The Theory of Selection Functions

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(based on joint work with M. Escardó)

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Outline

1. Quantifiers and Selection Functions

2. Finite and Infinite Products

3. Sequential Games
Outline

1. Quantifiers and Selection Functions
2. Finite and Infinite Products
3. Sequential Games
Quantifiers

\[ \phi : (X \to R) \to R \]
Quantifiers

\[ \phi : (X \to R) \to R \]

**For instance:**

<table>
<thead>
<tr>
<th>Operation</th>
<th>( \phi : (X \to R) \to R )</th>
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<tr>
<td>Quantifiers</td>
<td>( \forall X, \exists X : (X \to \mathbb{B}) \to \mathbb{B} )</td>
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<td>Supremum</td>
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## Quantifiers

\[ \phi : (X \to R) \to R \quad (\equiv K_RX) \]

### For instance:

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Quantifiers (Multi-valued)

\[ \phi : (X \to R) \to 2^R \quad (\equiv K_R X) \]

For instance:

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<td>Supremum-(i)</td>
<td>( \sup_{[0,1]}^i : ([0, 1] \to \mathbb{R}^n) \to 2^{\mathbb{R}^n} )</td>
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Theorem (Witness Theorem)

For any $p : X \rightarrow \mathbb{B}$ there is a point $a \in X$ such that

$$p(a) \iff \exists x^X p(x)$$

(similar to Hilbert's $\varepsilon$-term)
The Theory of Selection Functions

Quantifiers and Selection Functions

Theorem (Witness Theorem)
For any \( p: X \rightarrow \mathbb{B} \) there is a point \( a \in X \) such that
\[
p(a) \iff \exists x^X p(x)
\]
(similar to Hilbert’s \( \varepsilon \)-term)

Theorem (Counter-example Theorem)
For any \( p: X \rightarrow \mathbb{B} \) there is a point \( a \in X \) such that
\[
p(a) \iff \forall x^X p(x)
\]
(\( a \) is counter-example to \( p \) if one exists)
Theorem (Mean Value Theorem)

For any \( p \in C[0, 1] \) there is a point \( a \in [0, 1] \) such that

\[
p(a) = \int_{0}^{1} p
\]
Theorem (Mean Value Theorem)

For any \( p \in C[0, 1] \) there is a point \( a \in [0, 1] \) such that

\[
p(a) = \int_0^1 p
\]

Theorem (Maximum Value Theorem)

For any \( p \in [0, 1] \rightarrow \mathbb{R}^n \) there is a point \( a \in [0, 1] \) such that

\[
p(a) \in \sup^i p
\]
Selection Functions

\[ \varepsilon : (X \to R) \to X \]
Selection Functions

\[ \varepsilon : (X \rightarrow R) \rightarrow X \quad (\equiv J_R X) \]
Selection Functions

\[ \varepsilon : (X \rightarrow R) \rightarrow X \quad (\equiv J_{RX}) \]

For instance:

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Attainable Quantifiers

**Definition (Selection Functions for a Quantifier)**

$\varepsilon: JX$ is called a *selection function* for $\phi: KX$ if

$$p(\varepsilon p) \in \phi(p)$$

holds for all $p: X \to R$. 
Attainable Quantifiers

**Definition (Selection Functions for a Quantifier)**

$\varepsilon : JX$ is called a **selection function** for $\phi : KX$ if

$$p(\varepsilon p) \in \phi(p)$$

holds for all $p : X \rightarrow R$

**Definition (Attainable Quantifiers)**

A quantifier $\phi : KX$ is called **attainable** if it has a selection function $\varepsilon : JX$
Attainable Quantifiers: Examples

- $\sup: \mathcal{K}_\mathbb{R}[0, 1]$ is an attainable quantifier
  
  \[ \sup(p) = p(\arg\sup(p)) \]

  where $\arg\sup: \mathcal{J}_\mathbb{R}[0, 1]$. 
Attainable Quantifiers: Examples

- \( \sup: \mathbb{K}_R[0, 1] \) is an attainable quantifier
  \[
  \sup(p) = p(\text{argsup}(p))
  \]
  where \( \text{argsup}: \mathbb{J}_R[0, 1] \).

