Effective WKL Conservation in Feasible Analysis

Paulo Oliva
(joint work with Fernando Ferreira)

Queen Mary, University of London, UK
(pbo@dcs.qmul.ac.uk)

ALS Annual Meeting, Montreal, Canada

18 May 2006

Outline

Outline

Feasible Analysis

Cook and Urquhart's CPV$^\omega$

- Only polynomial-time construction allowed
- Induction for NP-predicates:

$$\forall x(A(\lfloor \frac{1}{2}x \rfloor) \to A(x)) \to (A(0) \to \forall x A(x))$$

where A is of the form $\exists y \leq t\,(s = u)$

Feasible Analysis

Cook and Urquhart's CPV$^\omega$

- Only polynomial-time construction allowed

- Induction for NP-predicates:

$$\forall x(A(\lfloor \frac{1}{2}x \rfloor) \to A(x)) \to (A(0) \to \forall x A(x))$$

  where A is of the form $\exists y \leq t \, (s = u)$

- **Feasible Analysis**

  CPV$^\omega$ plus quantifier-free choice AC$_{\mathsf{qf}}$

Feasible Analysis

Cook and Urquhart's CPV^ω

- Only polynomial-time construction allowed

- Induction for NP-predicates:

$$\forall x(A(\lfloor \frac{1}{2}x \rfloor) \to A(x)) \to (A(0) \to \forall x A(x))$$

  where A is of the form $\exists y \leq t\,(s = u)$

- **Feasible Analysis**
  CPV^ω plus quantifier-free choice $\mathsf{AC_{qf}}$ $(\mathsf{CPV}^\omega + \mathsf{AC_{qf}} \vdash \Delta_1^{\mathsf{pt}}\text{-}\mathsf{CA})$

Feasible Analysis

Cook and Urquhart's CPV^ω

- Only polynomial-time construction allowed

- Induction for NP-predicates:

$$\forall x(A(\lfloor \frac{1}{2}x \rfloor) \to A(x)) \to (A(0) \to \forall x A(x))$$

  where A is of the form $\exists y \leq t\,(s = u)$

- **Feasible Analysis**

  CPV^ω plus quantifier-free choice $\mathsf{AC}_{\mathsf{qf}}$ $\quad (\mathsf{CPV}^\omega + \mathsf{AC}_{\mathsf{qf}} \vdash \Delta^{\mathsf{pt}}_1\text{-}\mathsf{CA})$

$$\forall n(\exists k A_{\mathrm{qf}}(n,k) \leftrightarrow \forall k B_{\mathrm{qf}}(n,k)) \;\vdash\; \exists f \forall n(fn = 0 \leftrightarrow \exists k A_{\mathrm{qf}}(n,k))$$

Weak König's Lemma

- Every infinite binary tree has an infinite path
  Compactness of $\{0,1\}^\omega$ *under associated product topology*

- Formally

$$\underbrace{\mathsf{Bin}(T)}_{\Pi^0_1} \wedge \underbrace{\mathsf{Inf}(T)}_{\forall\Sigma^b_1} \to \exists\alpha^{\{0,1\}^\omega}\forall n^{\mathbb{N}}(\overline{\alpha}n \in T)$$

where T is a Σ^b_∞-formula

Weak König's Lemma

- Every infinite binary tree has an infinite path

  Compactness of $\{0,1\}^\omega$ *under associated product topology*

- Formally

$$\underbrace{\mathsf{Bin}(T)}_{\Pi^0_1} \wedge \underbrace{\mathsf{Inf}(T)}_{\forall\Sigma^b_1} \rightarrow \exists\alpha^{\{0,1\}^\omega}\forall n^{\mathbb{N}}(\overline{\alpha}n \in T)$$

  where T is a Σ^b_∞-formula

- Equivalent (over RCA_0) to:
  - Heine-Borel covering theorem for unit interval
  - Σ^0_1-separation
  - Boundedness theorem
  - Extreme value theorem
  - Gödel's completeness theorem for countable languages

Outline

WKL Conservation

RCA_0

- As used in 'reverse mathematics'
- Σ^0_1-induction and Δ^0_1-comprehension

Theorem (Friedman)

$\mathsf{RCA}_0 + \mathsf{WKL}$ *is* Π^0_2*-conservative over* PRA, *i.e.*

$$\mathsf{RCA}_0 + \mathsf{WKL} \vdash \forall n \exists m A_{\mathrm{qf}}(n, m) \quad \Rightarrow \quad \mathsf{PRA} \vdash A_{\mathrm{qf}}(n, t[n])$$

for some term t.

