Realizability Interpretations of Linear Logic

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Outline

1. Realizability (a reformulation)
2. Linear Logic (a model)
3. Functional Interpretations of LL
4. Functional Interpretations of ILL
Outline

1. Realizability (a reformulation)
2. Linear Logic (a model)
3. Functional Interpretations of LL
4. Functional Interpretations of ILL
Realizability

\[ \langle x, y \rangle \text{ mr } A \land B \quad \equiv \quad (x \text{ mr } A) \land (y \text{ mr } B) \]
\[ \langle x, y, i \rangle \text{ mr } A \lor B \quad \equiv \quad (x \text{ mr } A) \Diamond_i (y \text{ mr } B) \]
\[ f \text{ mr } A \rightarrow B \quad \equiv \quad \forall x((x \text{ mr } A) \rightarrow (fx \text{ mr } B)) \]
\[ \langle x, n \rangle \text{ mr } \exists z A \quad \equiv \quad x \text{ mr } A[n/z] \]
\[ f \text{ mr } \forall z A \quad \equiv \quad \forall z(fz \text{ mr } A) \]

where \( A \Diamond_i B \quad \equiv \quad (i = 0 \rightarrow A) \land (i = 1 \rightarrow B) \).
Realizability

Realizability associates a formula $A$ to a **set** of functionals (e.g. in Gödel’s $T$)

$$S_A \equiv \{ t : (t \in T) \land (t \text{ mr } A) \}$$

such that $A$ is provable iff $S_A$ is non-empty.
Realizability

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such that $A$ is provable iff $S_A$ is non-empty.

Realizability is a **proof interpretation**:

$$\vdash_{\pi} A \Rightarrow t_{\pi} \in S_A$$
Pointwise realizability

Can also be viewed as associating formulas to relations

\[
\langle x, v \rangle \text{ pmr}_{y,w} A \land B \quad :\equiv \quad (x \text{ pmr}_y A) \land (v \text{ pmr}_w B)
\]

\[
\langle x, v, i \rangle \text{ pmr}_{y,w} A \lor B \quad :\equiv \quad (x \text{ pmr}_y A) \bowtie_i (v \text{ pmr}_w B)
\]

\[
f \text{ pmr}_{x,w} A \rightarrow B \quad :\equiv \quad \forall y (x \text{ pmr}_y A) \rightarrow (fx \text{ pmr}_w B)
\]

\[
\langle x, n \rangle \text{ pmr}_y \exists z A \quad :\equiv \quad x \text{ pmr}_y A[n/z]
\]

\[
f \text{ pmr}_{z,y} \forall z A \quad :\equiv \quad fz \text{ pmr}_y A.
\]
Pointwise realizability

Can also be viewed as associating formulas to relations

\[ \langle x, v \rangle \mathrel{\text{pmr}}_{y,w} A \land B \equiv (x \mathrel{\text{pmr}}_y A) \land (v \mathrel{\text{pmr}}_w B) \]

\[ \langle x, v, i \rangle \mathrel{\text{pmr}}_{y,w} A \lor B \equiv (x \mathrel{\text{pmr}}_y A) \lozenge_i (v \mathrel{\text{pmr}}_w B) \]

\[ f \mathrel{\text{pmr}}_{x,w} A \rightarrow B \equiv \forall y (x \mathrel{\text{pmr}}_y A) \rightarrow (f x \mathrel{\text{pmr}}_w B) \]

\[ \langle x, n \rangle \mathrel{\text{pmr}}_y \exists z A \equiv x \mathrel{\text{pmr}}_y A[n/z] \]

\[ f \mathrel{\text{pmr}}_{z,y} \forall z A \equiv f z \mathrel{\text{pmr}}_y A. \]

An actual realiser refutes all possible challenges.

