserting $P \land Q \rightarrow Q \land P$

(1) $O$ attacks (0) asserting $P \land Q$

(2) $P$ attacks (1) asserting $P$ AND 1

(3) $O$ responds (2) asserting $P$

(4) $P$ attacks (1) asserting $P$ AND 2

(5) $O$ responds (4) asserting $Q$

(6) $P$ responds (1) asserting $Q \land P$

(7) $O$ attacks (6) asserting $P$ AND 1

(8) $P$ responds (7) asserting $Q$
**Lorenzen Games – E.g.** \( P \land Q \rightarrow Q \land P \)

Possible play in this game:

(0) **P** starts by asserting \( P \land Q \rightarrow Q \land P \)

(1) **O** attacks (0) asserting \( P \land Q \)

(2) **P** attacks (1) asserting \( P \land Q \)

(3) **O** responds (2) asserting \( P \land Q \)

(4) **P** attacks (1) asserting \( P \land Q \)

(5) **O** responds (4) asserting \( Q \)

(6) **P** responds (1) asserting \( Q \land P \)

(7) **O** attacks (6) asserting \( P \land Q \)

(8) **P** responds (7) asserting \( Q \)
Lorenzen Games – E.g. $P \land Q \rightarrow Q \land P$

Possible play in this game:

(0) $P$ starts by asserting $P \land Q \rightarrow Q \land P$

(1) $O$ attacks (0) asserting $P \land Q$

(2) $P$ attacks (1) asserting $\land_1$

(3) $O$ responds (2) asserting $P$

(4) $P$ attacks (1) asserting $\land_2$

(5) $O$ responds (4) asserting $Q$

(6) $P$ responds (1) asserting $Q \land P$

(7) $O$ attacks (6) asserting $\land_1$
Lorenzen Games – E.g. $P \land Q \rightarrow Q \land P$

Possible play in this game:

(0) **P** starts by asserting $P \land Q \rightarrow Q \land P$

(1) **O** attacks (0) asserting $P \land Q$

(2) **P** attacks (1) asserting $\land_1$

(3) **O** responds (2) asserting $P$
Lorenzen Games – E.g. $P \land Q \rightarrow Q \land P$

Possible play in this game:

1. $P$ starts by asserting $P \land Q \rightarrow Q \land P$
2. $O$ attacks (0) asserting $P \land Q$
3. $P$ attacks (1) asserting $\land_1$
4. $O$ responds (2) asserting $P$
5. $P$ attacks (1) asserting $\land_2$
Lorenzen Games – E.g. \( P \land Q \rightarrow Q \land P \)

Possible play in this game:

(0) \( P \) starts by asserting \( P \land Q \rightarrow Q \land P \)

(1) \( O \) attacks (0) asserting \( P \land Q \)

(2) \( P \) attacks (1) asserting \( \land_1 \)

(3) \( O \) responds (2) asserting \( P \)

(4) \( P \) attacks (1) asserting \( \land_2 \)

(5) \( O \) responds (4) asserting \( Q \)
Lorenzen Games – E.g. $P \land Q \rightarrow Q \land P$

Possible play in this game:

(0) $P$ starts by asserting $P \land Q \rightarrow Q \land P$

(1) $O$ attacks (0) asserting $P \land Q$

(2) $P$ attacks (1) asserting $\land_1$

(3) $O$ responds (2) asserting $P$

(4) $P$ attacks (1) asserting $\land_2$

(5) $O$ responds (4) asserting $Q$

(6) $P$ responds (1) asserting $Q \land P$
Lorenzen Games – E.g.  \( P \land Q \rightarrow Q \land P \)

Possible play in this game:

(0) **P** starts by asserting  \( P \land Q \rightarrow Q \land P \)

(1) **O** attacks (0) asserting  \( P \land Q \)

(2) **P** attacks (1) asserting  \( \land_1 \)

(3) **O** responds (2) asserting  \( P \)

(4) **P** attacks (1) asserting  \( \land_2 \)

(5) **O** responds (4) asserting  \( Q \)

(6) **P** responds (1) asserting  \( Q \land P \)

(7) **O** attacks (6) asserting  \( \land_1 \)
Lorenzen Games – E.g. $P \land Q \rightarrow Q \land P$

Possible play in this game:

1. **P** starts by asserting $P \land Q \rightarrow Q \land P$
2. **O** attacks (0) asserting $P \land Q$
3. **P** attacks (1) asserting $\land_1$
4. **O** responds (2) asserting $P$
5. **P** attacks (1) asserting $\land_2$
6. **O** responds (4) asserting $Q$
7. **P** responds (1) asserting $Q \land P$
8. **O** attacks (6) asserting $\land_1$
9. **P** responds (7) asserting $Q$
General organisation of the game:

**S1** P may only assert atomic formulas already asserted by O
General organisation of the game:

S1 P may only assert atomic formulas already asserted by O
S2 A player can only respond the latest open attack
Lorenzen Games – Structural Rules

General organisation of the game:

S1 P may only assert atomic formulas already asserted by O
S2 A player can only respond the latest open attack
S3 An attack may be responded at most once
General organisation of the game:

