Modified Realizability of Linear Logic

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Mathematics, Algorithms and Proofs

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Truth semantics

$A \wedge B$	A and B are true
$\forall z A(z)$	$A(z)$ is true for all z
$A \rightarrow B$	if A is true then B is true

Construction semantics

$A \wedge B$	constructions for A and B
$\forall z A(z)$	construction turning z into a const. for $A(z)$
$A \to B$	construction turning const. for A into const. for B

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- Mathematics is like a game, mathematicians are always winners because they plays both roles
- But, imagine two mathematicians are betting on a conjecture

$$\text{(PE)} \quad \forall n \geq 2\, \exists x, y, z (\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z})$$

- **Players**

  Eloise and Abelard

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- **Adjudication of Winner**

  Relation between players' moves

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  Relation between players' moves

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- **For instance**

1) $|(\mathsf{BF})|_y^x \;\equiv\; x > y$

2) $|(\mathsf{BE})|_y^f \;\equiv\; f(y) > y$

3) $|(\mathsf{PE})|_n^{f,g,h} \;\equiv\; \frac{4}{n} = \frac{1}{fn} + \frac{1}{gn} + \frac{1}{hn}$

Game semantics

$$|A \wedge B|^{x,v}_{f,g} \quad \equiv \quad |A|^{x}_{fv} \wedge |B|^{v}_{gx}$$

Game semantics

$$
\begin{aligned}
|A \wedge B|^{x,v}_{f,g} &\equiv |A|^{x}_{fv} \wedge |B|^{v}_{gx} \\
|\forall z A(z)|^{f}_{y,z} &\equiv |A(z)|^{fz}_{y}
\end{aligned}
$$

THE ROYAL SOCIETY

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|\forall z A(z)|^{f}_{y,z} &\equiv |A(z)|^{fz}_{y} \\
|A \to B|^{f,g}_{x,w} &\equiv |A|^{x}_{fw} \to |B|^{gx}_{w}
\end{aligned}
$$

Eloise has no winning strategy for game

$$A \to A \wedge A$$

Game semantics

$$|A \otimes B|^{x,v}_{f,g} \equiv |A|^{x}_{fv} \otimes |B|^{v}_{gx}$$

$$|\forall z A(z)|^{f}_{y,z} \equiv |A(z)|^{fz}_{y}$$

$$|A \multimap B|^{f,g}_{x,w} \equiv |A|^{x}_{fw} \multimap |B|^{gx}_{w}$$

Bringing "Infinity" Back

$$!A$$

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$$(3) \quad |!A|^x_f \equiv !|A|^x_{fx}$$

Soundness

Theorem

$$\Gamma \vdash A \quad \Rightarrow \quad |\Gamma|^{v}_{r[y]} \vdash |A|^{t[v]}_{y}$$

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$$\Gamma \vdash A \quad \stackrel{(??)}{\Leftarrow} \quad |\Gamma|^{v}_{r[y]} \vdash |A|^{t[v]}_{y}$$

Completeness

$$|\Gamma|^{v}_{r[y]} \vdash |A|^{t[v]}_{y}$$

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$$|\Gamma|^{v}_{r[y]} \vdash |A|^{t[v]}_{y}$$

$$\forall w \exists v |\Gamma|^{v}_{w} \vdash \exists x \forall y |A|^{x}_{y}$$

Completeness

$$|\Gamma|^{v}_{r[y]} \vdash |A|^{t[v]}_{y}$$

cut rule

$$\forall w \exists v |\Gamma|^{v}_{w} \vdash \exists x \forall y |A|^{x}_{y}$$

Completeness

$$|\Gamma|^{v}_{r[y]} \vdash |A|^{t[v]}_{y}$$

cut rule

$$\forall w \exists v |\Gamma|^{v}_{w} \vdash \exists x \forall y |A|^{x}_{y}$$

$$\exists v \forall w |\Gamma|^{v}_{w} \vdash \forall y \exists x |A|^{x}_{y}$$

Completeness

$$|\Gamma|^{v}_{r[y]} \vdash |A|^{t[v]}_{y}$$

cut rule

axiom

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Completeness

$$|\Gamma|^{v}_{r[y]} \vdash |A|^{t[v]}_{y}$$

cut rule

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$$\forall w \exists v |\Gamma|^{v}_{w} \vdash \exists x \forall y |A|^{x}_{y}$$

$$\exists v \forall w |\Gamma|^{v}_{w} \vdash \forall y \exists x |A|^{x}_{y}$$

$$\exists\!\!\!\forall^{v}_{w} |\Gamma|^{v}_{w} \vdash \exists\!\!\!\forall^{x}_{y} |A|^{x}_{y}$$

Summary

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- Interpretation of (classical) linear logic
- Stronger existence property for LL
- Conservation results for choice principles
- Refinement and better understanding of
  - $\exists$-free formulas
  - Markov and IP principles
  - Choice principles
- Interesting (well-behaved) branching quantifier

