Theorems and Symmetric Games

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BMC'07, Swansea
18 April 2007

THE ROYAL SOCIETY

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Mathematics is like a game,
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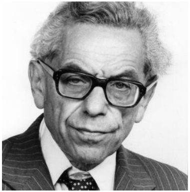
View a mathematical statement as the description of a game

$$\forall n \geq 2 \, \exists x, y, z (\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z})$$



Paul Erdös

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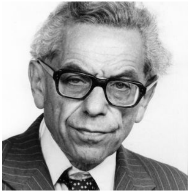

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$$f_0, f_1, f_2 : \mathbb{N} \to \mathbb{N}^*$$

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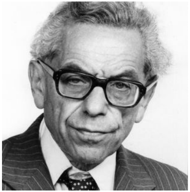

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$$\frac{4}{n} = \frac{1}{f_0(n)} + \frac{1}{f_1(n)} + \frac{1}{f_2(n)}$$

Games: Formal Description

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- **Adjudication of Winner**

  Relation $R(x, y)$ between players' moves

  (*usually* $|G|^x_y$)

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$x \in \mathbb{N}$	$y \in \mathbb{N}$	$x \geq y$

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$x \in \mathbb{N}$	$y \in \mathbb{N}$	$x \geq y$
$f \in \mathbb{N} \to \mathbb{N}$	$y \in \mathbb{N}$	$f(y) \geq y$

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$x \in \mathbb{N}$	$y \in \mathbb{N}$	$x \geq y$
$f \in \mathbb{N} \to \mathbb{N}$	$y \in \mathbb{N}$	$f(y) \geq y$
$f_i \in \mathbb{N} \to \mathbb{N}^*$	$n \geq 2$	$\frac{4}{n} = \frac{1}{f_0 n} + \frac{1}{f_1 n} + \frac{1}{f_2 n}$

Goal

A is true (is provable)
iff
Eloise has winning move in game $|A|_y^x$

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$$A \quad \text{iff} \quad \exists x \forall y |A|_y^x$$

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Game $\neg A$ should be game A with roles reversed

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Games Semantics

$$|A \wedge B|^{x,v}_{f,g} \quad \equiv \quad |A|^{x}_{fv} \text{ and } |B|^{v}_{gx}$$

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Games Semantics

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\begin{aligned}
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|A \rightarrow B|^{f,g}_{x,w} &\equiv |A|^{x}_{fw} \text{ implies } |B|^{gx}_{w} \\
|\forall z A(z)|^{f}_{y,z} &\equiv |A(z)|^{fz}_{y} \\
|\exists z A(z)|^{x,z}_{f} &\equiv |A(z)|^{x}_{fz}
\end{aligned}
$$

Games Semantics

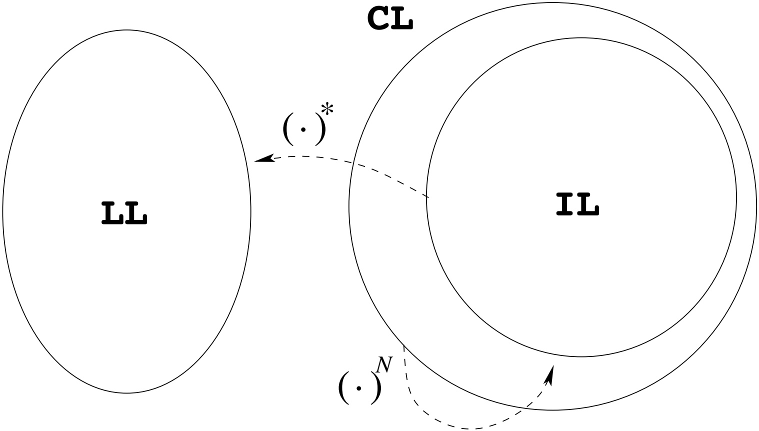
$$|A \otimes B|^{x,v}_{f,g} \equiv |A|^{x}_{fv} \text{ and } |B|^{v}_{gx}$$

$$|A \multimap B|^{f,g}_{x,w} \equiv |A|^{x}_{fw} \text{ implies } |B|^{gx}_{w}$$

$$|\forall z A(z)|^{f}_{y,z} \equiv |A(z)|^{fz}_{y}$$

$$|\exists z A(z)|^{x,z}_{f} \equiv |A(z)|^{x}_{fz}$$

Linear Logic



Uniform Move (Winning Move)

$$!A \qquad \qquad ?A$$

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$$(1) \quad |!A|^x \equiv \forall y |A|^x_y$$

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(1) \quad |!A|^x &\equiv \forall y |A|^x_y \\
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(2) \quad |!A|^x_f &\equiv \forall y \in f x \, |A|^x_y \\
|?A|^f_y &\equiv \exists x \in f y \, |A|^x_y
\end{aligned}
$$

