Functional Interpretations

Lecture 2: Classical Logic, Linear Logic and Arithmetic

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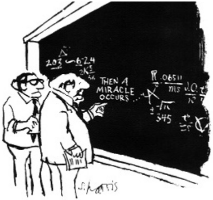

"I think you should be more explicit here in step two."

Munich, 29 March 2006

The interpretation

$$|A \wedge B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} \quad :\equiv \quad |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \wedge |B|^{\boldsymbol{v}}_{\boldsymbol{w}}$$

$$|A \vee B|^{\boldsymbol{x},\boldsymbol{v},b}_{\boldsymbol{y},\boldsymbol{w}} \quad :\equiv \quad \mathsf{if}(b, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}, |B|^{\boldsymbol{v}}_{\boldsymbol{w}})$$

$$|A \rightarrow B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} \quad :\equiv \quad \forall \boldsymbol{y} \lhd \boldsymbol{g}\boldsymbol{x}\boldsymbol{w}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \rightarrow |B|^{\boldsymbol{f}\boldsymbol{x}}_{\boldsymbol{w}}$$

$$|\forall z A(z)|^{\boldsymbol{f}}_{\boldsymbol{y},z} \quad :\equiv \quad |A(z)|^{\boldsymbol{f}z}_{\boldsymbol{y}}$$

$$|\exists z A(z)|^{\boldsymbol{x},z}_{\boldsymbol{y}} \quad :\equiv \quad |A(z)|^{\boldsymbol{x}}_{\boldsymbol{y}}$$

$$|\neg A|^{\boldsymbol{g}}_{\boldsymbol{x}} \quad :\equiv \quad \neg \forall \boldsymbol{y} \lhd \boldsymbol{g}\boldsymbol{x}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$$

Exercise

Find witnesses for the (parametrised) interpretation of

A. $\neg\neg\exists n(P(n) \to \forall m P(m))$

B. $\forall n(P(n) \to P(n+1)) \to (P(0) \to P(3))$

C. $\forall x \exists y A(x, y) \to \exists f \forall x A(x, fx)$ (Axiom of Choice)

D. $\neg\forall x A_{\mathsf{qf}}(x) \to \exists x \neg A_{\mathsf{qf}}(x)$ (Markov Principle)

Outline

1. Classical logic
   - Negative translation
   - A-translation
2. Linear logic
   - Shirahata's interpretation
3. Extensions of basic interpretation
   - Interpretable principles
   - Arithmetic

Outline

Kuroda's translation

Classical logic is obtained with the *stability rule*

$$\frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}$$

Theorem (Negative translation, Kuroda)

For each formula of A associate a new formula $A^\dagger$ by placing $\neg\neg$ after each universal quantifier. Let $A^N \equiv \neg\neg A^\dagger$. If

$$\Gamma \vdash_{\mathsf{CL}} A$$

the

$$\Gamma^N \vdash_{\mathsf{IL}} A^N$$

N-translation simplifications

Useful facts:

(i) $(\neg\neg A \to \neg\neg B) \Leftrightarrow \neg\neg(A \to B) \Leftrightarrow (A \to \neg\neg B)$

(ii) $\neg\neg\forall\neg\neg A \Leftrightarrow \forall\neg\neg A$

For instance:

$$
\begin{array}{lcl}
A & \equiv & \forall(\exists\forall\exists P \to \forall\exists Q) \\
A^N & \equiv & \neg\neg\forall\neg\neg(\exists\forall\neg\neg\exists P \to \forall\neg\neg\exists Q) \\
 & \overset{(ii)}{\Leftrightarrow} & \forall\neg\neg(\exists\forall\neg\neg\exists P \to \forall\neg\neg\exists Q) \\
 & \overset{(i)}{\Leftrightarrow} & \forall(\exists\forall\neg\neg\exists P \to \neg\neg\forall\neg\neg\exists Q) \\
 & \overset{(ii)}{\Leftrightarrow} & \forall(\exists\forall\neg\neg\exists P \to \forall\neg\neg\exists Q) \\
 & \Leftrightarrow & A \quad \text{(given Markov's principle)}
\end{array}
$$

Application 1: Relative consistency

- N-translation of $\bot$ is $(\bot \to \bot) \to \bot$ which is equivalent to $\bot$.
- Adding classical logic cannot make intuitionistic theory inconsistent.