S1 P may only assert atomic formulas already asserted by O
S2 A player can only respond the latest open attack
S3 An attack may be responded at most once
S4 A P-assertion may be attacked at most once
Outline

1. Lorenzen Games
2. Blass Games
3. Strategic-form Games
4. Extensive-form Games (Generalised)
Blass Games

Blass’1992

Games for affine logic (linear logic plus weakening)

Based on operations on infinite games devised in 1972
Blass Games

Blass’1992

Games for affine logic (linear logic plus weakening)

Based on operations on infinite games devised in 1972

Two main differences to Lorenzen games:
  - Infinitely long plays (means not all games are determined)
  - Two kinds of connectives (only one re-attackable)
Blass Games

Blass’1992

Games for **affine logic** (linear logic plus weakening)

Based on operations on infinite games devised in 1972

Two main differences to Lorenzen games:
- Infinitely long plays (means not all games are determined)
- Two kinds of connectives (only one re-attackable)

Can dispense with structural rule!
Two players \( P \) and \( O \)

A **Blass game** is a triple \( G = (M, p, G) \) where

- \( M \) is the set of **possible moves** at each round
- \( p \in \{P, O\} \) is the **starting player**
  (from then on players take turns)
- \( G : M^\omega \to \{P, O\} \) is the **outcome function**
Blass Games – Conjunctions

Given games $G_0 = (M_0, s_0, G_0)$ and $G_1 = (M_1, s_1, G_1)$
Blass Games – Conjunctions

Given games $G_0 = (M_0, s_0, G_0)$ and $G_1 = (M_1, s_1, G_1)$

The game $G_0 \& G_1$. Defined as

- O starts and chooses $i \in \{0, 1\}$
- Game $G_i$ is then played
Blass Games – Conjunctions

Given games $G_0 = (M_0, s_0, G_0)$ and $G_1 = (M_1, s_1, G_1)$

The game $G_0 \& G_1$. Defined as
- $O$ starts and chooses $i \in \{0, 1\}$
- Game $G_i$ is then played

The game $G_0 \otimes G_1$. Defined as
- play both games **interleaved**
- $O$’s turn in $G_0 \otimes G_1$ if it’s his turn in both $G_0$ and $G_1$
  He chooses one of the games and makes a move there
- $P$’s turn in $G_0 \otimes G_1$ if his turn in one of $G_0$ or $G_1$
  He must play on the sub-game where it’s his turn
- $O$ wins iff he wins in at least one of $G_0$ or $G_1$
Blass Games

- The dual of a game is simply a swapping of roles

\[ \mathcal{G}^\perp = (M, s, \overline{G}) \]

- Given game interpretation of atomics \( P \mapsto \mathcal{G}_P \)
  extend to game interpretation \( \mathcal{G}_A \) for all formulas \( A \)
Blass Games

- The dual of a game is simply a swapping of roles
  \[ \mathcal{G}^\perp = (M, \bar{s}, \bar{G}) \]
- Given game interpretation of atomics \( P \mapsto \mathcal{G}_P \)
  extend to game interpretation \( \mathcal{G}_A \) for all formulas \( A \)

Theorem (Blass, 1992)

\( A \) is provable in affine logic \( \Rightarrow \) \( P \) has winning strategy in \( \mathcal{G}_A \)
(Completeness only for additive fragment)
Blass Games

- The dual of a game is simply a swapping of roles
  \[ G_\perp = (M, \bar{s}, \bar{G}) \]

- Given game interpretation of atomics \( P \mapsto G_P \) extend to game interpretation \( G_A \) for all formulas \( A \)

Theorem (Blass, 1992)

\( A \) is provable in affine logic \( \Rightarrow \) \( \textbf{P} \) has winning strategy in \( G_A \)
(Completeness only for additive fragment)

- Abramsky and Jagadeesan’1992
  Soundness and completeness for MLL + mix rule

- Hyland and Ong’1993
  Soundness and completeness for MLL
Outline

1. Lorenzen Games
2. Blass Games
3. Strategic-form Games
4. Extensive-form Games (Generalised)
It is my thesis that game-theoretically inspired conceptualizations have much to offer in other parts of logical studies as well. An especially neat case in point is offered by Gödel's functional interpretation of first-order arithmetic. As Dana Scott first pointed out, by far the most natural way of looking at it is in game-theoretical terms.

It is my thesis that game-theoretically inspired conceptualizations have much to offer in other parts of logical studies as well. An especially neat case in point is offered by Gödel's functional interpretation of first-order arithmetic. As Dana Scott first pointed out, by far the most natural way of looking at it is in game-theoretical terms.


Our category of games is a special case of a general construction in the appendix to Barr’s book [1]. It is closely related to de Paiva’s *dialectica categories* [10,11].

Lafont/Streicher, *Games semantics for LL*, 1991
It is my thesis that game-theoretically inspired conceptualizations have much to offer in other parts of logical studies as well. An especially neat case in point is offered by Gödel's functional interpretation of first-order arithmetic. As Dana Scott first pointed out, by far the most natural way of looking at it is in game-theoretical terms.


Our category of games is a special case of a general construction in the appendix to Barr's book [1]. It is closely related to de Paiva's *dialectica categories* [10,11].

