Functional Interpretations

Lecture 3: Mathematical Analysis

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Outline

1. Introduction
2. Analysis
   - Weak König lemma
   - Countable choice
3. Proof mining
   - Fixed point theorems
   - Uniqueness theorems

Outline

Extensions

IL	Intuitionistic logic
CL	Classical logic
CL^ω	Higher-order classical logic
PA^ω	Higher-order Peano arithmetic
PA^2	classical analysis

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   - Countable choice
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Comprehension principles

- Analysis is obtained by adding set/functional-existence principles
  E.g. *comprehension* $\exists f \forall n^o (fn = 0 \leftrightarrow A(n))$

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- Subsystem via restrictions on *induction* and *comprehension* formulas

Comp\Ind	Σ^b_1	Σ^0_1	full
$\emptyset$	CPV	PRA	PA
Δ^0_1	$\mathsf{CPV}^\omega + \mathsf{cAC}_{\mathsf{qf}}$	RCA_0	RCA
WKL	+WKL	WKL_0	WKL
arithmetical	$\times$	ACA_0	ACA
full	$\times$	$\times$	PA^2

Countable choice implies comprehension

Theorem

Classically, comprehension follows from countable choice

$$\forall n^o \exists y^\tau A(n, y) \rightarrow \exists f \forall n A(n, fn)$$

Proof.

Apply countable choice to the classical theorem

$$\forall n \exists b (b = 0 \leftrightarrow A(n))$$

□

Classical analysis

- As formal system of classical analysis we take $\mathsf{PA}^\omega + \mathsf{cAC}$
- Subsystem using countable choice:

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$\emptyset$	CPV	PRA	PA
Δ^0_1	$\mathsf{CPV}^\omega + \mathsf{cAC}_{\mathsf{qf}}$	$\mathsf{PRA}^\omega + \mathsf{cAC}_{\mathsf{qf}}$	$\mathsf{PA}^\omega + \mathsf{cAC}_{\mathsf{qf}}$
WKL	+WKL	+WKL	+WKL
arithmetical	$\times$	$+\mathsf{cAC}_{\mathsf{ar}}$	$+\mathsf{cAC}_{\mathsf{ar}}$
full	$\times$	$\times$	$\mathsf{PA}^\omega + \mathsf{cAC}$

Weak König lemma

Let

- $\overline{\alpha}n$ stand for $\langle \alpha(0), \ldots, \alpha(n) \rangle$
- $s \in f$ stand for $f(s) = 0$
- $s' \succeq s$ stand for $|s'| \geq |s| \wedge \forall i < |s| (s_i = s'_i)$
- bTree(f) stand for $\forall s, s'(s' \succeq s \wedge s' \in f \to s \in f)$
- Inf(f) stand for $\forall n \exists s(|s| = n \wedge s \in f)$

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Definition (WKL)

Every infinite binary tree has an infinite path. Formally:

$$\forall f(\mathsf{bTree}(f) \wedge \mathsf{Inf}(f) \to \exists \alpha \forall n(\overline{\alpha}n \in f))$$

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$$\forall f (\mathsf{bTree}(f) \wedge \mathsf{Inf}(f) \rightarrow \exists \alpha \forall n (\overline{\alpha}n \in f))$$

Essential part $\quad \forall n \exists s (|s| = n \wedge s \in f) \rightarrow \exists \alpha \forall n (\overline{\alpha}n \in f)$

Classically $\quad \exists \alpha \forall n (\exists s (|s| = n \wedge s \in f) \rightarrow (\overline{\alpha}n \in f))$

Weak König lemma

The relevance of WKL comes from the following theorem:

Theorem (Harrington)

WKL_0 *is* Π^1_1*-conservative over* RCA_0

Moreover, WKL is equivalent (over RCA) to the following principles:

- Heine-Borel covering theorem for unit interval
- Σ^0_1-separation
- Boundedness theorem
- Extreme value theorem
- Gödel's completeness theorem for countable languages

Dealing with WKL

Three approaches to dealing with (negative translation of) WKL

1. Interpret WKL using Dialectica interpretation and a new recursion
2. Trivialise WKL via the bounded functional interpretation
3. Weaken WKL via monotone functional interpretation