- \( \text{fix}: \mathbb{K}_X X \) is an attainable quantifier
  \[
  \text{fix}(p) = p(\text{fix}(p))
  \]
  where \( \text{fix}: \mathbb{J}_X X \ (= \mathbb{K}_X X) \).
From Selection Functions to Quantifiers

Every selection function $\varepsilon : JX$ defines a quantifier $\overline{\varepsilon} : KX$

$$\overline{\varepsilon}(p) = p(\varepsilon(p))$$
From Selection Functions to Quantifiers

$\varepsilon : J X \rightarrow \overline{\varepsilon} : K X$

Not all quantifiers are attainable, e.g. $R = \{0, 1\}$

$\phi(p) = 0$
From Selection Functions to Quantifiers

Different $\varepsilon$ might define same $\phi$, e.g. $X = [0, 1]$ and $R = \mathbb{R}$

$$
\varepsilon_0(p) = \mu x. \sup p = p(x)
$$

$$
\varepsilon_1(p) = \nu x. \sup p = p(x)
$$
The Theory of Selection Functions

Finite and Infinite Products

Outline

1 Quantifiers and Selection Functions

2 Finite and Infinite Products

3 Sequential Games
Nested quantifiers \( \equiv \) single quantifier on \textbf{product space}
Nested quantifiers \( \equiv \) single quantifier on **product space**

\[ \exists x^X \forall y^Y p(x, y) \]
Nested quantifiers $\equiv$ single quantifier on **product space**

$$\exists x^X \forall y^Y p(x, y) \equiv (\exists X \otimes \forall Y)(p^{X \times Y} \rightarrow \mathbb{B})$$
Nested quantifiers $\equiv$ single quantifier on \textbf{product space}

\[
\exists x^X \forall y^Y p(x, y) \quad \equiv \quad (\exists X \otimes \forall Y)(p^{X \times Y} \to \mathbb{B})
\]

\[
\text{sup}_x \int_0^1 p(x, y) dy \quad \equiv \quad (\text{sup} \otimes \int)(p^{[0,1]^2} \to \mathbb{R})
\]
Nested quantifiers \(\equiv\) single quantifier on **product space**

\[
\exists x^X \forall y^Y p(x, y) \equiv (\exists x \otimes \forall y)(p^{X \times Y \to \mathbb{B}})
\]

\[
\sup_x \int_0^1 p(x, y) dy \equiv (\sup \otimes \int)(p^{[0,1]^2 \to \mathbb{R}})
\]

**Definition (Product of Single-valued Quantifiers)**

Given \(\phi: KX\) and \(\psi: KY\) define \(\phi \otimes \psi: K(X \times Y)\)

\[
(\phi \otimes \psi)(p) :\equiv \phi(\lambda x^X. \psi(\lambda y^Y. p(x, y)))
\]

where \(p: X \times Y \to \mathbb{R}\).
Nested quantifiers \( \equiv \) single quantifier on \textbf{product space}

\[
\exists x^X \forall y^Y p(x, y) \quad \equiv \quad (\exists X \otimes \forall Y)(p^{X \times Y} \rightarrow \mathbb{B})
\]

\[
\sup_x \int_0^1 p(x, y) \, dy \quad \equiv \quad (\sup \otimes \int)(p^{[0,1]^2} \rightarrow \mathbb{R})
\]

**Definition (Product of Single-valued Quantifiers)**

Given \( \phi : KX \) and \( \psi : KY \) define \( \phi \otimes \psi : K(X \times Y) \)

\[
(\phi \otimes \psi)(p) \quad \overset{R}{=} \quad \phi(\lambda x^X. \psi(\lambda y^Y.p(x, y)))
\]

where \( p : X \times Y \rightarrow \mathbb{R} \).

Does not work with multi-valued quantifiers!
Quantifier Elimination

Suppose $X$ and $Y$ are such that for some $\varepsilon$ and $\delta$

$$
\exists x^X p(x) = p(\varepsilon p)
$$
$$
\forall y^Y p(y) = p(\delta p).
$$
Suppose $X$ and $Y$ are such that for some $\varepsilon$ and $\delta$
\begin{align*}
\exists x^X p(x) &= p(\varepsilon p) \\
\forall y^Y p(y) &= p(\delta p).
\end{align*}
Then
\begin{equation*}
\exists x^X \forall y^Y p(x, y) = \exists x p(x, b(x))
\end{equation*}
where
\begin{equation*}
b(x) = \delta(\lambda y.p(x, y))
\end{equation*}
Quantifier Elimination

Suppose $X$ and $Y$ are such that for some $\varepsilon$ and $\delta$

\[
\exists x^X p(x) = p(\varepsilon p) \\
\forall y^Y p(y) = p(\delta p).
\]

Then

\[
\exists x^X \forall y^Y p(x, y) = \exists x p(x, b(x)) \\
= p(a, b(a))
\]

where

\[
b(x) = \delta(\lambda y.p(x, y)) \\
a = \varepsilon(\lambda x.p(x, b(x))).
\]
Product of Selection Functions