**Lemma**

\[(x \mathrel{\text{mr}} A) \iff \forall y (x \mathrel{\text{pmr}}_y A)\]
Embeddings IL into LL

\[(A \land B)^* \equiv A^* \& B^*\]

\[(A \lor B)^* \equiv \neg A^* \oplus \neg B^*\]

\[(A \rightarrow B)^* \equiv \neg A^* \lozenge B^*\]

\[(\forall x A)^* \equiv \forall x A^*\]

\[(\exists x A)^* \equiv \exists x \neg A^*\]
Embeddings IL into LL

\[(A \land B)^* \equiv A^* \& B^*\]
\[(A \lor B)^* \equiv !A^* \oplus !B^*\]
\[(A \to B)^* \equiv !A^* \multimap B^*\]
\[(\forall x A)^* \equiv \forall x A^*\]
\[(\exists x A)^* \equiv \exists x !A^*\]

\[(A \land B)^\circ \equiv A^\circ \otimes B^\circ\]
\[(A \lor B)^\circ \equiv A^\circ \oplus B^\circ\]
\[(A \to B)^\circ \equiv !(A^\circ \multimap B^\circ)\]
\[(\forall x A)^\circ \equiv !\forall x A^\circ\]
\[(\exists x A)^\circ \equiv \exists x A^\circ\]
Embeddings IL into LL

\[(A \land B)^* :\equiv A^* \land B^*\]
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\[(A \land B)^\circ :\equiv A^\circ \otimes B^\circ\]
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\[(A \rightarrow B)^\circ :\equiv !(A^\circ \rightarrow B^\circ)\]
\[(\forall x A)^\circ :\equiv !\forall x A^\circ\]
\[(\exists x A)^\circ :\equiv \exists x A^\circ\]

**Lemma**

\[A^\circ \circ \circ !A^*\]
Realizability and LL

\[(A \rightarrow B)^* \equiv !A^* \rightarrow B^*\]

\[(A \rightarrow B)^\circ \equiv !(A^\circ \rightarrow B^\circ)\]

\[f \text{ pmr}_{x,w} A \rightarrow B \equiv \forall y(x \text{ pmr}_y A) \rightarrow (fx \text{ pmr}_w B)\]

\[f \text{ mr} A \rightarrow B \equiv \forall x((x \text{ mr} A) \rightarrow (fx \text{ mr} B))\]
Realizability and LL

\[(A \to B)^* \equiv !A^* \to B^*\]

\[(A \to B)^\circ \equiv !(A^\circ \to B^\circ)\]

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\[f \text{ mr} A \to B \equiv \forall x ((x \text{ mr} A) \to (fx \text{ mr} B))\]
Realizability and LL

\[(A \rightarrow B)^* \equiv !A^* \rightarrow B^*\]

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\[f \text{ pmr}_{x,w} A \rightarrow B \equiv \forall y (x \text{ pmr}_y A) \rightarrow (fx \text{ pmr}_w B)\]

\[f \text{ mr} A \rightarrow B \equiv \forall x ((x \text{ mr} A) \rightarrow (fx \text{ mr} B))\]
Realizability and LL

\[(A \to B)^* \equiv !A^* \to B^*\]
\[(A \to B)\circ \equiv !(A\circ \to B\circ)\]
\[f \text{ pmr}_{x,w} A \to B \equiv \forall y (x \text{ pmr}_y A) \to (fx \text{ pmr}_w B)\]
\[f \text{ mr} A \to B \equiv \forall x ((x \text{ mr} A) \to (fx \text{ mr} B))\]

**Lemma**

\[A\circ \circ \to !A^*\]

**Lemma**

\[(x \text{ mr} A) \iff \forall y (x \text{ pmr}_y A)\]
Realizability and LL
Realizability and LL

\[
\begin{align*}
\text{IL} \xrightarrow{(\cdot)^*} \text{IL} & \quad \text{pointwise realizability} \\
\text{IL} \xrightarrow{(\cdot)^\circ} \text{IL} & \\
\text{LL} \xrightarrow{??} \text{LL} & \\
\text{IL} \xrightarrow{(\cdot)^\circ} \text{IL} & \\
\text{LL} \xrightarrow{??} \text{LL} & \\
\text{IL} \xrightarrow{(\cdot)^*} \text{IL} & \\
\end{align*}
\]
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1. Realizability (a reformulation)
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A model of LL

