A Game called Mathematics

A game-theoretic take on functional interpretations

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Mathematics is like a game, mathematicians are always winners because they play both roles

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View a mathematical statement as the description of a game

Outline

1. Logic

2. Mathematics

Outline

Notation: Quantifiers

For each natural number n greater than 2 there exist positive natural numbers x, y, z such that

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

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$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

$$\forall n^{\mathbb{N}} \geq 2 \; \exists x^{\mathbb{N}^*}, y^{\mathbb{N}^*}, z^{\mathbb{N}^*} (\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z})$$

Notation: Connectives

If a_0 *and* a_1 *are solutions of an equation* $f(x) = 0$ *then* $a_0 = a_1$*.*

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$$(f(a_0) = 0) \land (f(a_1) = 0) \rightarrow (a_0 = a_1)$$

Notation: Summary

$A \wedge B$	A and B
$A \rightarrow B$	If A then B
$\forall z^D A(z)$	For each z in D $A(z)$
$\exists z^D A(z)$	There exists z in D such that $A(z)$

$$\forall n \geq 2\, \exists x, y, z (\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z})$$

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$$f_0, f_1, f_2 : \mathbb{N} \to \mathbb{N}^* \qquad n \in \{2, \ldots\}$$

$$\frac{4}{n} = \frac{1}{f_0(n)} + \frac{1}{f_1(n)} + \frac{1}{f_2(n)}$$

Game Semantics (intuitively)

$$A \wedge B$$

$$A \rightarrow B$$

$$\forall z A(z)$$

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$A \wedge B$

P1 : Strategies for A and B

P2 : Strategy for $\neg A$ or $\neg B$

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P1 : Transform strategy for A into one for B

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Game Semantics (intuitively)

$A \wedge B$

P1 : Strategies for A and B

P2 : Strategy for $\neg A$ or $\neg B$

$A \rightarrow B$

P1 : Transform strategy for A into one for B

P2 : Strategies for A and $\neg B$

$\forall z A(z)$

P1 : Strategies for $A(z)$, for each z

P2 : z and strategy for $\neg A(z)$

Games: Formal Description

- Game $G \equiv (D_1, D_2, R \subseteq D_1 \times D_2)$
- **Two players**

  Eloise and Abelard

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  $x \in D_1$ and $y \in D_2$

Games: Formal Description

- Game $G \equiv (D_1, D_2, R \subseteq D_1 \times D_2)$
- **Two players**
  Eloise and Abelard
- **Two domains of moves**
  $x \in D_1$ and $y \in D_2$
- **Adjudication of Winner**
  Relation $R(x, y)$ between players' moves
  (*usually* $|G|_y^x$)

Games: Examples

Domain 1	Domain 2	Adjudication
$x \in \{0, 1, 2\}$	$y \in \{0, 1, 2\}$	$x + 1 = y \bmod 3$

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$x \in \{0, \ldots, 5\}$	$y \in \{0, \ldots, 5\}$	$x + y$ is even

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Domain 1	Domain 2	Adjudication
$x \in \{0, 1, 2\}$	$y \in \{0, 1, 2\}$	$x + 1 = y \bmod 3$
$x \in \{0, \ldots, 5\}$	$y \in \{0, \ldots, 5\}$	$x + y$ is even
$x \in \mathbb{N}$	$y \in \mathbb{N}$	$x \geq y$

Games: Examples

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$x \in \{0, 1, 2\}$	$y \in \{0, 1, 2\}$	$x + 1 = y$ mod 3
$x \in \{0, \ldots, 5\}$	$y \in \{0, \ldots, 5\}$	$x + y$ is even
$x \in \mathbb{N}$	$y \in \mathbb{N}$	$x \geq y$
$f \in \mathbb{N} \to \mathbb{N}$	$y \in \mathbb{N}$	$f(y) \geq y$

Games: Examples

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$x \in \mathbb{N}$	$y \in \mathbb{N}$	$x \geq y$
$f \in \mathbb{N} \to \mathbb{N}$	$y \in \mathbb{N}$	$f(y) \geq y$
$f_i \in \mathbb{N} \to \mathbb{N}^*$	$n \geq 2$	$\frac{4}{n} = \frac{1}{f_0 n} + \frac{1}{f_1 n} + \frac{1}{f_2 n}$