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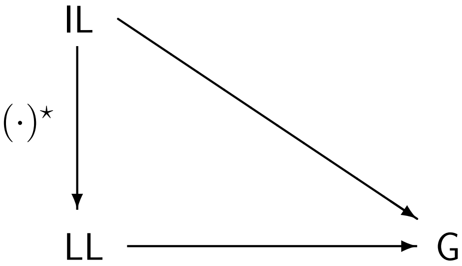
(2)\quad |!A|^x_f &\equiv \forall y \in f x\, |A|^x_y \\
|?A|^f_y &\equiv \exists x \in f y\, |A|^x_y \\
(3)\quad |!A|^x_f &\equiv |A|^x_{fx} \\
|?A|^f_y &\equiv |A|^{fy}_y
\end{aligned}
$$

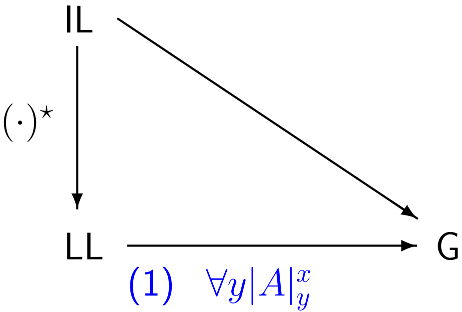
LL $\Rightarrow$ G

Theorem

If A is provable in linear logic
then
the game $|A|^x_y$ has a winning move

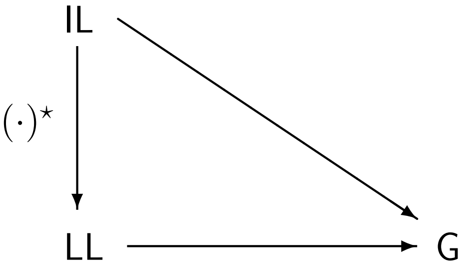

IL
$(\cdot)^\star$
LL
G

Modified realizability (Kreisel'1959)



Diller-Nahm interpretation (Diller-Nahm'1974)

Modified realizability (Kreisel'1959)



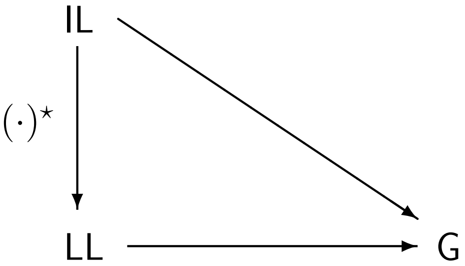
(1) $\forall y |A|^x_y$

(2) $\forall y \in f x \, |A|^x_y$

Dialectica interpretation (Gödel'1956)

Diller-Nahm interpretation (Diller-Nahm'1974)

Modified realizability (Kreisel'1959)



(1) $\forall y |A|^x_y$

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(3) $|A|^x_{fx}$

Completeness

$$|\Gamma|^{v}_{r[y]} \vdash |A|^{t[v]}_{y}$$

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$$\forall w \exists v |\Gamma|^{v}_{w} \vdash \exists x \forall y |A|^{x}_{y}$$

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cut rule

$$\forall w \exists v |\Gamma|^{v}_{w} \vdash \exists x \forall y |A|^{x}_{y}$$

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Completeness

$$|\Gamma|^{v}_{r[y]} \vdash |A|^{t[v]}_{y}$$

cut rule ↙ ↘ axiom

$$\forall w \exists v |\Gamma|^{v}_{w} \vdash \exists x \forall y |A|^{x}_{y} \qquad\qquad \exists v \forall w |\Gamma|^{v}_{w} \vdash \forall y \exists x |A|^{x}_{y}$$

Completeness

$$|\Gamma|^{v}_{r[y]} \vdash |A|^{t[v]}_{y}$$

cut rule (↙)

$$\forall w \exists v |\Gamma|^{v}_{w} \vdash \exists x \forall y |A|^{x}_{y}$$

axiom (↘)

$$\exists v \forall w |\Gamma|^{v}_{w} \vdash \forall y \exists x |A|^{x}_{y}$$

(↓)

$$\exists\!\!\!\forall^{v}_{w} |\Gamma|^{v}_{w} \vdash \exists\!\!\!\forall^{x}_{y} |A|^{x}_{y}$$

Simultaneous quantifier

New principles validated:

- Sequential choice

$$\forall z \exists\!\!\!\forall^{x}_{y} A(x, y, z) \multimap \exists\!\!\!\forall^{f}_{y,z} A(fz, y, z)$$

- Parallel choice

$$\exists\!\!\!\forall^{x}_{y} A(x) \parr \exists\!\!\!\forall^{v}_{w} B(v) \multimap \exists\!\!\!\forall^{f,g}_{y,w} (A(fw) \parr B(gy))$$

- Trump advantage

$$!\exists\!\!\!\forall^{x}_{y} A \multimap \exists x ! \forall y A$$

Summary

- Functional interpretations of linear logic

  Nice "game" flavour

  Usual interpretations of IL derivable

- Interesting new branching quantifier

- Sound extensions of linear logic