Interpret formulas $A$ of linear logic as **bipartite graphs**

- $(A^+, A^-, |A|_x^y)$
- two sets of nodes $A^+, A^-$
- edge relation $|A|_x^y$
A model of LL

Interpret formulas $A$ of linear logic as **bipartite graphs**

- $(A^+, A^-, |A|_x^y)$ (simultaneous game)
- two sets of nodes $A^+, A^-$ (sets of moves)
- edge relation $|A|_x^y$ (adjudication relation)
A model of LL

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- $(A^+, A^-, |A|_x)$ (simultaneous game)
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$\mathcal{B}(X, Y) \equiv$ bipartite graphs between $X$ and $Y$
(set of possible games with move-sets $X, Y$)
A model of LL

Interpret formulas $A$ of linear logic as **bipartite graphs**

- $(A^+, A^-, |A|^x_y)$ (simultaneous game)
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$\mathcal{B}(X, Y) \equiv$ bipartite graphs between $X$ and $Y$
(set of possible games with move-sets $X, Y$)

$\mathcal{B}_f(X, Y) \equiv$ functional bipartite graphs between $X$ and $Y$
(set of strategies in sequential version of game)
Some simple games

1 :≡ ({{*}, {*}, {{*, *}}})

⊥ :≡ ({{*}, {*}, {}})

0 :≡ ({}, {*}, {}})

⊤ :≡ ({{*}, {}, {}}).
Dual of a game

Given bipartite graph $A \equiv (A^+, A^-, |A|)$ define

$$A^\perp \equiv (A^-, A^+, \neg |A|).$$
Dual of a game

Given bipartite graph $A \equiv (A^+, A^-, |A|)$ define

$$A^\perp : \equiv (A^-, A^+, \neg|A|).$$

**Lemma**

- $A \sim (A^\perp)^\perp$
- $1 \sim \bot^\perp$
- $0 \sim \top^\perp$

where $\sim$ denotes graph isomorphism.
Sum of games

Play two games but only count outcome of one

\[ |A \oplus B|_{\langle y,w \rangle}^{\text{inj}_i x} := \begin{cases} |A|_y^x & \text{if } i = 0 \\ |B|_w^x & \text{if } i = 1 \end{cases} \]

\[ |A \& B|_{\langle x,v \rangle}^{\text{inj}_i y} := \begin{cases} |A|_y^x & \text{if } i = 0 \\ |B|_y^v & \text{if } i = 1 \end{cases} \]

where \((A \oplus B)^+ = A^+ \uplus B^+\) and \((A \oplus B)^- = A^- \times B^-\).
Sum of games

Play two games but only count outcome of one

\[ |A \oplus B|_{\langle y,w \rangle}^{\text{inj}_i x} \equiv \begin{cases} |A|_y^x & \text{if } i = 0 \\ |B|_w^x & \text{if } i = 1 \end{cases} \]

\[ |A \& B|_{\langle x,v \rangle}^{\text{inj}_i y} \equiv \begin{cases} |A|_y^x & \text{if } i = 0 \\ |B|_y^v & \text{if } i = 1 \end{cases} \]

where \((A \oplus B)^+ = A^+ \cup B^+\) and \((A \oplus B)^- = A^- \times B^-\).
Product of games

Play two games in parallel

\[ |A \otimes B|^{(S,T)}_{(y,w)} \equiv |A|^{Sw}_{y} \text{ or } |B|^{Ty}_{w} \]

\[ |A \otimes B|^{(x,v)}_{(S,T)} \equiv |A|^{x}_{Sv} \text{ and } |B|^{v}_{Tx} \]

where

1. \((A \otimes B)^+ = \mathcal{B}_f(B^-, A^+) \times \mathcal{B}_f(A^-, B^+)\)
2. \((A \otimes B)^- = A^- \times B^-\).
Product of games

Play two games in parallel

\[ |A \otimes B|_{\langle S,T \rangle}^{\langle y,w \rangle} :\equiv |A|_y^{Sw} \text{ or } |B|_w^{Ty} \]

\[ |A \otimes B|_{\langle S,T \rangle}^{\langle x,v \rangle} :\equiv |A|_{Sv}^x \text{ and } |B|_{Tx}^v \]

where

\[ (A \otimes B)^+ = B_f(B^-, A^+) \times B_f(A^-, B^+) \]
\[ (A \otimes B)^- = A^- \times B^- \]