Goal

A is true (is provable)
iff
player 1 has a winning move in game $|A|_y^x$

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A is true (is provable)

iff

player 1 has a winning move in game $|A|^x_y$

$$A \text{ iff } \exists x \forall y |A|^x_y$$

Symmetry

Game $\neg A$ should be game A with roles reversed

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$$|\neg A|_y^x \equiv \neg |A|_x^y$$

$$|\neg\neg A|_y^x \equiv |A|_y^x$$

Games Semantics (revisited)

$$|A \wedge B|^{x,v}_{f,g} \quad \equiv \quad |A|^{x}_{fv} \text{ and } |B|^{v}_{gx}$$

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$$|A \wedge B|^{x,v}_{f,g} \equiv |A|^{x}_{fv} \text{ and } |B|^{v}_{gx}$$

$$|A \rightarrow B|^{f,g}_{x,w} \equiv |A|^{x}_{fw} \text{ implies } |B|^{gx}_{w}$$

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$$|A \rightarrow B|^{f,g}_{x,w} \equiv |A|^{x}_{fw} \text{ implies } |B|^{gx}_{w}$$

$$|\forall z A(z)|^{f}_{y,z} \equiv |A(z)|^{fz}_{y}$$

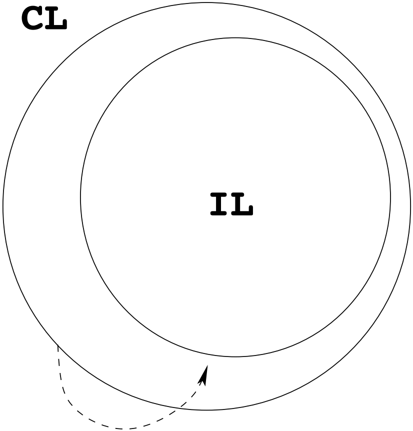
Games Semantics (revisited)

$$
\begin{aligned}
|A \wedge B|^{x,v}_{f,g} &\equiv |A|^{x}_{fv} \text{ and } |B|^{v}_{gx} \\
|A \to B|^{f,g}_{x,w} &\equiv |A|^{x}_{fw} \text{ implies } |B|^{gx}_{w} \\
|\forall z A(z)|^{f}_{y,z} &\equiv |A(z)|^{fz}_{y} \\
|\exists z A(z)|^{x,z}_{f} &\equiv |A(z)|^{x}_{fz}
\end{aligned}
$$

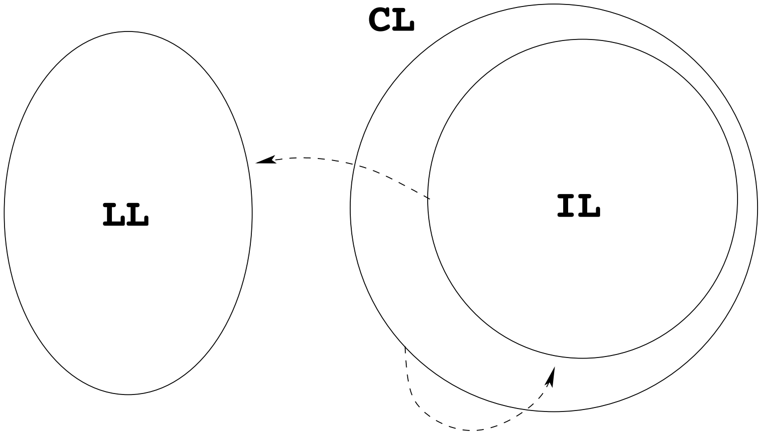
Games Semantics (revisited)

$$
\begin{aligned}
|A \otimes B|^{x,v}_{f,g} &\equiv |A|^{x}_{fv} \text{ and } |B|^{v}_{gx} \\
|A \multimap B|^{f,g}_{x,w} &\equiv |A|^{x}_{fw} \text{ implies } |B|^{gx}_{w} \\
|\forall z A(z)|^{f}_{y,z} &\equiv |A(z)|^{fz}_{y} \\
|\exists z A(z)|^{x,z}_{f} &\equiv |A(z)|^{x}_{fz}
\end{aligned}
$$

Linear Logic



Linear Logic



Uniform Move (Winning Move)