Dialectica interpretation	(Gödel'1956)
Diller-Nahm interpretation	(Diller-Nahm'1974)
Modified realizability	(Kreisel'1959)

$$\mathrm{IL} \longrightarrow \mathrm{IL}$$

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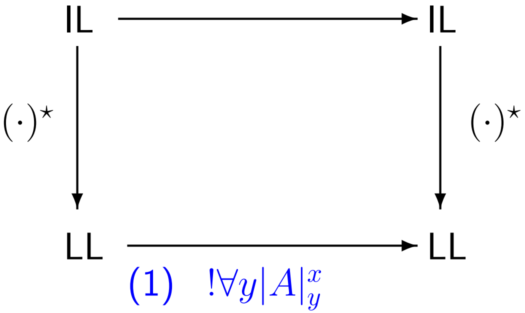

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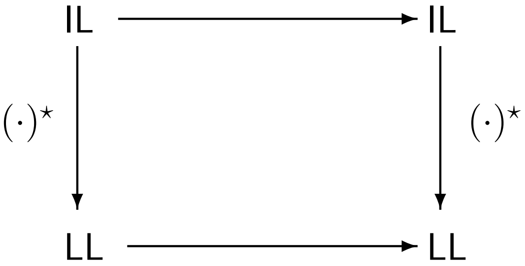

(1) $!\forall y |A|^x_y$

(2) $!\forall y \in f x \, |A|^x_y$

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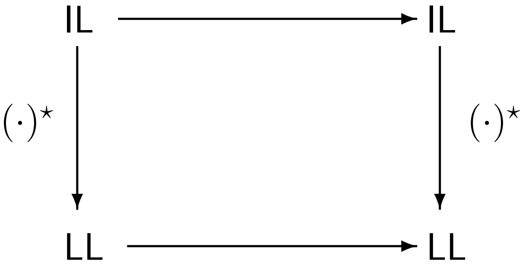

(1) $!\forall y |A|^x_y$

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(3) $!|A|^x_{fx}$

New principles validated

Trump advantage

(TA) $!\exists\!\!\!\forall^{x}_{y} A \multimap \exists x !\forall y A$

New principles validated

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(TA) $!\exists\!\!\!\forall_y^x A \multimap \exists x ! \forall y A$

Sequential choice

(AC_s) $\forall z \exists\!\!\!\forall_y^x A(x, y, z) \multimap \exists\!\!\!\forall_{y,z}^f A(fz, y, z)$

New principles validated

Trump advantage

(TA) $!\exists\!\!\!\forall^x_y A \multimap \exists x !\forall y A$

Sequential choice

(AC$_s$) $\forall z \exists\!\!\!\forall^x_y A(x, y, z) \multimap \exists\!\!\!\forall^f_{y,z} A(fz, y, z)$

Parallel choice

(AC$_p$) $\exists\!\!\!\forall^{x,v}_{f,g}(A(fv) \otimes B(gx)) \multimap \exists\!\!\!\forall^x_y A(y) \otimes \exists\!\!\!\forall^v_w B(w)$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ (\otimes) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ (\multimap)$$

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash\ !A}\ (!) \qquad \frac{\Gamma \vdash A}{\Gamma \vdash\ ?A}\ (?)$$

$$\frac{\Gamma \vdash A[z^\rho]}{\Gamma \vdash \forall z^\rho A[z]}\ (\forall) \qquad \frac{\Gamma \vdash A[t^\rho]}{\Gamma \vdash \exists z^\rho A[z]}\ (\exists)$$

$$\frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\ \textsf{(cut)} \qquad A_{\textsf{at}} \vdash A_{\textsf{at}} \quad \textsf{(id)}$$

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ \textsf{(con)} \qquad \frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ \textsf{(wkn)}$$

$$\frac{\Gamma, A \vdash B}{\Gamma, B^{\perp} \vdash A^{\perp}}\ (\perp) \qquad \frac{\Gamma \vdash A}{\pi\{\Gamma\} \vdash A}\ \textsf{(per)}$$

$$A \multimap A \otimes A$$

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$$|A|^{x}_{\Phi fg} \multimap |A|^{\Delta_0 x}_{f(\Delta_1 x)} \otimes |A|^{\Delta_1 x}_{g(\Delta_0 x)}$$

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$$|A|^{x}_{\Phi f g} \multimap |A|^{\Delta_0 x}_{f(\Delta_1 x)} \otimes |A|^{\Delta_1 x}_{g(\Delta_0 x)}$$

$$|A|^{x}_{\Phi f g} \multimap |A|^{x}_{f x} \otimes |A|^{x}_{g x}$$

$$A \multimap A \otimes A$$

$$|A|^{x}_{\Phi f g} \multimap |A|^{\Delta_0 x}_{f(\Delta_1 x)} \otimes |A|^{\Delta_1 x}_{g(\Delta_0 x)}$$

$$|A|^{x}_{\Phi f g} \multimap |A|^{x}_{f x} \otimes |A|^{x}_{g x}$$

$$(x \geq \Phi f g) \multimap (x \geq f x) \otimes (x \geq g x)$$