Lemma

\[ A \otimes \bot \sim A \]
\[ A \otimes 1 \sim A \]
Relative games

Let $A \multimap B :\equiv A^\perp \otimes B$

In particular we have that

$|A \multimap B|^{\langle S,T \rangle}_{\langle x,w \rangle} \equiv \text{if } |A|^{x}_{S,w} \text{ then } |B|^{T_x}_{w}$

where

- $(A \multimap B)^+ = B_f(A^+, B^+) \times B_f(B^-, A^-)$
- $(A \multimap B)^- = A^+ \times B^-$. 
Duplication of games

Play several copies of a game in parallel

\(|?A|_{y}^{*} : \equiv \exists x A^{+}|A|_{y}^{x}\)

\(|!A|_{x}^{*} : \equiv \forall y A^{-}|A|_{y}^{x}\)

where \((?A)^{+} = \{\ast\}\) and \((?A)^{-} = A^{-}\).
Duplication of games

Play several copies of a game in parallel

\[ |?A|_y^* :\equiv \exists x A^+ |A|_y^x \]
\[ |!A|_x^* :\equiv \forall y A^- |A|_y^x \]

where \((?A)^+ = \{\ast\}\) and \((?A)^- = A^-\).

**Lemma**

- \(?0 \sim \bot\)
- \(!\top \sim 1\)
Soundness

**Theorem**

*If* $A$ *is provable in linear logic then the bipartite graph* $A$ *has a covering point, i.e. there exists an* $x^A^+$ *such that* $\forall y^A^- | A |_y$. 
Soundness

Theorem

If $A$ is provable in linear logic then the bipartite graph $A$ has a covering point, i.e. there exists an $x^A^+$ such that $\forall y^A^- |A|_y$.

$A$ is provable $\Rightarrow$ first player has a winning move in game $A$
Intuitionistic truth via linear logic

Via $(\cdot)^\circ$: IL $\leftrightarrow$ LL we can model an intuitionistic formula $A$ as the bipartite graph $A^\circ$

More precisely, let $x \vdash A \iff \forall y (A^\circ)^- | A^\circ | x$

$A$ intuitionistically true if $\exists x (x \vdash A)$
Intuitionistic truth via linear logic

Via $(\cdot)^\circ$: $\text{IL} \leftrightarrow \text{LL}$ we can model an intuitionistic formula $A$ as the bipartite graph $A^\circ$.

More precisely, let

$$x \vdash A \equiv \forall y (A^\circ)^{-}_y | A^\circ|_x$$

$A$ intuitionistically true if $\exists x (x \vdash A)$

---

**Theorem**

$$\langle x, y \rangle \vdash A \land B \iff (x \vdash A) \land (y \vdash B)$$

$$\text{inj}_i x \vdash A \lor B \iff (x \vdash A) \Diamond_i (x \vdash B)$$

$$S \vdash A \rightarrow B \iff \forall x ((x \vdash A) \rightarrow (Sx \vdash B)).$$
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1. Realizability (a reformulation)

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Functional interpretation of LL

Four changes from previous interpretation:

1. Work with infinite bipartite graphs
   \( X, Y \) sets of functionals of finite type
   (strategies = functionals)

2. Define an interpretation of LL inside LL
   Adjudication relation as a formula of LL

3. Interpret quantifiers

4. Look at different interpretations of exponentials
Finite types

Assume a couple of basic types like $\mathbb{B}$ and $\mathbb{N}$

Close under

- Function type $\rho \rightarrow \tau$
- Product type $\rho \times \tau$
- List type $\rho^*$
Functional interpretation of LL

Additives

Play both games $|A|_y^x$ and $|B|_w^v$

One of the players chooses which game will count

$$|A \oplus B|_{y,w,z}^{x,v,z} \equiv |A|_y^x \Diamond z |B|_w^v$$
$$|A \& B|_{y,w,z}^{x,v,z} \equiv |A|_y^x \Diamond z |B|_w^v$$

where $A \Diamond z B \equiv (! (z = \text{tt}) \rightarrow A) \& (! (z = \text{ff}) \rightarrow B)$. 
Functional interpretation of LL

