Programs from Proofs IV

Programs from classical proofs via Gödel's dialectica interpretation

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Outline

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1. Motivation
2. The *dialectica* Interpretation
3. Interpretation at Work
   - Classical Predicate Logic
   - Classical Arithmetic
   - Classical Analysis

Inverse of a Function

Theorem

For any $H\colon X \to \mathbb{N}$ *there exists* $\alpha\colon \mathbb{N} \to X$ *such that*

$$H(\alpha(k)) = k \qquad \textit{whenever} \quad k \in \mathrm{img}(H)$$

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$$\forall k(\exists x(Hx = k) \to \exists x'(Hx' = k))$$

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and invoke the **axiom of (countable) choice**

$$\exists \alpha \forall k(\exists x(Hx = k) \to (H(\alpha k) = k))$$

No Injection from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$

Theorem

For any $H : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ *there exist* $f, g : \mathbb{N} \to \mathbb{N}$ *such that*

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Let $f_\alpha = \lambda n.\alpha(n)(n) + 1$ and $g_\alpha = \alpha(k_\alpha)$ where $k_\alpha = H(f_\alpha)$

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Clearly $f_\alpha(k_\alpha) \neq g_\alpha(k_\alpha)$ and $H(f_\alpha) = k_\alpha \stackrel{(*)}{=} H(g_\alpha)$ □

Interpreting Classical Theorems

How to "witness" a theorem like this:

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Q: *What does it mean to computationally interpret this?*

Herbrand Theorem

Theorem (Σ_1-formulas)

If ones proves $\exists x Q(x)$ classically then one can also prove

$$Q(t_0) \vee \ldots \vee Q(t_n)$$

for a finite family of terms $(t_i)_{i \leq n}$.

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If ones proves $\exists x \forall y Q(x, y)$ classically then one can also prove

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look at its Herbrand normal form

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Proof. Either $Q_n(p0)$ in which case we have (by weakening)

$$Q_n(p(p0)) \to Q_n(p0)$$

or $\neg Q_n(p0)$ in which case we have (by efq)

$$Q_n(p0) \to Q_n(0).$$

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enough to consider $t_0 = 0$ and $t_1 = p0$, i.e.

$$(Q_n(p0) \to Q_n(0)) \vee (Q_n(p(p0)) \to Q_n(p0))$$

Can even produce **single witness** if able to check $Q_n(x)$

$$\varepsilon_n p = \begin{cases} 0 & \text{if } \neg Q_n(p0) \\ p0 & \text{otherwise} \end{cases}$$

is such that $Q_n(p(\varepsilon_n p)) \to Q_n(\varepsilon_n p)$

Herbrand's Theorem

- Herbrand theorem only works for **prenex formulas**
- **Not modular** (as cut elimination)

  witnesses for A *and* $A \to B$ *doesn't give one for* B
- Similar to Kreisel's n.c.i. (which has same problems)

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- A modular generalisation that works for all formulas?

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- A modular generalisation that works for all formulas:

  Gödel's *dialectica* interpretation!

Outline

Gödel's *dialectica* Interpretation

Map every formula to the $\exists\forall$-fragment. For instance:

$$\exists x \forall y P(x,y) \qquad \mapsto \quad \exists x \; \forall y \; P(x,y)$$

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$$\begin{aligned} \exists x \forall y P(x,y) &\mapsto \exists x\ \forall y\ P(x,y) \\ \forall x \exists y P(x,y) &\mapsto \exists f\ \forall x\ P(x, fx) \end{aligned}$$

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Map every formula to the ∃∀-fragment. For instance:

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\exists x P(x) \wedge \forall y Q(y) & \mapsto & \exists x \; \forall y \; (P(x) \wedge Q(y))
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\exists x P(x) \wedge \forall y Q(y) & \mapsto & \exists x\ \forall y\ (P(x) \wedge Q(y)) \\
\exists x P(x) \rightarrow \exists y Q(y) & \mapsto & \exists f\ \forall x\ (P(x) \rightarrow Q(fx))
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\neg\neg \exists x \forall y P(x,y) & \mapsto & \exists \varepsilon\ \forall p\ \neg\neg P(\varepsilon p, p(\varepsilon p))
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Gödel's *dialectica* Interpretation

Can think of the mapping

$$A \quad \mapsto \quad \exists x \forall y A_D(x, y)$$

as associating a set of **functionals** to each formula

$$A \quad \mapsto \quad W_A \equiv \{\, t \in \mathsf{T} \,:\, \forall y A_D(t, y) \}$$

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Theorem (Soundness – Intuitionistic Version)

If A is HA-provable then W_A is non-empty.