Lafont/Streicher, *Games semantics for LL*, 1991

In developing a category-theoretic approach to the Dialectica interpretation, de Paiva [3] found a connection with linear logic. This connection suggests looking at the Dialectica interpretation, in de Paiva's category-theoretic version, from the point of view of game semantics, and this is the purpose of the present section.

Functional Moves (Strategies)

What if we could allow for higher-order moves?

∀ \( x \) \( \exists y \) \( Q(x,y) \) \( \Rightarrow \exists f \) \( \forall x \) \( Q(x,fx) \)

Repeated applications turns long games into two-round games

\( \exists f_0 \ldots f_n \) \( \forall x_0 \ldots x_n \) \( Q(x_0,f_0(x_0),\ldots,x_n,f_n(\vec{x})) \)

\( P \) chooses \( t = \langle t_0 \ldots t_n \rangle \), then \( O \) chooses \( s = \langle s_0 \ldots s_n \rangle \). 
\( P \) wins iff \( Q(s_0,t_0(s_0),\ldots,s_n,t_n(\vec{s})) \)
Functional Moves (Strategies)

What if we could allow for higher-order moves?
Can make use of Skolemisation

$$\forall x \exists y Q(x, y) \implies \exists f \forall x Q(x, fx)$$

Repeated applications turns long games
$$\forall x_0 \exists y_0 \ldots \forall x_n \exists y_n Q(x_0, y_0, \ldots, x_n, y_n)$$

into two-round games
$$\exists f_0 \ldots f_n \forall x_0 \ldots x_n Q(x_0, f_0(x_0), \ldots, x_n, f_n(\vec{x}))$$

P chooses $$t = \langle t_0 \ldots t_n \rangle$$, then O chooses $$s = \langle s_0 \ldots s_n \rangle$$
P wins iff
$$Q(s_0, t_0(s_0), \ldots, s_n, t_n(\vec{s}))$$
What if we could allow for higher-order moves?

Can make use of Skolemisation

$$\forall x \exists y Q(x, y) \Rightarrow \exists f \forall x Q(x, f x)$$

Repeated applications turns long games

$$\forall x_0 \exists y_0 \ldots \forall x_n \exists y_n Q(x_0, y_0, \ldots, x_n, y_n)$$

into two-round games

$$\exists f_0 \ldots f_n \forall x_0 \ldots x_n Q(x_0, f_0(x_0), \ldots, x_n, f_n(\vec{x}))$$
Functional Moves (Strategies)

What if we could allow for higher-order moves?
Can make use of Skolemisation

$$\forall x \exists y Q(x, y) \implies \exists f \forall x Q(x, fx)$$

Repeated applications turns long games

$$\forall x_0 \exists y_0 \ldots \forall x_n \exists y_n Q(x_0, y_0, \ldots, x_n, y_n)$$

into two-round games

$$\exists f_0 \ldots f_n \forall x_0 \ldots x_n Q(x_0, f_0(x_0), \ldots, x_n, f_n(\vec{x}))$$

\(P\) chooses \(t = \langle t_0 \ldots t_n \rangle\), then \(O\) chooses \(s = \langle s_0 \ldots s_n \rangle\)

\(P\) wins iff \(Q(s_0, t_0(s_0), \ldots, s_n, t_n(\vec{s}))\)
Finite Types and System T

**Finite types** generated by

\[ X, Y \equiv \mathbb{B} \mid \mathbb{N} \mid X \times Y \mid X \cup Y \mid Y^X \]
Finite Types and System T

**Finite types** generated by

\[ X, Y :\equiv \mathbb{B} | \mathbb{N} | X \times Y | X \sqcup Y | Y^X \]

Gödel primitive recursor

\[ R(x, f, n) \overset{X}{=} \begin{cases} x & \text{if } n = 0 \\ f(n - 1, R(x, f, n - 1)) & \text{if } n > 0 \end{cases} \]

where \( X \) is an any finite type
Finite Types and System T

Finite types generated by

\[ X, Y \equiv \mathbb{B} \mid \mathbb{N} \mid X \times Y \mid X \cup Y \mid Y^X \]

Gödel primitive recursor

\[
R(x, f, n) \overset{X}{=} \begin{cases} 
  x & \text{if } n = 0 \\
  f(n - 1, R(x, f, n - 1)) & \text{if } n > 0
\end{cases}
\]

where \( X \) is an any finite type

Gödel’s system T: Primitive recursive functionals
Finite Types and System T

**Finite types** generated by

\[ X, Y \equiv \mathbb{B} \mid \mathbb{N} \mid X \times Y \mid X \cup Y \mid Y^X \]

Gödel primitive recursor

\[ R(x, f, n) \equiv \begin{cases} x & \text{if } n = 0 \\ f(n - 1, R(x, f, n - 1)) & \text{if } n > 0 \end{cases} \]

where \( X \) is an any finite type

**Gödel’s system T**: Primitive recursive functionals

**Remark**: Ackermann function definable using \( X = \mathbb{N}^\mathbb{N} \)
Strategic-form Games

Identify $\mathbb{B} = \{P, O\}$

Formula $A$ assigned a game with outcome function

$$|A| : X \times Y \to \mathbb{B}$$

where $X, Y$ are finite types (Gödel’s *dialectica* interpretation)
Strategic-form Games