Quantifiers (Generalised additives)

Play all games $|A_z|_y^x$

One player chooses which game will count

Other player is allowed to know which game was chosen

$|\exists z A_z|_y^x : \equiv |A_z|_y^f$

$|\forall z A_z|_y^f : \equiv |A_z|_y^z$
Functional interpretation of LL

**Multiplicatives**

Play games $|A|_y^x$ and $|B|_w^v$ in parallel

One of the players can play copycat

$|A \otimes B|_{f,g}^{x,y,w} \equiv |A|_y^f \otimes |B|_w^g$

$|A \otimes B|_{f,g}^{x,v} \equiv |A|_v^x \otimes |B|_g^x$
Exponentials (Generalised multiplicatives)

Play several copies of game $|A|^x_y$

One player must choose a uniform move

$|?A|^y_y :\equiv ?\exists x |A|^x_y$

$|!A|^x :\equiv !\forall y |A|^x_y$

Other player plays second (break of symmetry)

Other player plays a set of moves
Functional interpretation of LL

Exponentials (Generalised multiplicatives)

Play several copies of game $|A|^x_y$

One player must choose a uniform move

$|?A|^f_y : \equiv ?\exists x \sqsubseteq fy \; |A|^x_y$

$|!A|^x_g : \equiv !\forall y \sqsubseteq gx \; |A|^x_y$

Other player plays second (break of symmetry)

Other player plays a set of moves
Exponentials: Conditions

The kind of move-sets need to satisfy:

There exists terms $\eta$, $\epsilon$ and $\mu$ such that

(I) Every element $x$ belongs to a set $\eta x$

(II) The sets $y_i$ are contained in the set $\epsilon y_0 y_1$

(III) For each $x \sqsubseteq b$ the set $hx$ is contained in $\mu hb$
Exponentials: Conditions

The kind of move-sets need to satisfy:

There exists terms $\eta$, $\epsilon$ and $\mu$ such that

(I) Every element $x$ belongs to a set $\eta x$

$$\forall y \sqsubseteq \eta x \ A \vdash A[x/y]$$

(II) The sets $y_i$ are contained in the set $\epsilon y_0 y_1$

(III) For each $x \sqsubseteq b$ the set $h x$ is contained in $\mu h b$
Exponentials: Conditions

The kind of move-sets need to satisfy:

There exists terms $\eta$, $\epsilon$ and $\mu$ such that

(I) Every element $x$ belongs to a set $\eta x$
$$\forall y \sqsubseteq \eta x \quad A \vdash A[x/y]$$

(II) The sets $y_i$ are contained in the set $\epsilon y_0 y_1$
$$\forall y \sqsubseteq \epsilon y_0 y_1 \quad A \vdash \forall y \sqsubseteq y_i A \quad (i \in \{0, 1\})$$

(III) For each $x \sqsubseteq b$ the set $hx$ is contained in $\mu hb$
Exponentials: Conditions

The kind of move-sets need to satisfy:

There exists terms $\eta$, $\epsilon$ and $\mu$ such that

(I) Every element $x$ belongs to a set $\eta x$
$$\forall y \sqsubseteq \eta x \vdash A[x/y]$$

(II) The sets $y_i$ are contained in the set $\epsilon y_0 y_1$
$$\forall y \sqsubseteq \epsilon y_0 y_1 \vdash \forall y \sqsubseteq y_i A \quad (i \in \{0, 1\})$$

(III) For each $x \sqsubseteq b$ the set $hx$ is contained in $\mu hb$
$$\forall y \sqsubseteq \mu hb \vdash \forall x \sqsubseteq b \forall y \sqsubseteq hx A.$$
Soundness

**Theorem**

*Assuming (I, II, III). If*

\[ \text{LL} \vdash A \]

*there exists a closed simply typed } \lambda \text{-term } t \text{ such that}*

\[ \text{LL}^\omega \vdash \forall y |A|^t_y. \]
Instances satisfying (I, II, III)