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Theorem (Soundness – Intuitionistic Version)

If A is HA-provable then W_A is non-empty. That is, if

(1) A is provable in Heyting arithmetic

then

(2) $A_D(t, y)$ is provable in the quantifier-free calculus T,

for some term $t \in$ T.

Negative Translation

Extending dialectica interpretation to **classical** logic

Compose with embedding of CL into IL.

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\begin{array}{lcl}
(P)^N & \mapsto & P \\
(A \wedge B)^N & \mapsto & A^N \wedge B^N \\
(A \vee B)^N & \mapsto & A^N \vee B^N \\
(A \to B)^N & \mapsto & A^N \to B^N \\
(\exists x A)^N & \mapsto & \exists x A^N \\
(\forall x A)^N & \mapsto & \forall x \neg\neg A^N
\end{array}
$$

Then CL $\vdash A$ implies IL $\vdash \neg\neg A^N$ (Kuroda'51)

Examples of Negative Translation

Classical		**Intuitionistic**
$\exists x P(x) \vee \neg \exists x P(x)$	$\mapsto$	$\neg\neg(\exists x P(x) \vee \neg \exists x P(x))$

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$\neg\neg \exists n P(n) \to \exists n P(n)$	$\mapsto$	$\neg\neg(\neg\neg \exists n P(n) \to \exists n P(n))$
$\exists n \forall k (fn \leq fk)$	$\mapsto$	$\neg\neg \exists n \forall k \neg\neg (fn \leq fk)$

Soundness (Peano Arithmetic)

Theorem (Classical Version)

Assume A^N interpreted as $\exists x \forall y A^N_D(x, y)$. If

(1) A *is provable in Peano arithmetic*

then

(2) $A^N_D(t, y)$ *is provable in the quantifier-free calculus T, for some term $t \in$ T.*

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We can prove (classically)

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$$\neg\neg\exists x \forall y A_i(x,y)$$

Whose dialectica interpretation is

$$\exists \varepsilon_i \forall p A_i(\varepsilon_i p, p(\varepsilon_i p))$$

which has witness

$$\varepsilon_i p = \begin{cases} 0 & \text{if } \neg Q_i(p0) \\ p0 & \text{if } \quad Q_i(p0) \end{cases}$$

Classical Arithmetic

We have

$$\forall i \leq n \exists x \forall y (\underbrace{Q_i(y) \to Q_i(x)}_{A_i(x,y)})$$

By finite choice (i.e. induction) we obtain

$$\exists s \forall i \leq n \forall y A_i(s_i, y)$$

Classical Arithmetic

We have

$$\forall i \leq n \exists x \forall y (\underbrace{Q_i(y) \to Q_i(x)}_{A_i(x,y)})$$

By finite choice (i.e. induction) we obtain

$$\exists s \forall i \leq n \forall y A_i(s_i, y)$$

Its (classical) dialectica interpretation is

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ε_i as in previous slide

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product of sel. fcts.

Selection Functions

Let $J_R X = (X \to R) \to X$

$\varepsilon \colon J_R X$ are called **selection functions**

Given sequence $\varepsilon \colon \Pi_{i \leq n} J_R X_i$, define (⊗ prod of sel. fct.)

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Theorem

Let $s = (\bigotimes_{i=0}^{n} \varepsilon_i)(q)$ *with* $q \colon \Pi_{i=0}^{n} X_i \to R$. *For* $0 \leq i \leq n$

$$\begin{aligned} s_i &\stackrel{X_i}{=} \varepsilon_i p_i \\ qs &\stackrel{R}{=} p_i(\varepsilon_i p_i) \end{aligned}$$

for some $p_i \colon X_i \to R$.