WKL Conservation

RCA_0

- As used in 'reverse mathematics'
- Σ^0_1-induction and Δ^0_1-comprehension

Theorem (Friedman)

$\mathsf{RCA}_0 + \mathsf{WKL}$ *is* Π^0_2*-conservative over* PRA, *i.e.*

$$\mathsf{RCA}_0 + \mathsf{WKL} \vdash \forall n \exists m A_{\mathrm{qf}}(n, m) \quad \Rightarrow \quad \mathsf{PRA} \vdash A_{\mathrm{qf}}(n, t[n])$$

for some term t.

- Friedman's proof is model-theoretic
- Effective proof by Kohlenbach [1992] using combination of Dialectica interpretation and majorizability

WKL Conservation: in Feasible Analysis

Theorem (Ferreira'1994)

$\mathsf{CPV}^\omega + \Delta_1^{\mathsf{pt}}\text{-}\mathsf{CA} + \mathsf{WKL}$ *is* Π_2^0*-conservative over* PV^ω.

WKL Conservation: in Feasible Analysis

Theorem (Ferreira'1994)

$\mathsf{CPV}^\omega + \Delta_1^{\mathsf{pt}}\text{-}\mathsf{CA} + \mathsf{WKL}$ *is* Π_2^0*-conservative over* PV^ω.

- Ferreira's proof is also model-theoretic (and ineffective)
- Effective version was not known

  (Kohlenbach's proof makes use of non-feasible constructions)

Avigad and Feferman's Conjecture

"It is reasonable to conjecture that functional interpretations can be used to obtain similar results for systems of polynomial-time-computable arithmetic, especially in view of the conservation result of Ferreira [1994], which showed that a suitable version of WKL *is conservative over* PV^{ω} *for* Π^0_2*-formulas"*

Handbook of Proof Theory, 1998

Avigad and Feferman's Conjecture

"It is reasonable to conjecture that functional interpretations can be used to obtain similar results for systems of polynomial-time-computable arithmetic, especially in view of the conservation result of Ferreira [1994], which showed that a suitable version of WKL *is conservative over* PV^{ω} *for* Π^0_2*-formulas"*

Handbook of Proof Theory, 1998

Problem: FI don't distinguish bounded and unbounded quantifiers

Outline

Goal

- Interpretation distinguishing $\exists x^\rho \leq t A(x)$ and $\exists x^\rho A(x)$.
- Need: bounded quantifiers for all finite types.
- Need: extended $\leq$ to all finite types.

The Bounded Quantifier

- Use Howard/Bezem's strong majorizability relation
- Extension of the $\leq$-relation to higher types:

$$
\begin{aligned}
x \trianglelefteq_{\mathbb{N}} y \quad &:= \quad x \leq_{\mathbb{N}} y \\
x \trianglelefteq_{\rho \to \sigma} y \quad &:= \quad \forall v^{\rho} \forall u \trianglelefteq_{\rho} v (\underbrace{xu \trianglelefteq_{\sigma} yv}_{\text{above}} \wedge \underbrace{yu \trianglelefteq_{\sigma} yv}_{\text{monotone}})
\end{aligned}
$$

The Bounded Quantifier

- Use Howard/Bezem's strong majorizability relation
- Extension of the $\leq$-relation to higher types:

$$
\begin{aligned}
x \trianglelefteq_{\mathbb{N}} y &:= x \leq_{\mathbb{N}} y \\
x \trianglelefteq_{\rho\to\sigma} y &:= \forall v^{\rho} \forall u \trianglelefteq_{\rho} v (\underbrace{xu \trianglelefteq_{\sigma} yv}_{\text{above}} \wedge \underbrace{yu \trianglelefteq_{\sigma} yv}_{\text{monotone}})
\end{aligned}
$$

- Bounded quantifiers (for all types) defined as

$$
\begin{aligned}
\mathsf{B}_{\forall} &: \forall x \trianglelefteq t A(x) \leftrightarrow \forall x (x \trianglelefteq t \to A(x)) \\
\mathsf{B}_{\exists} &: \exists x \trianglelefteq t A(x) \leftrightarrow \exists x (x \trianglelefteq t \wedge A(x))
\end{aligned}
$$

The Bounded Quantifier

- Use Howard/Bezem's strong majorizability relation
- Extension of the $\leq$-relation to higher types:

$$x \trianglelefteq_{\mathbb{N}} y \quad := \quad x \leq_{\mathbb{N}} y$$

$$x \trianglelefteq_{\rho \to \sigma} y \quad := \quad \forall v^{\rho} \forall u \trianglelefteq_{\rho} v (\underbrace{xu \trianglelefteq_{\sigma} yv}_{\text{above}} \wedge \underbrace{yu \trianglelefteq_{\sigma} yv}_{\text{monotone}})$$