- **Whole set**

\[ !A^x \equiv !\forall y A^y \]

- **Finite sets**

\[ !A^x \equiv !\forall y A^y \]

- **Singleton sets** (assuming decidability)

\[ !A^x \equiv !A^x \]

\[ f : \equiv \forall y \in fx A^y \]
Instances satisfying (I, II, III)

- **Whole set**
  \[ \forall y | A | y \]

- **Finite sets**
  \[ \forall y \in f x | A | y \]
Instances satisfying (I, II, III)

- **Whole set**
  \[ \text{Whole set} \]
  \[ |!A|^x \equiv !\forall y |A|_y^x \]

- **Finite sets**
  \[ \text{Finite sets} \]
  \[ |!A|^x_f \equiv !\forall y \in f x |A|_y^x \]

- **Singleton sets**
  \[ \text{Singleton sets} \]
  \[ |!A|^x_f \equiv |!A|_{fx}^x \]
Instances satisfying (I, II, III)

- **Whole set**
  \[ |!A|^x :\equiv |!\forall y | A|^y_x \]

- **Finite sets**
  \[ |!A|_f^x :\equiv |!\forall y \in f x | A|^y_x \]

- **Singleton sets** (assuming decidability)
  \[ |!A|_f^x :\equiv |!| A|^x_{fx} \]
Functional interpretation of LL

- Symmetric game $\Rightarrow$ branching quantifier

$$A \mapsto \forall x \exists y. |A|_{x y}$$

- Characterisation principles more complicated
- Games $!A$ and $?A$ correspond to a “double advantage”
Functional interpretation of LL

- Symmetric game $\Rightarrow$ branching quantifier
  
  \[ A \rightarrow \exists y \forall x |A|_y^y \]

- Characterisation principles more complicated
- Games $!A$ and $?A$ correspond to a “double advantage”
- Could we use sequential games?
- Can this “double advantage” be separated?
Functional interpretation of LL

- Symmetric game $\Rightarrow$ branching quantifier
  \[ A \rightarrow \forall x \exists y |A|_y \]

- Characterisation principles more complicated
- Games $!A$ and $?A$ correspond to a “double advantage”
- Could we use sequential games?
- Can this “double advantage” be separated?

Yes, in intuitionistic linear logic
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Simultaneous versus sequential games

Let us now work with **sequential games**

i.e. Eloise plays first, followed by Abelard’s move

\[ A \iff \exists x \forall y | A |^x_y \]
Simultaneous versus sequential games

Let us now work with **sequential games**

i.e. Eloise plays first, followed by Abelard’s move

\[ A \iff \exists x \forall y \mid A \upharpoonright x_y \]

No restriction, since Eloise’s move could be a function

\[ \exists f \forall y \mid A \upharpoonright f_y \equiv \forall y \exists x \mid A \upharpoonright x_y \]
Functional interpretation of ILL

\[ |A \oplus B|_{x,v,w,z} \equiv |A|_y \lozenge |B|_w \]

\[ |A \& B|_{x,v,y,w,z} \equiv |A|_y \lozenge |B|_w \]
Functional interpretation of ILL

\[ |A \oplus B|_{x,v,z} \quad : \equiv \quad |A|_{x} \diamond |B|_{v} \]

\[ |A \land B|_{x,v} \quad : \equiv \quad |A|_{x} \diamond |B|_{v} \]

\[ |\exists z A|_{x,z} \quad : \equiv \quad |A|_{x} \]

\[ |\forall z A|_{f,z} \quad : \equiv \quad |A|_{f} \]

\[ |A \otimes B|_{x,v,\omega} \quad : \equiv \quad |A|_{x} \otimes |B|_{\omega} \]

\[ |! A|_{x} \quad : \equiv \quad |\forall y \ominus a|_{x,y} \]

\[ |A \Rightarrow B|_{x,y,w} \quad : \equiv \quad |A|_{x} \Rightarrow |B|_{y,w} \]
Functional interpretation of ILL