$!A \qquad \qquad ?A$

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$$!A \qquad ?A$$

$$
\begin{aligned}
(1) \quad |!A|^x &\equiv \forall y |A|_y^x \\
|?A|^x &\equiv \exists x |A|_y^x
\end{aligned}
$$

Uniform Move (Winning Move)

$$!A \qquad ?A$$

$$
\begin{array}{lrcl}
(1) & |!A|^x & \equiv & \forall y |A|^x_y \\
 & |?A|^x & \equiv & \exists x |A|^x_y \\
(2) & |!A|^x_f & \equiv & \forall y \in f x \, |A|^x_y \\
 & |?A|^f_y & \equiv & \exists x \in f y \, |A|^x_y
\end{array}
$$

Uniform Move (Winning Move)

$$!A \qquad ?A$$

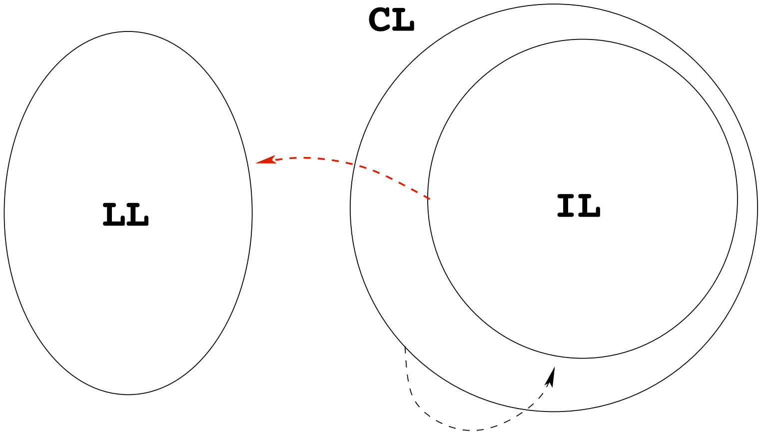
$$
\begin{array}{lrcl}
(1) & |!A|^x & \equiv & \forall y |A|^x_y \\
 & |?A|^x & \equiv & \exists x |A|^x_y \\
(2) & |!A|^x_f & \equiv & \forall y \in fx\, |A|^x_y \\
 & |?A|^f_y & \equiv & \exists x \in fy\, |A|^x_y \\
(3) & |!A|^x_f & \equiv & |A|^x_{fx} \\
 & |?A|^f_y & \equiv & |A|^{fy}_y
\end{array}
$$

LL $\Rightarrow$ G

Theorem

If A is provable in linear logic
then
the game $|A|_y^x$ has a winning move

IL ⇒ LL

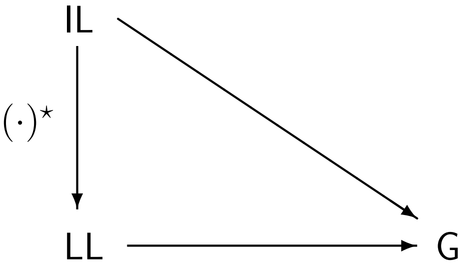


IL $\Rightarrow$ LL

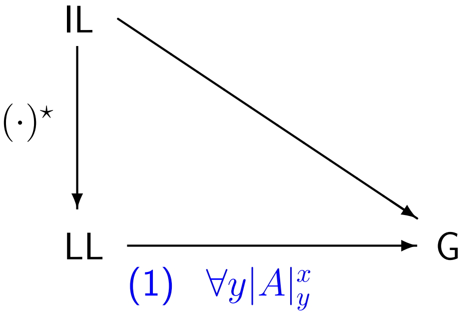
Intuitionistic logic	**Linear logic**
$(A \rightarrow B)^\star$	$!A^\star \multimap B^\star$
$(\forall x A)^\star$	$\forall x A^\star$
$(\exists x A)^\star$	$\exists x !A^\star$

Theorem (Girard'86)

A is provable in intuitionistic logic
iff
$A^\star$ is provable in linear logic

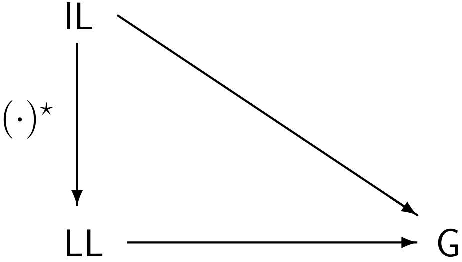

IL
$(\cdot)^\star$
LL
G

Modified realizability (Kreisel'1959)