Definition (Product of Selection Functions)

Given $\varepsilon: JX$ and $\delta: JY$ define $\varepsilon \otimes \delta: J(X \times Y)$ as

$$(\varepsilon \otimes \delta)(p^{X \times Y \to R})^{X \times Y} := (a, b(a))$$

where

$$b(x) = \delta(\lambda y. p(x, y))$$

$$a = \varepsilon(\lambda x. p(x, b(x)))$$.
Homomorphism Lemma

\[ \varepsilon \otimes \delta = \bar{\varepsilon} \otimes \bar{\delta} \]
Homomorphism Lemma

\[ \varepsilon \otimes \delta = \varepsilon \otimes \bar{\delta} \]

**Proof.**
\[
(\varepsilon \otimes \delta)(q) = q(a, b_a) = \varepsilon(\lambda x.q(x, b_x)) = \varepsilon(\lambda x.\bar{\delta}(\lambda y.q(x, y))) = (\varepsilon \otimes \bar{\delta})(q). \quad \square
\]
Homomorphism Lemma

Lemma

\[ \varepsilon \otimes \delta = \overline{\varepsilon} \otimes \overline{\delta} \]

Proof.

\[ (\varepsilon \otimes \delta)(q) = q(a, b_a) = \overline{\varepsilon}(\lambda x. q(x, b_x)) = \overline{\varepsilon}(\lambda x. \overline{\delta}(\lambda y. q(x, y))) = (\overline{\varepsilon} \otimes \overline{\delta})(q). \]

Corollary

If \( \phi: KX \) and \( \psi: KY \) are attainable single-valued quantifiers with sel. fct. \( \varepsilon: JX \) and \( \delta: JY \) then

\[ \overline{\varepsilon} \otimes \overline{\delta} = \phi \otimes \psi \]
Definition (Iterated Product – Finite)

Given $\varepsilon_i : JX_i$, $0 \leq i \leq n$, define $(\bigotimes_{i=k}^n \varepsilon_i) : \Pi_{i=k}^n X_i$ as

$$\left( \bigotimes_{i=k}^n \varepsilon_i \right) = \varepsilon_k \otimes \left( \bigotimes_{i=k+1}^n \varepsilon_i \right)$$
Definition (Iterated Product – Finite)

Given $\varepsilon_i : JX_i$, $0 \leq i \leq n$, define $(\bigotimes_{i=k}^n \varepsilon_i) : J\Pi_{i=k}^n\ X_i$ as

\[
\left( \bigotimes_{i=k}^n \varepsilon_i \right) = \varepsilon_k \otimes \left( \bigotimes_{i=k+1}^n \varepsilon_i \right)
\]

Definition (Iterated Product – Infinite)

Given $\varepsilon_i : JX_i$, $i \in \mathbb{N}$, define $(\bigotimes_{i\geq k} \varepsilon_i) : J\Pi_{i\geq k}\ X_i$ as

\[
\left( \bigotimes_{i\geq k} \varepsilon_i \right) = \varepsilon_k \otimes \left( \bigotimes_{i\geq k+1} \varepsilon_i \right)
\]

for $q : \Pi_i X_i \to R$ continuous and $R = \mathbb{N}$ (assumed henceforth)
Theorem (Idempotency)

Given $\varepsilon_i : JX_i$ and $q : \prod_i X_i \to R$, let

$$\alpha \equiv \prod_{i \geq 0} X_i \left( \bigotimes_{i \geq 0} \varepsilon_i \right) (q)$$

then, for all $k$,

$$\text{tail}^k (\alpha) \equiv \prod_{i \geq k} X_i \left( \bigotimes_{i \geq k} \varepsilon_i \right) (q[\alpha](k))$$
Theorem (Idempotency)

Given $\varepsilon_i: J X_i$ and $q: \Pi_i X_i \to R$, let

$$\alpha \equiv \prod_{i \geq 0} X_i \left( \bigotimes_{i \geq 0} \varepsilon_i \right) (q)$$

then, for all $k$,

$$\text{tail}^k (\alpha) \equiv \prod_{i \geq k} X_i \left( \bigotimes_{i \geq k} \varepsilon_i \right) (q_{\alpha}(k))$$

Proof.