\[
\begin{align*}
|A \oplus B|_{x,v,z} & :\equiv |A|_x \diamond z \quad |B|_v \\
|A \& B|_{x,v} & :\equiv |A|_x \diamond z \quad |B|_v \\
|\exists z A|_{x,z} & :\equiv |A|_x \\
|\forall z A|_{f,z} & :\equiv |A|_f \\
|A \rightarrow B|_{f,g} & :\equiv |A|_x \rightarrow fxw \quad |B|_{gx} \\
|A \otimes B|_{x,v} & :\equiv |A|_x \otimes |B|_v
\end{align*}
\]
Functional interpretation of ILL

\[
|A \oplus B|_{y,w}^{x,v,z} \equiv |A|_y^x \odot z |B|_w^v \\
|A \land B|_{y,w,z}^{x,v} \equiv |A|_y^x \odot z |B|_w^v \\
|\exists z A|_y^{x,z} \equiv |A|_y^x \\
|\forall z A|_{y,z}^f \equiv |A|_y^f z \\
|A \rightarrow B|_{x,w}^{f,g} \equiv |A|_x^f |B|_{gx}^{fw} \\
|A \otimes B|_{y,w}^{x,v} \equiv |A|_y^x \otimes |B|_w^v \\
|!A|_a^x \equiv !\forall y \sqsubseteq a |A|_y^x.
\]
Instances satisfying (I, II, III)

Same three conditions need to be satisfied, and we have:

- **Whole set**
  \[ !A \upharpoonright^x \equiv !\forall y \upharpoonright^x A \upharpoonright^y \]
Instances satisfying (I, II, III)

Same three conditions need to be satisfied, and we have:

- **Whole set**
  \[ |!A|^x :\equiv !\forall y |A|^y \]

- **Kreisel mod. realizability**
  \[ |A^\circ|^x \circ \circ (x \text{ mr } A)^\circ \]
Instances satisfying (I, II, III)

Same three conditions need to be satisfied, and we have:

- **Whole set**
  \[ !A^x : \equiv !\forall y A^y \]
  - Kreisel mod. realizability
  \[ A^y \circ \circ (x \text{ mr } A) \circ \]

- **Finite sets**
  \[ !A^x_\alpha : \equiv !\forall y \in \alpha A^y \]

Instances satisfying (I, II, III)

Same three conditions need to be satisfied, and we have:

- **Whole set**
  \[ |!A|^{x} \equiv !\forall y |A|^{y} \]
  Kreisel mod. realizability
  \[ |A^\circ|^{x} \circ \circ (x \ mr \ A)^\circ \]

- **Finite sets**
  \[ |!A|^{x}_{a} \equiv !\forall y \in a |A|^{y} \]
  Diller-Nahm inter.
  \[ |A^{*}|^{x}_{y} \circ \circ (A_{dn}(x; y))^{*} \]
Instances satisfying (I, II, III)

Same three conditions need to be satisfied, and we have:

- **Whole set**
  \[ |!A|^x :\equiv !\forall y |A|^y \]

- **Finite sets**
  \[ |!A|_a^x :\equiv !\forall y \in a |A|^y \]

- **Singleton sets**
  \[ |!A|_y^x :\equiv |!A|^x \]

Kreisel mod. realizability
\[ |A^\circ|^x \dashv\vdash (x \text{ mr } A)^\circ \]

Diller-Nahm inter.
\[ |A^*|^x \dashv\vdash (A_{dn}(x; y))^* \]
Instances satisfying (I, II, III)

Same three conditions need to be satisfied, and we have:

- **Whole set**
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  Kreisel mod. realizability
  \[ |A^\circ|^x \circ \circ (x \text{ mr } A)^\circ \]

- **Finite sets**
  \[ |!A|^x_\alpha :\equiv !\forall y \in \alpha |A|^y \]
  Diller-Nahm inter.
  \[ |A^*|^x_\gamma \circ \circ (A_{dn}(x; y))^* \]