Identify $\mathbb{B} = \{P, O\}$

Formula $A$ assigned a game with outcome function

\[ |A| : X \times Y \to \mathbb{B} \]

where $X, Y$ are finite types (Gödel’s *dialectica* interpretation)

Intuition:

- $P$ plays first choosing $t^X$
- $O$ then chooses $s^Y$
- $P$ wins iff $|A|^t_s$ is true
Strategic-form Games

Identify $\mathbb{B} = \{P, O\}$

Formula $A$ assigned a game with outcome function

$$|A| : X \times Y \to \mathbb{B}$$

where $X, Y$ are finite types (Gödel’s *dialectica* interpretation)

Intuition:
- $P$ plays first choosing $t^X$
- $O$ then chooses $s^Y$
- $P$ wins iff $|A|^t_s$ is true

**Theorem (Gödel, 1958)**

$$\text{HA} \vdash A \quad \exists t \in T \quad T \vdash \forall y |A|^t_y$$
Strategic-form Games

Assume $|A| : X \times Y \to \mathbb{B}$ and $|B| : V \times W \to \mathbb{B}$ defined.

Then:

$$|A \land B|_{\langle x,v \rangle} \equiv |A|_x \land |B|_v$$
Strategic-form Games

Assume $|A| : X \times Y \rightarrow \mathbb{B}$ and $|B| : V \times W \rightarrow \mathbb{B}$ defined.

Then:

$$|A \land B|_{\langle x,v \rangle} \equiv |A|_y \land |B|_w$$

$$|A \lor B|_{\langle y,w \rangle} \equiv \begin{cases} |A|_y \quad \text{if } b = l \\ |B|_w \quad \text{if } b = r \end{cases}$$
Strategic-form Games

Assume \( |A| : X \times Y \rightarrow \mathbb{B} \) and \( |B| : V \times W \rightarrow \mathbb{B} \) defined. Then:

\[
|A \land B|_{\langle x, v \rangle} \equiv |A|_y \land |B|_w \\
|A \lor B|_{\langle y, w \rangle} \equiv \begin{cases} 
|A|_y & \text{if } b = l \\
|B|_w & \text{if } b = r 
\end{cases}
\]

\[
\exists z A|_{\langle a, x \rangle} \equiv |A[a/z]|_{y} 
\]
Strategic-form Games

Assume $|A| : X \times Y \to \mathbb{B}$ and $|B| : V \times W \to \mathbb{B}$ defined.

Then:

$$|A \land B|_{\langle x,v \rangle} \equiv |A|_y \land |B|_w$$

$$|A \lor B|_{\langle y,w \rangle} \equiv \begin{cases} |A|_y & \text{if } b = l \\ |B|_w & \text{if } b = r \end{cases}$$

$$|\exists zA|_{a,x} \equiv |A[a/z]|_y^x$$

$$|\forall zA|_{a,y} \equiv |A[a/z]|_y^{fa}$$
Strategic-form Games

Assume $|A| : X \times Y \rightarrow \mathbb{B}$ and $|B| : V \times W \rightarrow \mathbb{B}$ defined.

Then:

$$|A \land B|_{\langle x,v \rangle} \equiv |A|_x \land |B|_v$$

$$|A \lor B|_{\langle y,w \rangle} \equiv \begin{cases} |A|_y & \text{if } b = l \\ |B|_w & \text{if } b = r \end{cases}$$

$$|\exists z A|_{\langle a,x \rangle} \equiv |A[a/z]|_x$$

$$|\forall z A|_{\langle a,y \rangle} \equiv |A[a/z]|_{fa}$$

$$|A \rightarrow B|_{\langle f,g \rangle} \equiv |A|_{gxw} \rightarrow |B|_{fx}$$
Functional interpretations

Strategic-form game above is **dialectica interpretation**

\[ |A|_y^x \equiv A_D(x; y) \]
Functional interpretations

Strategic-form game above is dialectica interpretation

\[ |A|_y^x \equiv A_D(x; y) \]

Variant where interpretation of implication is changed to

\[ |A \rightarrow B|_{\langle x, w \rangle}^f \equiv \forall y |A|_y^x \rightarrow |B|_{w}^{fx} \]

gives Kreisel's modified realizability

\[ \forall y |A|_y^x \equiv x \text{ mr } A \]
Functional interpretations – Linear logic

\[P\text{ and } O\text{ choose moves simultaneously!}\]