Diller-Nahm interpretation (Diller-Nahm'1974)

Modified realizability (Kreisel'1959)



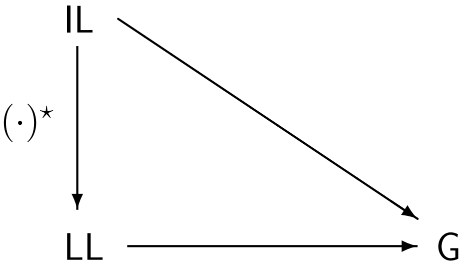
(1) $\forall y |A|^x_y$

(2) $\forall y \in f x \, |A|^x_y$

Dialectica interpretation (Gödel'1956)

Diller-Nahm interpretation (Diller-Nahm'1974)

Modified realizability (Kreisel'1959)



(1) $\forall y |A|^x_y$

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(3) $|A|^x_{fx}$

Dialectica interpretation (Gödel'1956)

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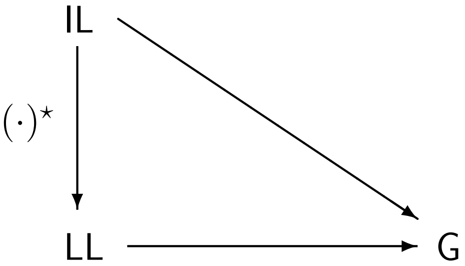


(1) $\forall y |A|^x_y$

(2) $\forall y \in f x\, |A|^x_y$

(3) $|A|^x_{fx}$ (Good for IL + MP)

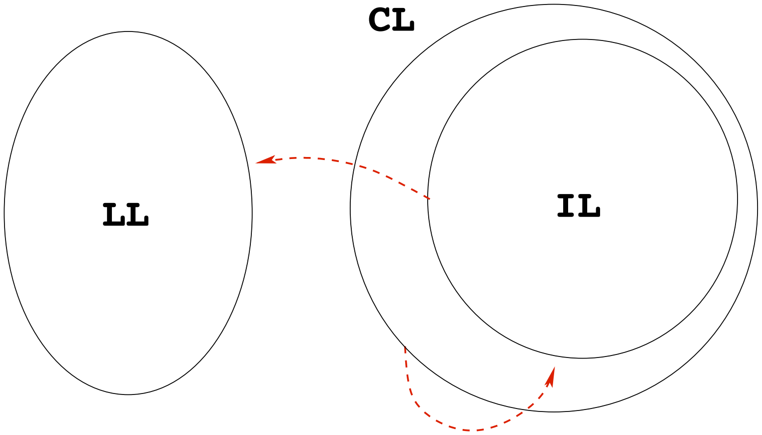
IL $\Rightarrow$ LL $\Rightarrow$ G

Theorem

If A is provable in intuitionistic logic
then
the game $|A^\star|^x_y$ has a winning move

$$\mathsf{IL} \xrightarrow{\ (\cdot)^\star\ } \mathsf{LL} \xrightarrow{\ \mathsf{Dial.}\ } \mathsf{G}$$

CL ⇒ IL



CL ⇒ IL

Classical logic	**Intuitionistic logic**
$(A \to B)^N$	$A^N \to B^N$
$(\forall x A)^N$	$\forall x \neg\neg A^N$
$(\exists x A)^N$	$\exists x A^N$

Theorem (Kuroda'51)

A is provable in classical logic
iff
$\neg\neg A^N$ is provable in intuitionistic logic

CL $\Rightarrow$ IL $\Rightarrow$ LL $\Rightarrow$ G

Theorem

If A is provable in classical logic

then

the game $|(\neg\neg A^N)^\star|^x_y$ has a winning move

$$\mathsf{CL} \xrightarrow{\neg\neg(\cdot)^N} \mathsf{IL} \xrightarrow{(\cdot)^\star} \mathsf{LL} \xrightarrow{\text{Dial.}} \mathsf{G}$$