Back to Example

Hence, given that

$$s_i \overset{X_i}{=} \varepsilon_i p_i$$
$$qs \overset{R}{=} p_i(\varepsilon_i p_i)$$

In order to produce s such that

$$\forall i \leq n \quad \underbrace{A_i(s_i, qs)}_{Q_i(qs) \to Q_i(s_i)}$$

we only need to find ε_i such that for all p_i

$$\forall i \leq n \; A_i(\varepsilon_i p_i, p_i(\varepsilon_i p_i))$$

(which is easy, as we have seen!)

Classical Analysis

What about infinitely many "uses" of classical logic? Given

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whose *dialectica* interpretation (of negative translation) is

$$\forall \psi \forall q \exists \alpha \forall n \leq \psi\alpha \, (\underbrace{Q_n(q\alpha) \to Q_n(\alpha(n))}_{A_n(\alpha(n),q\alpha)})$$

Controlled Iterated Product

This can be solved by a “controlled” iterated product

$$\left(\bigotimes_{s}^{\psi}\varepsilon\right)(q) \stackrel{R}{=} \begin{cases} \mathbf{0} & \psi(\hat{s}) < |s| \\ \left(\varepsilon_{|s|} \otimes \lambda x^{X_{|s|}}.\left(\bigotimes_{s*x}^{\psi}\varepsilon\right)\right)(q) & \text{otherwise} \end{cases}$$

Theorem

Let $\alpha = \left(\bigotimes_{\langle\rangle}^{\psi}\varepsilon\right)(q)$*. There exist* $p_i \colon X_i \to R$ *s.t.*

$$\begin{aligned} \alpha_i &\stackrel{X_i}{=} \varepsilon_i(p_i) \\ q\alpha &\stackrel{R}{=} p_i(\varepsilon_i p_i) \end{aligned}$$

for all $i \leq \psi(\alpha)$*.*

Back to $(\mathbb{N}^{\mathbb{N}} \to \mathbb{N})$-Example (I)

Theorem

For any $H : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ *there exist* $f, g : \mathbb{N} \to \mathbb{N}$ *such that*

$$f \neq g \quad \textit{and} \quad H(f) \stackrel{\mathbb{N}}{=} H(g)$$

Proof.

Let $\alpha^{\mathbb{N} \to \mathbb{N}^{\mathbb{N}}}$ be some inverse of H, i.e. for all f and k

$$(*) \quad H(\alpha(k)) = k \qquad \text{if} \;\; H(f) = k$$

(using **classical logic** and **countable choice**)

Let $f_\alpha = \lambda n.\alpha(n)(n) + 1$ and $g_\alpha = \alpha(k_\alpha)$ where $k_\alpha = H(f_\alpha)$

Clearly $f_\alpha(k_\alpha) \neq g_\alpha(k_\alpha)$ and $H(f_\alpha) = k_\alpha \stackrel{(*)}{=} H(g_\alpha)$ □

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Construct approximation to inverse of H, i.e. $\alpha^{\mathbb{N} \to \mathbb{N}^{\mathbb{N}}}$ s.t.

$$\forall k \leq H(f_\alpha) \left(\underbrace{H(f_\alpha) = k \ \to \ H(\alpha(k)) = k}_{A_k(\alpha(k), f_\alpha)} \right)$$

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Enough to produce ε_k such that for all p

$$\underbrace{H(p(\varepsilon_k p)) = k \; \to \; H(\varepsilon_k p) = k}_{A_k(\varepsilon_k p, p(\varepsilon_k p))}$$

We have just built such ε_k's!

Back to $(\mathbb{N}^{\mathbb{N}} \to \mathbb{N})$-Example (II)

Let ε_i as before and $f_\alpha := \lambda n.\alpha(n)(n) + 1$

Theorem

Fix $H : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$. Let $q\alpha = f_\alpha$ and $\psi\alpha = H(f_\alpha)$. Define

$$\alpha = \left(\bigotimes_{\langle\,\rangle}^{\psi} \varepsilon \right)(q)$$

and $f = f_\alpha$ and $g = \alpha(\psi\alpha)$. Then

$$Hf = Hg \qquad \text{and} \qquad f(\psi\alpha) \neq g(\psi\alpha)$$

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