Assume \(|A| : X \times Y \to \mathbb{B}\) and \(|B| : V \times W \to \mathbb{B}\) defined

\[
|A^\perp|_y^x \equiv \neg |A|_x^y
\]

\[
|A \& B|_{\text{inj}_b}_y^\langle x,v \rangle \equiv \begin{cases} 
|A|_y^x & \text{if } b = 0 \\
|B|_y^v & \text{if } b = 1 
\end{cases}
\]

\[
|A \otimes B|_{\langle f,g \rangle}^\langle x,v \rangle \equiv |A|_{fv}^x \land |B|_{gx}^v
\]

\[
\forall z A|_{\langle a,y \rangle}^f \equiv |A[a/z]|_{gy}^f
\]

\[
!A|_f^x \equiv |A|_{fx}^x
\]
**Functional interpretations – Linear logic**

**P** and **O** choose moves simultaneously!

Assume \(|A| : X \times Y \to \mathbb{B}\) and \(|B| : V \times W \to \mathbb{B}\) defined

\begin{align*}
|A^\perp|_x^y & \equiv \neg |A|_x^y \\
|A \& B|_{\text{inj}_b^y} & \equiv \begin{cases} 
|A|_y^x & \text{if } b = 0 \\
|B|_y^v & \text{if } b = 1 
\end{cases} \\
|A \otimes B|_{\langle f,v \rangle} & \equiv |A|_{fv}^x \land |B|_{gx}^v \\
\forall z |A|_{\langle a,y \rangle}^f & \equiv |A[a/z]|_{x}^{f^a} \\
|! A|_f^x & \equiv |A|_{fx}^x \quad (\text{Gödel dialectica})
\end{align*}
Functional interpretations – Linear logic

\( \textbf{P} \text{ and } \textbf{O} \text{ choose moves simultaneously!} \)

Assume \( |A| : X \times Y \rightarrow \mathbb{B} \) and \( |B| : V \times W \rightarrow \mathbb{B} \) defined

\[
|A\perp|_y^x \equiv \neg|A|_x^y \\
|A \& B|_{\text{inj}_b}^{\langle x,v \rangle} \equiv \begin{cases} 
|A|_y^x & \text{if } b = 0 \\
|B|_y^v & \text{if } b = 1 
\end{cases} \\
|A \otimes B|_{\langle f,g \rangle}^{\langle x,v \rangle} \equiv |A|_{f}^{x} \land |B|_{g}^{v} \\
|\forall z A|_{\langle a,y \rangle}^{f} \equiv |A[a/z]|_{y}^{f} \\
|!A|_{f}^{x} \equiv |A|_{f}^{x} \quad \text{(Gödel dialectica)} \\
\text{or } \forall y \in fx |A|_{y}^{x} \quad \text{(Diller-Nahm variant)}
\]
**Functional interpretations – Linear logic**

**P** and **O** choose moves simultaneously!

Assume $|A| : X \times Y \rightarrow \mathbb{B}$ and $|B| : V \times W \rightarrow \mathbb{B}$ defined

\[
|A^\perp|_x^y \quad \equiv \quad \neg |A|_x^y
\]

\[
|A \& B|_{\text{inj}_b}^{\langle x,v \rangle} \quad \equiv \quad \begin{cases} 
|A|_x^y & \text{if } b = 0 \\
|B|_v^y & \text{if } b = 1 
\end{cases}
\]

\[
|A \otimes B|_{\langle f,g \rangle}^{\langle x,v \rangle} \quad \equiv \quad |A|_f^{xv} \land |B|_g^v
\]

\[
\forall z A|_f^{\langle a,y \rangle} \quad \equiv \quad |A[a/z]|_y^f
\]

\[
|!A|_f^x \quad \equiv \quad |A|_f^{fx} \quad \quad \text{ (Gödel dialectica)}
\]

or $\forall y \in fx \, |A|_y^x$ (Diller-Nahm variant)

or $\forall y \, |A|_y^x$ (modified realizability)
Outline

1. Lorenzen Games
2. Blass Games
3. Strategic-form Games
4. Extensive-form Games (Generalised)
An extensive form game consists of

- A set of $n$ players
- A tree $T$, called the game tree
- A payoff function $q : T_{\text{leaf}} \rightarrow \mathbb{R}^n$
  ($T_{\text{leaf}} = \text{leaves of } T$)
- A partition of the non-terminal nodes into $n$ subsets
Generalising “Goal”

Usually:

\[ X = \text{set of choices} \]
\[ \mathbb{R} = \text{payoffs} \]

Maximise return

\[
\max \in (X \to \mathbb{R}) \to \mathbb{R}
\]
Generalising “Goal”

**Usually:**

\[
X = \text{set of choices} \quad \mathbb{R} = \text{payoffs}
\]

Maximise return

\[\max \in (X \to \mathbb{R}) \to \mathbb{R}\]

**More generally:**

\[
X = \text{set of possible moves} \quad R = \text{set of outcomes}
\]

“Quantifier”

\[\phi \in (X \to R) \to 2^R\]
Generalising “Goal”

**Usually:**

\[ X = \text{set of choices} \]
\[ \mathbb{R} = \text{payoffs} \]

Maximise return

\[ \max \in (X \to \mathbb{R}) \to \mathbb{R} \]

**More generally:**

\[ X = \text{set of possible moves} \]
\[ R = \text{set of outcomes} \]

“Quantifier”

\[ \phi \in (X \to R) \to 2^R \]

**Other Quantifiers:** \( \exists, \forall, \sup, \inf, \min, \max, \int_0^1, \text{fix} \)
Extensive-form Game (Generalised)

No players! (at least not explicitly)
No players! (at least not explicitly)

An extensive form game is described by

- A labelled tree $T$, called the game tree
  $(X_s = \text{labels on branching at position } s)$
- A set of outcomes $R$
- Quantifiers $\phi_s : K_R X_s$ for each position $s$
- An outcome function $q : T_{\text{leaf}} \rightarrow R$
  $(T_{\text{leaf}} = \text{leaves of } T)$
Definition (Strategy)