By course-of-values induction on $k$
Theorem (Product Quantifier)

Given attainable $\phi_i : K X_i$, with sel. func. $\varepsilon_i : J X_i$, and $q : \prod_i X_i \to R$, there exist $p_i : X_i \to R$ such that

$$
q(\alpha) = \left( \bigotimes_{i \geq 0} \varepsilon_i \right)(q) \in \bigcap_i \phi_i(p_i)
$$

($\alpha$ as before)
Theorem (Product Quantifier)

Given attainable $\phi_i : KX_i$, with sel. func. $\varepsilon_i : JX_i$, and $q : \prod_i X_i \to R$, there exist $p_i : X_i \to R$ such that

$$q(\alpha) = \left( \bigotimes_{i \geq 0} \varepsilon_i \right)(q) \in \bigcap_i \phi_i(p_i)$$

($\alpha$ as before)

Proof.

Take $p_i = \lambda y_i \cdot (\bigotimes_{k \geq i} \varepsilon_k)(q[\alpha](i) * y_i)$

Recall that $p_i(\varepsilon_i(p_i)) \in \phi_i(p_i)$

Then $p_i(\varepsilon_i(p_i)) = p_i(\alpha(i)) = q(\alpha)$ (Idempotency theorem)
Corollary (Spector Equation)

Given attainable quantifiers $\phi_i : KX_i$, with selection functions $\varepsilon_i : JX_i$, and $q : \Pi X_i \rightarrow R$, there exist $\alpha$ and $p_i$ such that

$$
\alpha(i) = \varepsilon_i(p_i)
$$

$$
q(\alpha) \in \phi_i(p_i) \quad \text{(for all } i)\n$$
Corollary (Spector Equation)

*Given attainable quantifiers* $\phi_i : K X_i$, *with selection functions* $\varepsilon_i : J X_i$, *and* $q : \Pi X_i \rightarrow R$, *there exist* $\alpha$ *and* $p_i$ *such that*

\[
\alpha(i) = \varepsilon_i(p_i)
\]

\[
q(\alpha) \in \phi_i(p_i) \quad (\text{for all } i)
\]

Proof.

Take $\alpha$ and $p_i$ as before, i.e.

\[
p_i = \lambda y_i . (\bigotimes_{k \geq i} \varepsilon_k)(q[\alpha](i) \ast y_i)
\]

\[
\alpha = (\bigotimes_{k \geq i} \varepsilon_k)(q)
\]
Theorem (Optimal Strategy)

**Given attainable** \( \phi_i : KX_i \), with sel. func. \( \varepsilon_i : JX_i \), and
\( q : \Pi_i X_i \rightarrow R \), **there exist** \( \alpha_k : \Pi_{i<k} X_i \rightarrow X_k \) **such that**

\[
q(\alpha^{\vec{x}}) \in \phi_k(\lambda y_k. q(\alpha^{\vec{x}}, y_k)) \quad (\vec{x} = x_0, \ldots, x_{k-1})
\]

where \( \alpha^{\vec{x}}(i) = x_i \) **if** \( i < k \) **and** \( \alpha_i([\alpha^{\vec{x}}](i)) \) **otherwise**
Theorem (Optimal Strategy)

*Given attainable $\phi_i: KX_i$, with sel. func. $\varepsilon_i: JX_i$, and $q: \Pi_i X_i \to R$, there exist $\alpha_k: \Pi_{i<k} X_i \to X_k$ such that*

$$q(\alpha^{\vec{x}}) \in \phi_k(\lambda y_k. q(\alpha^{\vec{x}}, y_k)) \quad (\vec{x} = x_0, \ldots, x_{k-1})$$

*where $\alpha^{\vec{x}}(i) = x_i$ if $i < k$ and $\alpha_i([\alpha^{\vec{x}}](i))$ otherwise*

Proof.

*Take $\alpha_k(\vec{x}) = \pi_0((\bigotimes_{k \geq i} \varepsilon_i)(q\vec{x}, y_i))$*

*We have $\alpha^{\vec{x}} = (\bigotimes_{k \geq i} \varepsilon_i)(q\vec{x})$ (Idempotency thm)*

*Use Product Quantifier theorem*
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3. Sequential Games
Sequential Games

**Definition**

A Game is a tuple \((R, (X_i)_{i \in \mathbb{N}}, (\phi_i)_{i \in \mathbb{N}}, q)\) where

- \(R\) is the set of **possible outcomes**
- \(X_i\) is the set of **available moves** at round \(i\)
- \(\phi_i: K_R X_i\) is the **goal (mul.-val.) quantifier** for round \(i\)
- \(q: \prod_{i \in \mathbb{N}} X_i \to R\) is the **outcome function**

with \(q\) determined after **finitely** many moves
Definition (Strategy)

Family of mappings $\text{next}_k : \prod_{i=0}^{k-1} X_i \rightarrow X_k$
Definition (Strategy)