Outline

Example 1

CL	:	$\forall n \exists k P(n,k)$
IL	:	$\neg\neg\forall n \neg\neg\exists k P(n,k)$
IL + MP	:	$\forall n \exists k P(n,k)$
LL	:	$\forall n \exists k ! P(n,k)$
G	:	$(f \in \mathbb{N} \to \mathbb{N}, k \in \mathbb{N}, P(fk,k))$

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Example 2

CL	:	$\exists n P(n) \rightarrow \exists k Q(k)$
IL	:	$\exists n P(n) \rightarrow \neg\neg\exists k Q(k)$
IL + MP	:	$\exists n P(n) \rightarrow \exists k Q(k)$
LL	:	$!\exists n !P(n) \multimap \exists k !Q(k)$
G	:	$(f^{\mathbb{N}\rightarrow\mathbb{N}}, n^{\mathbb{N}}, P(n) \rightarrow Q(fn))$

Example 2

CL	:	$\exists n P(n) \to \exists k Q(k)$
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Example 3

CL	:	$\forall n P(n) \to \forall k Q(k)$
IL	:	$\forall n \neg\neg P(n) \to \forall k \neg\neg Q(k)$
IL + MP	:	$\forall n P(n) \to \forall k Q(k)$
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Mathematical Objects (Data types)

Finite types

$$\mathbb{N} \qquad \rho \times \tau \qquad \rho \to \tau$$

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Finite types

$$\mathbb{N} \qquad \rho \times \tau \qquad \rho \to \tau$$

Definable:

- Integers and Rationals: pairs of naturals
- Reals: Cauchy sequences of rationals
- Polynomials: Tuples of reals

Mathematical Objects (Data types)

Notation	**Representation**
$x >_{\mathbb{R}} 0$	$\exists k(x_k > 2^{-k})$
$x =_{\mathbb{R}} 0$	$\forall k(\lvert x_k \rvert \leq 2^{-k})$

Mathematical Objects (Data types)

Notation	**Representation**
$x >_{\mathbb{R}} 0$	$\exists k(x_k > 2^{-k})$
$x =_{\mathbb{R}} 0$	$\forall k(\lvert x_k \rvert \leq 2^{-k})$
$x >_{\mathbb{R}} y$	$\exists k(x_k - y_k > 2^{-k})$
$x =_{\mathbb{R}} y$	$\forall k(\lvert x_k - y_k \rvert \leq 2^{-k})$

Simple Example

$$\forall x^{\mathbb{R}} \exists n^{\mathbb{N}} (n - 1 < x < n + 1)$$

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$$\forall x^{\mathbb{R}} (\tilde{x}_0 - 1 < x < \tilde{x}_0 + 1)$$

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$$\forall x^{\mathbb{R}} \exists n^{\mathbb{N}} (n - 1 < x < n + 1)$$

$$\forall x^{\mathbb{R}} (\tilde{x}_0 - 1 < x < \tilde{x}_0 + 1)$$

$$\forall x \leq K \; \exists n \leq K \; (n - 1 < x < n + 1)$$

(Strong) Majorizability Relation

Definition (Howard'73, Bezem'85)

$$
\begin{aligned}
n \geq^*_{\mathbb{N}} m \quad &\equiv \quad n \geq m \\
f \geq^*_{\rho\to\tau} g \quad &\equiv \quad \forall x \forall y \geq^*_{\rho} x \, (f y \geq^*_{\tau} g x \wedge f y \geq^*_{\tau} f x)
\end{aligned}
$$

Monotone Interpretations

Theorem (Kohlenbach'92)

If

$$\Delta \vdash \forall x^1 \forall y \leq_1 t(x) \exists k^{\mathbb{N}} A_0(x, y, k)$$

then

$$\Delta_\varepsilon \vdash \forall x^1 \forall y \leq_1 t(x) \exists k^{\mathbb{N}} \leq \phi(x) A_0(x, y, k)$$

Monotone Interpretations

Theorem (Kohlenbach'92)

If

$$\Delta \vdash \forall x^1 \forall y \leq_1 t(x) \exists k^{\mathbb{N}} A_0(x, y, k)$$

then

$$\Delta_\varepsilon \vdash \forall x^1 \forall y \leq_1 t(x) \exists k^{\mathbb{N}} \leq \phi(x) A_0(x, y, k)$$

$$\Delta \equiv \forall a^1 \exists b \leq_1 r(a) \forall c^{\mathbb{N}} B_0(a, b, c)$$

$$\Delta_\varepsilon \equiv \forall a^1 \forall \varepsilon \exists b_\varepsilon \leq_1 r(a) \forall c^{\mathbb{N}} \leq \varepsilon B_0(a, b_\varepsilon, c)$$