Choice of move for each position, i.e.

\[
\text{next}: \prod_{s \in T} X_s
\]
Definition (Strategy)
Choice of move for each position, i.e.

\[ \text{next: } \prod_{s \in T} X_s \]

Definition (Strategic Play)
Any strategy and position \( s \) determines a play \( \alpha^s \), which we call the **strategic extension** of \( s \)
Definition (Strategy)
Choice of move for each position, i.e.

\[
\text{next: } \Pi_{s \in T} X_s
\]

Definition (Strategic Play)
Any strategy and position \(s\) determines a play \(\alpha^s\), which we call the \textbf{strategic extension} of \(s\)

Definition (Optimal Strategy)
A strategy is \textbf{optimal} if for any position \(s\) we have

\[
q(s * \alpha^s) \in \phi_s(\lambda x. q(s * x * \alpha^{s*x}))
\]
Functionals $\varepsilon: (X \to R) \to X$ are called selection functions.
Quantifiers and Selection Functions

Functionals $\varepsilon : (X \to R) \to X$ are called selection functions $J_R X$.

A quantifier $\phi : K_R X$ is attainable if for some $\varepsilon : J_R X$,

$$p(\varepsilon p) \in \phi p$$

for all $p : X \to R$. 
Quantifiers and Selection Functions

Functionals $\varepsilon : (X \rightarrow R) \rightarrow X$ are called selection functions

A quantifier $\phi : K_RX$ is attainable if for some $\varepsilon : J_RX$

$$p(\varepsilon p) \in \phi p$$

for all $p : X \rightarrow R$

$J_R$ and $K_R$ are strong monads, so we have $F \in \{J_R, K_R\}$

$$\otimes : FX \times (X \rightarrow FY) \rightarrow F(X \times Y)$$

product operations on selection functions and quantifiers
Iterated Products

Iterated product of quantifiers

\[
(\otimes^T_s \varphi) (q) \overset{R}{=} \begin{cases} 
q(\langle \rangle) & \text{if } T_{\text{leaf}}(s) \\
(\varphi_s \otimes \lambda x. (\otimes^{T}_{s\ast x} \varphi))(q) & \text{otherwise}
\end{cases}
\]

where \(q\) is the outcome function of sub-game at position \(s\)
Iterated Products

Iterated product of quantifiers

\[
\left( \bigotimes_s T \phi \right) (q) \overset{R}{=} \begin{cases} 
q([]) & \text{if } T_{\text{leaf}}(s) \\
\left( \phi_s \otimes \lambda x. \left( \bigotimes_{s^x} T \phi \right) \right)(q) & \text{otherwise}
\end{cases}
\]

where \( q \) is the outcome function of sub-game at position \( s \)

Iterated product of selection functions

\[
\left( \bigotimes_s T \varepsilon \right) (q) = \begin{cases} 
[] & \text{if } T_{\text{leaf}}(s) \\
\left( \varepsilon_s \otimes \lambda x. \left( \bigotimes_{s^x} T \phi \right) \right)(q) & \text{otherwise}
\end{cases}
\]

Spector's BR \( \equiv \) Restricted BR, over system \( T \) [O./Powell'12]
Iterated Products

Iterated product of quantifiers ($\sim$ Spector’s bar recursion)

$$(\bigotimes_s^T \phi)(q) \overset{R}{=} \begin{cases} q(\emptyset) & \text{if } T_{\text{leaf}}(s) \\ \left(\phi_s \otimes \lambda x. \left(\bigotimes_{s \times x}^T \phi\right)\right)(q) & \text{otherwise} \end{cases}$$

where $q$ is the outcome function of sub-game at position $s$

Iterated product of selection functions

$$(\bigotimes_s^T \varepsilon)(q) = \begin{cases} \emptyset & \text{if } T_{\text{leaf}}(s) \\ \left(\varepsilon_s \otimes \lambda x. \left(\bigotimes_{s \times x}^T \phi\right)\right)(q) & \text{otherwise} \end{cases}$$
Iterated Products

Iterated product of quantifiers ($\sim$ Spector’s bar recursion)

\[
\left( \bigotimes_s^T \phi \right) (q) \overset{R}{=} \begin{cases} 
q([]) & \text{if } T_{\text{leaf}}(s) \\
\left( \varepsilon_s \otimes \lambda x. \left( \bigotimes_{s \ast x}^T \phi \right) \right) (q) & \text{otherwise}
\end{cases}
\]

where $q$ is the outcome function of sub-game at position $s$

Iterated product of selection functions ($\sim$ Restricted BR)

\[
\left( \bigotimes_s^T \varepsilon \right) (q) = \begin{cases} 
[] & \text{if } T_{\text{leaf}}(s) \\
\left( \varepsilon_s \otimes \lambda x. \left( \bigotimes_{s \ast x}^T \phi \right) \right) (q) & \text{otherwise}
\end{cases}
\]
Iterated Products