- **Singleton sets**
  \[ |!A|^x_y :\equiv !|A|^x_y \]
  Gödel Dialectica inter.
  \[ |A^*|^x_\gamma \circ \circ (A_D(x; y))^* \]
Realizability and LL

\[ \text{Kreisel mr} \]

\[ !A^x \equiv !\forall y A^y \]

\[ (\cdot)^\circ \]
Realizability and LL

Realizability Interpretations of Linear Logic

Functional Interpretations of ILL

\[ (!A)_f^x \equiv !\forall y^x f^x | A^x_y \]

Diller-Nahm

QLL → LL → QLL

QLL → IL → QLL
Realizability and LL

\[(\cdot)^* : !A^x_f \equiv !A^x_{fx}\]

Dialectica
Question

Modified realizability interprets full extensionality

$$\forall x (fx = gx) \rightarrow Ff = Fg$$

Dialectica interprets **Markov principle**

$$\neg\forall x A_{qf} \rightarrow \exists x \neg A_{qf}$$

Can we combine both?
Question

Modified realizability interprets full extensionality

\[ \forall x (fx = gx) \rightarrow Ff = Fg \]

Dialectica interprets **Markov principle**

\[ \neg \forall x A_{qf} \rightarrow \exists x \neg A_{qf} \]

Can we combine both?

**Yes** *(thanks to the fact that ! is not cannonical)*
Multi-modal ILL

Add **three** different modalities \( !_k A, !_d A \) and \( !_g A \) with rules

\[
\begin{align*}
!_X \Gamma \vdash A & \quad & \frac{\Gamma, A \vdash B}{\Gamma, !_Y A \vdash B} \quad (\text{!}_r) \quad \frac{\Gamma, !_Y A \vdash B}{\Gamma, A \vdash B} \quad (\text{!}_l) \\
\Gamma, !_Z A, !_Z A \vdash B & \quad & \frac{\Gamma, !_Y A \vdash B}{\Gamma, !_Y A \vdash B} \quad (\text{con,} \star) \\
\Gamma, !_Z A, !_Z A \vdash B & \quad & \frac{\Gamma \vdash B}{\Gamma, !_Y A \vdash B} \quad (\text{wkn})
\end{align*}
\]

where \( X, Y, Z_i \in \{k > d > g\} \) and \( X \geq Y \geq Z_i \)
Multi-modal ILL

Add **three** different modalities $!_k A$, $!_d A$ and $!_g A$ with rules

\[
\begin{align*}
!_X \Gamma \vdash A & \quad \text{(}_r) \\
!_X \Gamma \vdash !_Y A & \\
\Gamma, !_Z_0 A, !_Z_1 A \vdash B & \quad \text{(con,⋆)} \\
\Gamma, !_Y A \vdash B & \\
\Gamma, !_Y A \vdash B & \quad \text{(wkn)}
\end{align*}
\]

where $X, Y, Z_i \in \{k > d > g\}$ and $X \geq Y \geq Z_i$

(⋆) Syntactic condition ensuring decidability when $Y = g$
Hybrid functional interpretation

Kreisel bang

\[ \Vert_{k} A^{x} \equiv \forall y A^{y} \]

Diller-Nahm bang

\[ \Vert_{d} A^{x}_{f} \equiv \forall y \in f \, x \, A^{y} \]

Gödel bang

\[ \Vert_{g} A^{x}_{f} \equiv \Vert A^{x}_{f \, x} \]
Hybrid functional interpretation

Kreisel bang
\[ !_k A^x := ! \forall y A^y \]

Diller-Nahm bang
\[ !_d A^x_f := ! \forall y \in f x A^y \]

Gödel bang
\[ !_g A^x_f := ! A^x_{fx} \]

Let a colouring algorithm decide the optimal/desired labelling
Hybrid functional interpretation

- Enriched Theorem
- Verification Proof
- Theorem A
- Proof of A
- CL translation
- Hybrid Functional Interpretation
- LL translation
- Colour proof given colouring of theorem
- Proof of A in LL
- Theorem A in LL
References

Modified realizability interpretation of classical linear logic
LICS 2007

Hybrid functional interpretations

Functional interpretations of linear and intuitionistic logic
To appear in I&C

Hybrid functional interpretations of linear and IL
To appear in JoL&C

Functional interpretations of intuitionistic linear logic
with G. Ferreira, in preparation