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Options 1. and 2. are particularly suitable for feasible systems

1. via Dialectica interpretation

Lets look at Dialectica interpretation of (negative translation of) WKL

The negative translation of WKL is equivalent (up to MP) to

$$\forall n \exists s (|s| = n \wedge s \in f) \rightarrow \neg\neg \exists \alpha \forall n (\overline{\alpha} n \in f)$$

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The Dialectica interpretation asks for witnesses for

$$\forall g, Y \exists \alpha, n \big((|g_n| = n \wedge g_n \in f) \rightarrow (\overline{\alpha}(Y\alpha) \in f) \big)$$

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Possible with the following recursion schema:

$$\mathsf{bB}(n) = \begin{cases} n & (Y(\hat{g}_n) < |g_n|) \vee (|g_n| \neq n) \\ \mathsf{bB}(n+1) & \text{otherwise} \end{cases}$$

taking $n = \mathsf{bB}(0)$ and $\alpha = \hat{g}_n$.

2. via Bounded functional interpretation

The following principle is interpreted for b.f.i.

$$\forall b^{\tau} \exists z \leq^{*} c \forall y \leq^{*} b A_{\mathrm{b}}(y, z) \rightarrow \exists z \leq^{*} c \forall y A_{\mathrm{b}}(y, z)$$

so-called *Bounded Contra Collection Principle*.

Theorem

WKL *follows from* BCC.

Proof.

Consider the following instance of BCC:

$$\forall k \exists \alpha \leq^{*} 1 \forall n \leq^{*} k (\overline{\alpha} n \in f) \rightarrow \exists \alpha \leq^{*} 1 \forall n (\overline{\alpha} n \in f)$$

□

3. via Monotone functional interpretation

1. Start with a proof $\exists \alpha \forall n (\ldots) \to \forall x \exists y A_{\mathsf{qf}}(x, y)$
2. Monotone f.i. will produce bounds t, q such that

$$\forall x, \alpha (\forall n \leq q[x, \alpha](\ldots) \to \exists y \leq t[x, \alpha] A_{\mathsf{qf}}(x, y))$$

3. Given that $\alpha \leq^* 1$ and $q \leq^* q^*$ and $t \leq^* t^*$ we have

$$\forall x, \alpha (\forall n \leq q^*[x, 1](\ldots) \to \exists y \leq t^*[x, 1] A_{\mathsf{qf}}(x, y))$$

4. But it's easy to find α such that $\forall n \leq q^*[x, 1](\ldots)$.

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$$\forall n^o \neg\neg\exists y^\tau A^\dagger(n, y) \rightarrow \neg\neg\exists f \forall n A^\dagger(n, fn)$$

- Consider cAC for universal formulas.

$$\forall n^o \neg\neg\exists y \forall x A_{\mathsf{qf}}(n, y, x) \rightarrow \neg\neg\exists f \forall n, x A_{\mathsf{qf}}(n, fn, x)$$

Interpretation asks for functionals n, g, f depending on Φ, Ψ, Δ s.t.

$$\neg\neg A_{\mathsf{qf}}(n, \Phi ng, g(\Phi ng)) \rightarrow \neg\neg A_{\mathsf{qf}}(\Psi f, f(\Psi f), \Delta f)$$

Interpreting countable choice

How to produce n, g, f witnessing

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Enough to satisfy the equations:

$$\left\{ \begin{array}{lcl} n & \stackrel{o}{=} & \Psi f \\ fn & \stackrel{\tau}{=} & \Phi n g \\ g(fn) & \stackrel{\sigma}{=} & \Delta f \end{array} \right\}$$

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Enough to satisfy the equations:

$$\left\{ \begin{array}{lcl} n & \stackrel{o}{=} & \Psi f \\ f n & \stackrel{\tau}{=} & \Phi n g \\ g(fn) & \stackrel{\sigma}{=} & \Delta f \end{array} \right\} \quad \Rightarrow \quad \left\{ \begin{array}{lcl} i & \leq & \Psi \hat{s} \\ s_i & \stackrel{\tau}{=} & \Phi i g_i \\ g_i(s_i) & \stackrel{\sigma}{=} & \Delta \hat{s} \end{array} \right\}$$

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Solution can be produced as follows:

$$\begin{array}{lcl} G_0[y_1] & := & \lambda y_0 . \Delta(\langle y_0, y_1, ... \rangle) \\ S_0[y_1] & := & \Phi 0 G_0[y_1] \\ g_1 & := & \lambda y_1 . \Delta(\langle S_0[y_1], y_1, ... \rangle) \\ s_1 & := & \Phi 1 g_1 \end{array}$$

Finally, take $g_0 = G_0[s_1]$ and $s_0 = S_0[s_1]$.