Iterated product of quantifiers ($\sim$ Spector’s bar recursion)

\[
\left( \bigotimes_{s}^{T} \phi \right) (q) \overset{R}{=} \begin{cases} 
q([]) & \text{if } T_{\text{leaf}}(s) \\
\left( \phi_{s} \otimes \lambda x. \left( \bigotimes_{s \ast x}^{T} \phi \right) \right) (q) & \text{otherwise}
\end{cases}
\]

where $q$ is the outcome function of sub-game at position $s$

Iterated product of selection functions ($\sim$ Restricted BR)

\[
\left( \bigotimes_{s}^{T} \varepsilon \right) (q) = \begin{cases} 
[] & \text{if } T_{\text{leaf}}(s) \\
\left( \varepsilon_{s} \otimes \lambda x. \left( \bigotimes_{s \ast x}^{T} \phi \right) \right) (q) & \text{otherwise}
\end{cases}
\]

Spector’s BR $\equiv$ Restricted BR, over system $T$ [O./Powell’12]
Sequential Games – Main Result

Fix an unbounded game \( G = (T, R, \phi, q) \)

Assume \( \phi_s : K_R X_s \) attainable with selection fcts \( \varepsilon_s : J_R X_s \)
Sequential Games – Main Result

Fix an unbounded game $G = (T, R, \phi, q)$

Assume $\phi_s : K_RX_s$ attainable with selection fcts $\varepsilon_s : J_RX_s$

**Theorem (Escardo/O.’2010)**

An optimal strategy for $G$ can be calculated as

$$\text{next}(s) \overset{X_s}{=} \left( \left( \begin{array}{c} T \\ \bigotimes_s \varepsilon \\ \bigotimes_s q \end{array} \right)(q) \right)_0$$
Sequential Games – Main Result

Fix an unbounded game $G = (T, R, \phi, q)$
Assume $\phi_s : K_R X_s$ attainable with selection fcts $\varepsilon_s : J_R X_s$

**Theorem (Escardo/O.’2010)**
An optimal strategy for $G$ can be calculated as

$$\text{next}(s) \overset{X_s}{=} \left( \left( \left( \underbrace{T}_{s} \prod_{\varepsilon} s \right) (q) \right) \right)_0$$

*Backward induction* @ Game Theory $(\phi = \sup)$
*Bekič’s lemma* @ Fixed Point Theory $(\phi = \text{fix})$
*Backtracking* @ Algorithms $(\phi = \exists)$
*Bar recursion* @ Proof Theory
Countable Choice (dialectica interpretation)

Let us look at negative translation of countable choice:

\[ \Pi_1^0 \text{-AC}^N : \forall n \neg \exists x A_n(x) \rightarrow \neg \exists \alpha \forall n A_n(\alpha n) \]
Countable Choice (dialectica interpretation)

Let us look at negative translation of countable choice:

$$\Pi_1^1-\text{AC}_0^N : \forall n \neg \exists x A_n(x) \rightarrow \neg \exists \alpha \forall n A_n(\alpha n)$$

Assuming interpretation of $A_n(x)$ is $|A_n(x)|_y$ we have

$$\forall n \neg \exists x \forall y |A_n(x)|_y \rightarrow \neg \exists \alpha \forall n \forall y |A_n(\alpha n)|_y$$
Countable Choice (dialectica interpretation)

Let us look at negative translation of countable choice:

\[ \Pi_1^1-AC_0^N : \forall n \forall x A_n(x) \rightarrow \neg \exists \alpha \forall n A_n(\alpha n) \]

Assuming interpretation of \( A_n(x) \) is \( |A_n(x)|_y \) we have

\[ \forall n \forall x \forall y |A_n(x)|_y \rightarrow \neg \exists \alpha \forall n \forall y |A_n(\alpha n)|_y \]

and then

\[ \exists \varepsilon \forall n \forall p |A_n(\varepsilon np)|_{p(\varepsilon np)} \rightarrow \forall q, \omega \exists \alpha \forall n \leq \omega \alpha |A_n(\alpha n)|_{q\alpha} \]
Countable Choice (dialectica interpretation)

Let us look at negative translation of countable choice:

$$\Pi_1^1-AC_0^N : \forall n \neg \exists x A_n(x) \rightarrow \neg \exists \alpha \forall n A_n(\alpha n)$$

Assuming interpretation of $A_n(x)$ is $|A_n(x)|_y$ we have

$$\forall n \neg \exists x \forall y |A_n(x)|_y \rightarrow \neg \exists \alpha \forall n \forall y |A_n(\alpha n)|_y$$

and then

$$\exists \varepsilon \forall n \forall p |A_n(\varepsilon np)|_{p(\varepsilon np)} \rightarrow \forall q, \omega \exists \alpha \forall n \leq \omega \alpha |A_n(\alpha n)|_{q\alpha}$$

Finally

$$\forall \varepsilon, q, \omega \exists \alpha \left( \forall n \forall p |A_n(\varepsilon np)|_{p(\varepsilon np)} \rightarrow \forall n \leq \omega \alpha |A_n(\alpha n)|_{q\alpha} \right)$$
Countable Choice (dialectica interpretation)

Let us look at negative translation of countable choice:

\[ \Pi_1^1-\text{AC}^N_0 : \forall n \neg \exists x A_n(x) \rightarrow \neg \forall \alpha \forall n A_n(\alpha n) \]