- Bounded quantifiers (for all types) defined as

$$\mathsf{B}_{\forall} \;:\; \forall x \trianglelefteq t A(x) \leftrightarrow \forall x (x \trianglelefteq t \to A(x))$$
$$\mathsf{B}_{\exists} \;:\; \exists x \trianglelefteq t A(x) \leftrightarrow \exists x (x \trianglelefteq t \wedge A(x))$$

- Let the theory $\mathsf{T}^{\omega}_{\trianglelefteq}$ be an extension of T^{ω} with axioms/rule for $\trianglelefteq$, $\mathsf{B}_{\forall}$ and $\mathsf{B}_{\exists}$. (*need to be careful in axiomatisation*)

The Interpretation

- Usual Dialectica interpretation

  A assigned to *quantifier-free* formula $|A|^{\vec{x}}_{\vec{y}}$ $(A \leftrightarrow \exists \vec{x} \forall \vec{y} |A|^{\vec{x}}_{\vec{y}})$

The Interpretation

- Usual Dialectica interpretation

  A assigned to *quantifier-free* formula $|A|^{\vec{x}}_{\vec{y}}$ $\quad (A \leftrightarrow \exists \vec{x} \forall \vec{y} |A|^{\vec{x}}_{\vec{y}})$

- **Bounded Dialectica Interpretation**

  A associated to *bounded* formula $|A|^{\vec{x}}_{\vec{y}}$ with *monotonicity* on $\vec{x}$

$$|A|^{\vec{x}}_{\vec{y}} \wedge \vec{x} \trianglelefteq \vec{w} \rightarrow |A|^{\vec{w}}_{\vec{y}}$$

The Interpretation

- Usual Dialectica interpretation

  A assigned to *quantifier-free* formula $|A|^{\vec{x}}_{\vec{y}}$ $\quad (A \leftrightarrow \exists \vec{x} \forall \vec{y} |A|^{\vec{x}}_{\vec{y}})$

- **Bounded Dialectica Interpretation**

  A associated to *bounded* formula $|A|^{\vec{x}}_{\vec{y}}$ with *monotonicity* on $\vec{x}$

$$|A|^{\vec{x}}_{\vec{y}} \wedge \vec{x} \trianglelefteq \vec{w} \to |A|^{\vec{w}}_{\vec{y}}$$

- Some cases:

$$\begin{aligned} |\forall x A(x)|^{f}_{a,c} \quad &:\equiv \quad \forall x \trianglelefteq a |A(x)|^{fa}_{c} \\ |A \to B|^{f,g}_{b,e} \quad &:\equiv \quad \forall c \trianglelefteq gbe |A|^{b}_{c} \to |B|^{fb}_{e} \end{aligned}$$

Soundness Theorem

Let P[$\trianglelefteq$] denote the (family of) *bounded version* of

- axiom choice
- independence of premise
- Markov principle
- contra collection (*generalisation of* WKL)
- majorizability axiom, $\forall x \exists y (x \trianglelefteq y)$

Soundness Theorem

Let P[$\trianglelefteq$] denote the (family of) *bounded version* of

- axiom choice
- independence of premise
- Markov principle
- contra collection (*generalisation of* WKL)
- majorizability axiom, $\forall x \exists y (x \trianglelefteq y)$

Theorem (Ferreira and Oliva'2006)

If

$$\mathsf{IPV}^{\omega}_{\trianglelefteq} + \mathsf{P}[\trianglelefteq] \vdash A(z)$$

then there are closed monotone terms t of appropriate types such that

$$\mathsf{IPV}^{\omega}_{\trianglelefteq} \vdash \forall a \forall z \trianglelefteq a \, \forall c |A(z)|^{ta}_{c}$$

Corollary

Theorem (Ferreira and Oliva'2006)

If

$$\mathsf{CPV}^\omega + \Delta_1^{\mathsf{pt}}\text{-}\mathsf{CA} + \mathsf{WKL} \vdash \forall n \exists m A_{\mathrm{qf}}(n,m),$$

where A_{qf} *is a quantifier-free formula, then*

$$\mathsf{PV}^\omega \vdash A_{\mathrm{qf}}(n, t[n])$$

for a closed term t *of* CPV^ω.