Quiz: solution

Consider the following game with 3 people.

1. Each person i builds a function g_i which given her number $x_i > 0$ should give the (predicted) sum of all numbers $x_1 + x_2 + x_3$.
2. Person $i \in \{1, 2, 3\}$ is then assigned the number $x_i := g_i(i)$
3. It should be the case that $g_i(x_i) = x_1 + x_2 + x_3$

How should the participants proceed in choosing g_i?

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$$G_1[x_2, x_3] := \lambda x_1 . x_1 + x_2 + x_3$$

$$X_1[x_2, x_3] := 1 + x_2 + x_3$$

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$$x_3 := X_1[X_2[3]] + X_2[3] + 3$$

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$$X_2[x_3] := X_1[2, x_3] + 2 + x_3$$

$$g_3 := \lambda x_3 . X_1[X_2[x_3], x_3] + X_2[x_3] + x_3 \qquad = \lambda x_3 . (6x_3 + 11)$$

$$x_3 := X_1[X_2[3]] + X_2[3] + 3 \qquad = 29$$

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$$G_1[x_2, x_3] := \lambda x_1.x_1 + x_2 + x_3$$

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$$G_2[x_3] := \lambda x_2.X_1[x_2, x_3] + x_2 + x_3 \qquad = \lambda x_2.(2x_2 + 59)$$

$$X_2[x_3] := X_1[2, x_3] + 2 + x_3 \qquad = 63$$

$$g_3 := \lambda x_3.X_1[X_2[x_3], x_3] + X_2[x_3] + x_3 \qquad = \lambda x_3.(6x_3 + 11)$$

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How should the participants proceed in choosing g_i?

$$G_1[x_2, x_3] := \lambda x_1 . x_1 + x_2 + x_3 \qquad = \lambda x_1 .(x_1 + 92)$$

$$X_1[x_2, x_3] := 1 + x_2 + x_3 \qquad = 93$$

$$G_2[x_3] := \lambda x_2 . X_1[x_2, x_3] + x_2 + x_3 \qquad = \lambda x_2 .(2x_2 + 59)$$

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$$g_3 := \lambda x_3 . X_1[X_2[x_3], x_3] + X_2[x_3] + x_3 \qquad = \lambda x_3 .(6x_3 + 11)$$

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Finite bar recursion

General case (with a fixed bound k)

$$\begin{aligned} i &\leq k \\ s_i &\stackrel{\tau}{=} \Phi i g_i \\ g_i(s_i) &\stackrel{\sigma}{=} \Delta \hat{s} \end{aligned}$$

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General solution can be constructed as follows:

$$\mathsf{fB}(s) = \begin{cases} s & k < |s| \\ \mathsf{fB}(s * X_s) & \text{otherwise} \end{cases}$$

where $X_s := \Phi(|s|, G_s)$ and $G_s := \lambda y.\Delta(\mathsf{fB}(s * y))$.

Then take $s := \mathsf{fB}(\langle\ \rangle)$ and $g_i := G_{\overline{s}i}$.

Spector's bar recursion

Finally, the full problem

$$\begin{array}{lcl} i & \leq & \Psi \hat{s} \\ s_i & \stackrel{\tau}{=} & \Phi i g_i \\ g_i(s_i) & \stackrel{\sigma}{=} & \Delta \hat{s} \end{array}$$

can be solved with

$$\mathsf{BR}(s) = \begin{cases} s & \Psi \hat{s} < |s| \\ \mathsf{BR}(s * X_s) & \text{otherwise} \end{cases}$$

where $X_s := \Phi(|s|, G_s)$ and $G_s := \lambda y.\Delta(\mathsf{BR}(s * y))$.