Assuming interpretation of \( A_n(x) \) is \( |A_n(x)|_y \) we have

\[ \forall n \neg \exists x \forall y |A_n(x)|_y \rightarrow \neg \forall \alpha \forall n \forall y |A_n(\alpha n)|_y \]

and then

\[ \exists \varepsilon \forall n \forall p |A_n(\varepsilon n p)|_{p(\varepsilon n p)} \rightarrow \forall q, \omega \exists \alpha \forall n \leq \omega \alpha |A_n(\alpha n)|_{q\alpha} \]

Finally

\[ \forall \varepsilon, q, \omega \exists \alpha \left( \forall n \forall p |A_n(\varepsilon n p)|_{p(\varepsilon n p)} \rightarrow \forall n \leq \omega \alpha |A_n(\alpha n)|_{q\alpha} \right) \]

quantifier at round \( n \)
Countable Choice (dialectica interpretation)

Let us look at negative translation of countable choice:

$$\Pi_1^{-AC}_0^N : \forall n \neg \neg \exists x A_n(x) \rightarrow \neg \neg \exists \alpha \forall n A_n(\alpha n)$$

Assuming interpretation of $A_n(x)$ is $|A_n(x)|_y$ we have

$$\forall n \neg \neg \exists x \forall y |A_n(x)|_y \rightarrow \neg \neg \exists \alpha \forall n \forall y |A_n(\alpha n)|_y$$

and then

$$\exists \varepsilon \forall n \forall p |A_n(\varepsilon np)|_{p(\varepsilon np)} \rightarrow \forall q, \omega \exists \alpha \forall n \leq \omega \alpha |A_n(\alpha n)|_{q\alpha}$$

Finally

$$\forall \varepsilon, q, \omega \exists \alpha \left( \forall n \forall p |A_n(\varepsilon np)|_{p(\varepsilon np)} \rightarrow \forall n \leq \omega \alpha |A_n(\alpha n)|_{q\alpha} \right)$$
Countable Choice (dialectica interpretation)

Let us look at negative translation of countable choice:

\[ \Pi_1^0-\mathrm{AC}^N : \forall n \neg \exists x A_n(x) \rightarrow \neg \exists \alpha \forall n A_n(\alpha n) \]

Assuming interpretation of \( A_n(x) \) is \( |A_n(x)|_y \) we have

\[ \forall n \neg \exists x \forall y |A_n(x)|_y \rightarrow \neg \exists \alpha \forall n \forall y |A_n(\alpha n)|_y \]

and then

\[ \exists \varepsilon \forall n \forall p |A_n(\varepsilon np)|_{p(\varepsilon np)} \rightarrow \forall q, \omega \exists \alpha \forall n \leq \omega \alpha |A_n(\alpha n)|_{q\alpha} \]

Finally

\[ \forall \varepsilon, q, \omega \exists \alpha \left( \forall n \forall p |A_n(\varepsilon np)|_{p(\varepsilon np)} \rightarrow \forall n \leq \omega \alpha |A_n(\alpha n)|_{q\alpha} \right) \]
Countable Choice (dialectica interpretation)

Interpretation of $\text{AC}_0 \equiv$ Game in extensive form
Countable Choice (dialectica interpretation)

Interpretation of AC_0 ≡ Game in extensive form

Given \(|A_n(x)|_y\) and selection fcts. \(\varepsilon_n\) define quantifiers

\[
\phi_n p \equiv \{y : |A_n(\varepsilon_n p)|_y\}
\]
Countable Choice (dialectica interpretation)

Interpretation of $\text{AC}_0 \equiv \text{Game in extensive form}$

Given $|A_n(x)|_y$ and selection fcts. $\varepsilon_n$ define quantifiers

$$\phi_n p \equiv \{y : |A_n(\varepsilon_n p)|_y\}$$

Premise of $|\text{AC}_0^N|$ says that $\phi_n$ are attainable with sel. fcts. $\varepsilon_n$
Countable Choice (dialectica interpretation)

Interpretation of $AC_0 \equiv$ Game in extensive form

Given $|A_n(x)|_y$ and selection fcts. $\varepsilon_n$ define quantifiers

$$\phi_n p \equiv \{y : |A_n(\varepsilon_n p)|_y\}$$

Premise of $|AC_0^N|$ says that $\phi_n$ are attainable with sel. fcts. $\varepsilon_n$

**Corollary**

Given $A_n(x)$, a witness $\alpha$ for dialectica interpretation of $\Pi_1-AC_0^N$ can be calculated as

$$\alpha = \left(\begin{array}{c|c}
T & \varepsilon \\
\hline
s & \end{array}\right)(q')$$

where $T_{leaf}(s) \equiv \omega(s \ast 0) < |s|$ and $q'(s) = q(s \ast 0)$
A. Blass
A game semantics for linear logic
*APAL*, 56:183-220, 1992

P. Oliva
Unifying functional interpretations

M. Escardó and P. Oliva
Selection functions, bar recursion and backward induction
*MSCS*, 20(2):127-168, 2010

M. Escardó and P. Oliva
Sequential games and optimal strategies