Family of mappings $\text{next}_k : \prod_{i=0}^{k-1} X_i \rightarrow X_k$

Definition (Strategic Play)

Given strategy $\text{next}_k$ and partial play $\vec{a} = a_0, \ldots, a_{k-1}$, the **strategic extension** of $\vec{a}$ is $b^{\vec{a}} = b_k^{\vec{a}}, \ldots, b_{n-1}^{\vec{a}}$ where

$$b_i^{\vec{a}} = \text{next}_i(\vec{a}, b_k^{\vec{a}}, \ldots, b_{i-1}^{\vec{a}})$$
Definition (Strategy)

Family of mappings \( \text{next}_k : \prod_{i=0}^{k-1} X_i \rightarrow X_k \)

Definition (Strategic Play)

Given strategy \( \text{next}_k \) and partial play \( \vec{a} = a_0, \ldots, a_{k-1} \), the strategic extension of \( \vec{a} \) is \( \vec{b} \vec{a} = b^\vec{a}_k, \ldots, b^\vec{a}_{n-1} \) where

\[
    b^\vec{a}_i = \text{next}_i(\vec{a}, b^\vec{a}_k, \ldots, b^\vec{a}_{i-1})
\]

Definition (Optimal Strategy)

Strategy \( \text{next}_k \) is optimal if for any partial play \( \vec{a} \)

\[
    q(\vec{a}, \vec{b}) \in \phi_k(\lambda x_k . q(\vec{a}, x_k, \vec{b}^\vec{a}, x_k))
\]
Product of selection functions computes optimal strategies!
Product of selection functions computes optimal strategies!

Corollary

For any game with attainable goal quantifiers $\phi_i: KX_i$ an optimal strategy can be computed as

$$\text{next}_k(\vec{x}) = \pi_0 \left( \left( \bigotimes_{i \geq k} \varepsilon_i \right) (q_{\vec{x}}) \right)$$
Product of selection functions computes optimal strategies!

**Corollary**

For any game with attainable goal quantifiers $\phi_i: KX_i$ an optimal strategy can be computed as

$$\text{next}_k(\vec{x}) = \pi_0 \left( \left( \bigotimes_{i \geq k} \varepsilon_i \right) (q_{\vec{x}}) \right)$$

**Proof.**

Follows directly from Optimal Strategy theorem
Standard Game Theory

When $R = \mathbb{R}^n$ and $\phi_i$ are $\max^i$ or $\sup^i$

(attainable quantifiers with selection functions $\text{argsup}^i$)

Generalised Game $\mapsto$ Standard Game
Optimal strategy $\mapsto$ Strategy in Nash equilibrium
Product of $\text{argsup}^i$ $\mapsto$ Backward induction!
Proof Theory

Computational interpretation

$$\exists i \leq n \forall x^{X_i} \exists^R A_i(x, r) \quad \leftrightarrow \quad \forall \varepsilon(.) \exists i \leq n \exists p A_i(\varepsilon i p, p(\varepsilon i p))$$
Proof Theory

Computational interpretation

$$\exists i \leq n \forall x^X_i \exists r^R A_i(x, r) \rightarrow \forall \varepsilon(.) \exists i \leq n \exists p A_i(\varepsilon_i p, p(\varepsilon_i p))$$

$\varepsilon$’s define quantifiers, which partially define a game
Proof Theory

Computational interpretation

$$\exists i \leq n \forall x \exists r A_i(x, r) \implies \forall \varepsilon(.) \exists i \leq n \exists p A_i(\varepsilon_i p, p(\varepsilon p))$$

$\varepsilon$'s define quantifiers, which partially define a game

Computational interpretation relies on completing the definition of the game so optimal strategy solves problem
Open Questions

1. Relation of product of selection functions $\otimes$ to the different product BBC (over system $T$)

$$BBC(\varepsilon)(q) = \lambda n.\varepsilon_n \left( \lambda x_n.\overline{BBC(\varepsilon)(q_{(n,x_n)})} \right)$$

(due to Berardi, Bezem, Coquand)
Open Questions

1. Relation of product of selection functions \( \otimes \) to the different product BBC (over system T)

\[
\text{BBC}(\varepsilon)(q) = \lambda n.\varepsilon_n \left( \lambda x_n.\overline{\text{BBC}(\varepsilon)}(q_{(n,x_n)}) \right)
\]

(due to Berardi, Bezem, Coquand)

2. Does BBC compute optimal strategies in some different (but also natural) notion of game
References

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Sequential games and optimal strategies