Finally, take $\mathsf{BR}(\langle\ \rangle)$.

Modified bar recursion

In general, one should solve

$$\forall n^o((\exists y^\tau A_B(n,y) \to B) \to B) \wedge (\exists f \forall n A_B(n, fn) \to B) \to B$$

Given realizers for

$$\forall n^o((\exists y^\tau A_B(n,y) \to B) \to B) \qquad \Phi \; : \; o \to ((\tau \to b) \to b)$$

$$\exists f \forall n A_B(n, fn) \to B \qquad \Psi \; : \; (o \to \tau) \to b$$

produce a realizer for B.

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produce a realizer for B. Also have $\Delta : b \to \tau$.

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produce a realizer for B. Also have $\Delta : b \to \tau$. Can be done was

$$\tilde{\mathsf{BR}}(s) = \begin{cases} \hat{s} & \omega_\Psi(\hat{s}) < |s| \\ \tilde{\mathsf{BR}}(s * X_s) & \text{otherwise} \end{cases}$$

where $X_s := \Delta(\Phi(|s|, G_s))$ and $G_s := \lambda y^\tau . \Psi(\tilde{\mathsf{BR}}(s * y))$.

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where $X_s := \Delta(\Phi(|s|, G_s))$ and $G_s := \lambda y^\tau . \Psi(\tilde{\mathsf{BR}}(s * y))$.

Finally, take $\Psi(\tilde{\mathsf{BR}}(\langle\,\rangle))$.

Modified bar recursion

Since we are only interested in $\Psi(\tilde{\mathsf{BR}}(\langle\,\rangle))$, we can simplify the recursion

$$\tilde{\mathsf{BR}}(s) = \begin{cases} \hat{s} & \omega_\Psi(\hat{s}) < |s| \\ \tilde{\mathsf{BR}}(s * X_s) & \text{otherwise} \end{cases}$$

eliminating the use of ω_Ψ as

$$\mathsf{MBR}(s) = \Psi(s * X_s * \mathsf{MBR}(s * X_s))$$

where $X_s := \Delta(\Phi(|s|, G_s))$ and $G_s := \lambda y^\tau . \Psi(s * y * \mathsf{MBR}(s * y))$.

Outline

Proof mining

- Extraction of computational information from (ineffective) proofs

- **Monotone interpretations**
  - Via Howard/Bezem majorizability relation,
    every witness $\boldsymbol{t}$ (i.e. $|A|^{\boldsymbol{t}}$) has a majorant $\boldsymbol{t}^*$
  - In particular, we have $\exists \boldsymbol{x} \leq^* \boldsymbol{t}^* |A|^{\boldsymbol{x}}$
  - Useful to get independence of "bounded" parameters
  - Idea can be inductively carried through

- Particularly useful for obtaining bounds in functional analysis

Representing reals, rationals, continuous fcts...

Object	Notation	Representation	Equality
Naturals $\mathbb{N}$	i, j	primitive	primitive
Integers $\mathbb{Z}$	n, m	$\mathbb{N} \times \mathbb{N}$	$\left\{ \begin{array}{c} p(n_0) =_{\mathbb{N}} p(m_0) \\ n_1 =_{\mathbb{N}} m_1 \end{array} \right\}$
Rationals $\mathbb{Q}$	δ, ϵ	$\mathbb{Z} \times \mathbb{Z}^*$	$\delta_0 \epsilon_1 =_{\mathbb{Z}} \delta_1 \epsilon_0$
Reals $\mathbb{R}$	x, y	Cauchy $\mathbb{N} \to \mathbb{Q}$	$\forall i(\lvert x(i) - y(i) \rvert \leq_{\mathbb{Q}} 2^{-i})$

Continuous functions
- restriction to rationals, $f_r : (\mathbb{Q} \cap [0,1]) \times \mathbb{N} \to \mathbb{Q}$
- $\forall d \in (\mathbb{Q} \cap [0,1]) \forall n (\lvert f(d) - f_r(d,n) \rvert \leq 2^{-n})$
- modulus of uniform continuity, $\omega_f : \mathbb{N} \to \mathbb{N}$
- $\forall x, y (\lvert x - y \rvert \leq 2^{-\omega_f(n)} \to \lvert fx - fy \rvert \leq 2^{-n})$