Pattern 1: $\forall\exists$

Monotone convergence

$$\forall x \in X; y \in K; \varepsilon \in \mathbb{Q}^*_+ \exists n \in \mathbb{N} \, (f(x, y, n) < \varepsilon)$$

Asymptotic regularity

$$\forall x \in K \forall \varepsilon \in \mathbb{Q}^*_+ \exists n \forall m \geq n (d(x_m, f(x_m)) < \varepsilon)$$

Pattern 1: $\forall\exists$

Monotone convergence

$$\forall x \in X; y \in K; \varepsilon \in \mathbb{Q}^*_+ \exists n \in \mathbb{N} \, (f(x,y,n) < \varepsilon)$$

$$\forall x \in X; y \in K; \varepsilon \in \mathbb{Q}^*_+ \forall m \geq \delta(x,\varepsilon)(f(x,y,m) < \varepsilon)$$

Asymptotic regularity

$$\forall x \in K \forall \varepsilon \in \mathbb{Q}^*_+ \exists n \forall m \geq n(d(x_m, f(x_m)) < \varepsilon)$$

Pattern 1: ∀∃

Monotone convergence

$$\forall x \in X; y \in K; \varepsilon \in \mathbb{Q}_+^* \exists n \in \mathbb{N} \, (f(x, y, n) < \varepsilon)$$

$$\forall x \in X; y \in K; \varepsilon \in \mathbb{Q}_+^* \forall m \geq \delta(x, \varepsilon)(f(x, y, m) < \varepsilon)$$

Asymptotic regularity

$$\forall x \in K \forall \varepsilon \in \mathbb{Q}_+^* \exists n \forall m \geq n(d(x_m, f(x_m)) < \varepsilon)$$

$$\forall x \in K \forall \varepsilon \in \mathbb{Q}_+^* \forall m \geq \kappa(\varepsilon)(d(x_m, f(x_m)) < \varepsilon)$$

Pattern 2: $\exists \rightarrow \exists$

Contractivity

$$\forall x, y \in K (x \neq y \rightarrow d(f(x), f(y)) < d(x, y))$$

Monotonicity

$$\forall x, y \in [0, 1](x - y > 0 \rightarrow f(x) - f(y) > 0)$$

Pattern 2: $\exists \to \exists$

Contractivity

$$\forall x, y \in K(x \neq y \to d(f(x), f(y)) < d(x, y))$$

$$\begin{cases} \forall x, y \in K; \varepsilon \in \mathbb{Q}_+^* \\ \qquad (d(x,y) > \varepsilon \to d(f(x), f(y)) + \eta(\varepsilon) < d(x,y)) \end{cases}$$

Monotonicity

$$\forall x, y \in [0,1](x - y > 0 \to f(x) - f(y) > 0)$$

Pattern 2: $\exists \to \exists$

Contractivity

$$\forall x, y \in K(x \neq y \to d(f(x), f(y)) < d(x, y))$$

$$\left\{ \begin{array}{l} \forall x, y \in K; \varepsilon \in \mathbb{Q}_+^* \\ \qquad (d(x, y) > \varepsilon \to d(f(x), f(y)) + \eta(\varepsilon) < d(x, y)) \end{array} \right.$$

Monotonicity

$$\forall x, y \in [0, 1](x - y > 0 \to f(x) - f(y) > 0)$$

$$\left\{ \begin{array}{l} \forall x, y \in [0, 1]; \varepsilon \in \mathbb{Q}_+^* \\ \qquad (x - y > \varepsilon \to f(x) - f(y) > \delta(\varepsilon)) \end{array} \right.$$

Pattern 3: $\forall \to \forall$

Uniqueness

$$\forall x \in X; y_i \in K(\bigwedge_{i=1}^{2} f(x, y_i) \overset{\mathbb{R}}{=} 0 \to d_K(y_1, y_2) \overset{\mathbb{R}}{=} 0)$$

Convexity

$$\forall x, y \in B(\|\tfrac{1}{2}(x + y)\| \overset{\mathbb{R}}{=} 1 \to \|x - y\| \overset{\mathbb{R}}{=} 0)$$