Proof Sketch

1. Assume

$$\mathsf{CPV}^{\omega} + \Delta_1^{\mathsf{pt}}\text{-}\mathsf{CA} + \mathsf{WKL} \vdash \forall n \exists m A_{\mathrm{qf}}(n, m)$$

Proof Sketch

1. Assume

$$\mathsf{CPV}^\omega + \Delta_1^{\mathsf{pt}}\text{-}\mathsf{CA} + \mathsf{WKL} \vdash \forall n \exists m A_{\mathrm{qf}}(n, m)$$

2. Then

$$\mathsf{CPV}^\omega_\trianglelefteq + \mathsf{AC}^b_{\mathsf{qf}} + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash \forall n \exists m A_{\mathrm{qf}}(n, m)$$

Proof Sketch

1. Assume
$$\mathsf{CPV}^\omega + \Delta^{\mathsf{pt}}_1\text{-}\mathsf{CA} + \mathsf{WKL} \vdash \forall n \exists m A_{\mathrm{qf}}(n, m)$$
2. Then
$$\mathsf{CPV}^\omega_\trianglelefteq + \mathsf{AC}^b_{\mathsf{qf}} + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash \forall n \exists m A_{\mathrm{qf}}(n, m)$$
3. By the negative translation
$$\mathsf{IPV}^\omega_\trianglelefteq + \mathsf{MP}_{\Sigma^b_1} + \mathsf{AC}^b_{\mathsf{qf}} + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash \forall n \exists m A_{\mathrm{qf}}(n, m)$$

Proof Sketch

1. Assume

$$\mathsf{CPV}^\omega + \Delta_1^{\mathsf{pt}}\text{-}\mathsf{CA} + \mathsf{WKL} \vdash \forall n \exists m A_{\mathrm{qf}}(n,m)$$

2. Then

$$\mathsf{CPV}^\omega_{\trianglelefteq} + \mathsf{AC}^b_{\mathsf{qf}} + \mathrm{P}_{\mathrm{bd}}[\trianglelefteq] \vdash \forall n \exists m A_{\mathrm{qf}}(n,m)$$

3. By the negative translation

$$\mathsf{IPV}^\omega_{\trianglelefteq} + \mathsf{MP}_{\Sigma_1^b} + \mathsf{AC}^b_{\mathsf{qf}} + \mathrm{P}_{\mathrm{bd}}[\trianglelefteq] \vdash \forall n \exists m A_{\mathrm{qf}}(n,m)$$

4. By bounded Dialectica interpretation

$$\mathsf{IPV}^\omega + \mathsf{MP}_{\Sigma_1^b} + \mathsf{AC}^b_{\mathsf{qf}} \vdash \forall n \exists m \leq t[n] A_{\mathrm{qf}}(n,m)$$

Proof Sketch

1. Assume

$$\mathsf{CPV}^\omega + \Delta_1^{\mathsf{pt}}\text{-}\mathsf{CA} + \mathsf{WKL} \vdash \forall n \exists m A_{\mathrm{qf}}(n, m)$$

2. Then

$$\mathsf{CPV}^\omega_{\trianglelefteq} + \mathsf{AC}^b_{\mathsf{qf}} + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash \forall n \exists m A_{\mathrm{qf}}(n, m)$$

3. By the negative translation

$$\mathsf{IPV}^\omega_{\trianglelefteq} + \mathsf{MP}_{\Sigma_1^b} + \mathsf{AC}^b_{\mathsf{qf}} + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash \forall n \exists m A_{\mathrm{qf}}(n, m)$$

4. By bounded Dialectica interpretation

$$\mathsf{IPV}^\omega + \mathsf{MP}_{\Sigma_1^b} + \mathsf{AC}^b_{\mathsf{qf}} \vdash \forall n \exists m \leq t[n] A_{\mathrm{qf}}(n, m)$$

5. By Cook and Urquhart Dialectica interpretation

$$\mathsf{PV}^\omega \vdash A_{\mathrm{qf}}(n, s[n])$$

Outline

Final remarks

More general result allowing:

- Extensionality
- Σ^0_1 Uniform Boundness

$$\forall f \leq_1 h \exists e^1 A_{\mathrm{b}}(f, h, e) \to \exists g^1 \forall f \leq_1 h \exists e \leq_1 g A_{\mathrm{b}}(f, h, e)$$

  for any parameter h^1 and Σ^b_∞-formula A_{b}.

- More general forms of choice

References

1992 W. Kohlenbach
Effective bounds from ineffective proofs in analysis
The Journal of Symbolic Logic, 57:1239–1273

1994 F. Ferreira
A feasible theory for analysis.
The Journal of Symbolic Logic, 59:1001–1011

1998 J. Avigad and S. Feferman
Gödel's functional ("Dialectica") interpretation.
Handbook of proof theory, 137:337–405

2005 F. Ferreira and P. Oliva
Bounded functional interpretation.
Annals of Pure and Applied Logic, 135, 73–112

2005 F. Ferreira and P. Oliva
Bounded functional interpretation and feasible analysis.
Submitted for publication