Proof mining in practise: Two important points

1. **No need to formalise the whole proof**

   A. *Some lemmas, like universal lemmas, do not contribute to final witness, hence its proof can be ignored (take them as axioms)*

   B. *Witnesses for the interpretation of some lemmas can be produced without the need of a proof (take the witnessed interpretation of the lemma as an axiom)*

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2. **No need to witness all information**

   Not all information will be of interest. It is possible (and often simpler) to carry out a (partial) interpretation, leaving the unwanted information behind. For instance, in $\exists \delta \in \mathbb{Q}^*_+ (x >_{\mathbb{R}} \delta)$*, the existential information in* $>_{\mathbb{R}}$ *is irrelevant.*

Pattern 1: $\exists \to \exists$

Contractivity $\forall x, y \in K(x \neq y \to d(f(x), f(y)) < d(x, y))$

Monotonicity $\forall x, y \in [0,1](x - y > 0 \to f(x) - f(y) > 0)$

Convergence $\forall x \in X; y \in K; \varepsilon \in \mathbb{Q}_+^* \exists n \in \mathbb{N}\,(f(x, y, n) < \varepsilon)$

Asy reg. $\forall x \in K \forall \varepsilon \in \mathbb{Q}_+^* \exists n \forall m \geq n(d(x_m, f(x_m)) < \varepsilon)$

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Pattern 2: $\forall \to \forall$

Uniqueness $\forall x \in X; y_i \in K(\bigwedge_{i=1}^{2} f(x, y_i) \stackrel{\mathbb{R}}{=} 0 \to d_K(y_1, y_2) \stackrel{\mathbb{R}}{=} 0)$

Convexity $\forall x, y \in B(\|\frac{1}{2}(x+y)\| \stackrel{\mathbb{R}}{=} 1 \to \|x-y\| \stackrel{\mathbb{R}}{=} 0)$

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Edelstein fixed point theorem

Let (K, d) be a compact metric space and $f : K \to K$ be contractive.

Theorem (Edelstein'62)

From any starting point $x \in K$, the iteration $(f^n(x))_{n\in\mathbb{N}}$ (also denoted by $(x_n)_{n\in\mathbb{N}}$) converges to the unique fixed point of f.

Proof.

Use three lemmas:

1. $\mathsf{CTN}(f) \to \forall x \mathsf{ASY}(x_n)$
2. $\mathsf{CTN}(f) \to \mathsf{UNI}(\mathsf{fix}(f))$
3. $\forall x \mathsf{ASY}(x_n) \wedge \mathsf{UNI}(\mathsf{fix}(f)) \to \forall x \mathsf{CVG}(d(x_n, c))$

□

Uniqueness theorem: L_1-Approximation

Lemma (Kro, stated)

If $f \in C[0,1]$ *has at most* n *roots and* $\|h \, \mathrm{sgn}(f)\|_1 = 0$ *then there exists* λ *such that* $\|f - \lambda h\|_1 < \|f\|_1$.

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Lemma (Kro, really used)

For all $f \in C[0,1]; x_1 \leq \ldots \leq x_n \in [0,1]; h \in P_n; \varepsilon \in \mathbb{Q}^*_+$, *if*

$$\forall y \in \bar{B}(fy \neq 0) \wedge \int_{\bar{B}} h\,\mathrm{sgn}(f) > \int_B |h|$$

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Lemma (Kro, computational version)

For all ..., $\delta, r \in \mathbb{Q}^*_+\ \exists l \in \mathbb{Q}^*_+$, *if*

$$\forall y \in \bar{B}(|f(y)| \geq \delta) \wedge \sum_{i=1}^{n+1} \sigma_i \int_{\bar{B}_i} h \geq \int_B |h| + 1$$

then there exists $\lambda \in \mathbb{R}$ *such that* $\|f - \lambda h\|_1 + l < \|f\|_1$

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- More general, formulas associated to sets (not just types)
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- Applicable to (intuitionistic, classical and linear) logic, arithmetic and analysis
- Yielding new results in mathematics!