Pattern 3: $\forall \to \forall$

Uniqueness

$$\forall x \in X; y_i \in K(\bigwedge_{i=1}^{2} f(x, y_i) \stackrel{\mathbb{R}}{=} 0 \to d_K(y_1, y_2) \stackrel{\mathbb{R}}{=} 0)$$

$$\begin{cases} \forall x \in X; y_1, y_2 \in K; \varepsilon \in \mathbb{Q}_+^* \\ \qquad (\bigwedge_{i=1}^{2} |f(x, y_i)| < \Phi(x, \varepsilon) \to d_K(y_1, y_2) < \varepsilon) \end{cases}$$

Convexity

$$\forall x, y \in B(\|\tfrac{1}{2}(x + y)\| \stackrel{\mathbb{R}}{=} 1 \to \|x - y\| \stackrel{\mathbb{R}}{=} 0)$$

Pattern 3: $\forall \to \forall$

Uniqueness

$$\forall x \in X; y_i \in K(\bigwedge_{i=1}^{2} f(x, y_i) \stackrel{\mathbb{R}}{=} 0 \to d_K(y_1, y_2) \stackrel{\mathbb{R}}{=} 0)$$

$$\begin{cases} \forall x \in X; y_1, y_2 \in K; \varepsilon \in \mathbb{Q}_+^* \\ \qquad (\bigwedge_{i=1}^{2} |f(x, y_i)| < \Phi(x, \varepsilon) \to d_K(y_1, y_2) < \varepsilon) \end{cases}$$

Convexity

$$\forall x, y \in B(\|\tfrac{1}{2}(x+y)\| \stackrel{\mathbb{R}}{=} 1 \to \|x - y\| \stackrel{\mathbb{R}}{=} 0)$$

$$\begin{cases} \forall x, y \in B; \varepsilon \in \mathbb{Q}_+^* \\ \qquad (\|\tfrac{1}{2}(x+y)\| > 1 - \eta(\varepsilon) \to \|x - y\| < \varepsilon) \end{cases}$$

Analysis

Analytical principles having the form Δ:

- Every $f \in C[0,1]$ is uniformly continuous
- Every $f \in C[0,1]$ is bounded
- Every $f \in C[0,1]$ attains a maximum
- Brouwer fixed point theorem
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Stripped of their mathematical structure become

Every infinite binary tree has an infinite path

L_1-Approximation

Theorem (Jackson'21)

For any fixed $n \in \mathbb{N}$ and continuous function $f \in C[0,1]$ there exists a unique polynomial $p_n \in P_n$ such that $\|f - p_n\|_1$ is minimal.

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Mathematically elementary proof (just 2 pages), but logically intricate. Use of classical logic and WKL. □

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Question: How to compute p_n given f and n?

- Partial results during the 1970's [Björnestål'1975 and Kroó'1978]
- Explicit algorithm extracted from Cheney's 1965 proof [Kohlenbach/O. 2001]

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Attainment of the infimum (WKL) used in proof of following lemma

Lemma (Original)

$\forall x \in A(f(x) \neq 0) \rightarrow \ldots$

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WKL used to obtain distance from zero

$$\forall x \in A(f(x) \neq 0) \to \exists \delta^{\mathbb{Q}^*_+} \forall x \in A(|f(x)| \geq \delta)$$

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Attainment of the infimum (WKL) used in proof of following lemma

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Theorem suggests a weaker version of lemma is sufficient

Lemma (Weakening)

$\exists \delta^{\mathbb{Q}^*_+} \forall x \in A(|f(x)| \geq \delta) \to \dots$

Further Reading

- Games, functional interpretation and linear logic
  Modified realizability of linear logic
  Oliva, Submitted for publication, 2007
- More on monotone interpretations and applications
  Proof mining: A systematic way of analyzing proofs in mathematics
  Kohlenbach/Oliva, Proc. Steklov Inst. Math, 2003
- Survey of recent applications
  Effective bounds from proofs in abstract functional analysis
  Kohlenbach, CiE 2005, Springer Publisher
- Generalisation of monotone interpretation
  General logical metatheorems for functional analysis
  Kohlenbach/Gerhardy, Trans. Amer. Math. Soc. (to appear)