Theorem

CL $\vdash \bot$ *if and only if* IL $\vdash \bot$

Application 2: Classical Herbrand theorem

Theorem (Herbrand, classical)

If

$$\mathsf{CL} \vdash \exists x A_{\mathsf{qf}}(x)$$

then, for some sequence of terms $t_0, \ldots, t_n$, *we have*

$$\mathsf{CL} \vdash A_{\mathsf{qf}}(t_0) \vee \ldots \vee A_{\mathsf{qf}}(t_n)$$

Proof.

1.	$\mathsf{CL} \vdash \exists x A_{\mathsf{qf}}(x)$	(assumption)
2.	$\mathsf{IL} \vdash \neg\forall x \neg A_{\mathsf{qf}}(x)$	(by n.t.)
3.	$\mathsf{IL} \vdash \neg(\neg A_{\mathsf{qf}}(t_0) \wedge \ldots \wedge \neg A_{\mathsf{qf}}(t_n))$	(by int. Herbrand)
4.	$\mathsf{CL} \vdash A_{\mathsf{qf}}(t_0) \vee \ldots \vee A_{\mathsf{qf}}(t_n)$	

□

A-translation

Negative translation as special case of the A-translation

- **Negative translation**

$$B^N \equiv (B^\dagger \to \bot) \to \bot, \text{ where } \begin{cases} P^\dagger & \equiv \quad P \\ (\forall x B(x))^\dagger & \equiv \quad \forall x((B^\dagger \to \bot) \to \bot) \end{cases}$$

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- **A-translation** (for fixed formula A)

$$B^A \equiv (B_A \to A) \to A, \text{ where } \begin{cases} P_A & \equiv & P \vee A \\ (\forall x B(x))_A & \equiv & \forall x((B_A \to A) \to A) \end{cases}$$

A-translation

Negative translation as special case of the A-translation

- **Negative translation**

$$B^N \equiv (B^\dagger \to \perp) \to \perp, \text{ where } \begin{cases} P^\dagger & \equiv \quad P \\ (\forall x B(x))^\dagger & \equiv \quad \forall x((B^\dagger \to \perp) \to \perp) \end{cases}$$

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Theorem (Friedman)

Assume A does not have free variables which are bounded in B.
If $\Gamma \vdash_{\mathsf{CL}} B$ then $\Gamma^A \vdash_{\mathsf{IL}} B^A$.

Application: Π_2^0-conservation

Observe that the A-translation of $\exists y P(x,y)$ is

$$(\exists y(P(x,y) \vee A) \to A) \to A$$

Taking $A \equiv \exists y P(x,y)$ we have that $\exists y P(x,y)^{\exists y P(x,y)}$ is

$$(\exists y(P(x,y) \vee \exists y P(x,y)) \to \exists y P(x,y)) \to \exists y P(x,y)$$

which is equivalent to $\exists y P(x,y)$, given that

$$\exists y(P(x,y) \vee \exists y P(x,y)) \to \exists y P(x,y)$$

is intuitionistically provable.

Theorem

CL $\vdash \forall x \exists y P(x,y)$ *if and only if* IL $\vdash \forall x \exists y P(x,y)$

Outline

Linear vs. non-linear negation

- Intuitionistic negation $A \to \bot$

  some set of consequences of A is inconsistent ($A \wedge \ldots \wedge A \to \bot$)
- Linear negation $A \multimap \bot$

  one single instance of A implies $\bot$

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- Linear Logic $\begin{cases} \text{identifies } A \text{ with } (A^\bot)^\bot \\ \text{has the existence property} \end{cases}$

Linear vs. non-linear negation

- Intuitionistic negation $A \to \perp$

  some set of consequences of A is inconsistent ($A \wedge \ldots \wedge A \to \perp$)
- Linear negation $A \multimap \perp$

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- Linear Logic $\begin{cases} \text{identifies } A \text{ with } (A^\perp)^\perp \\ \text{has the existence property} \end{cases}$

- Girard's comment:

  "This exceptional behaviour of 'nill' (the linear negation) comes from the fact that $A^\perp$ negates a single action of type A, whereas usual negation only negates some (unspecified) iteration of A, what usually leads to a Herbrand disjunction of unspecified length"

Classical linear logic

Connectives	Exponentials	Structural
$\dfrac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\,(\wedge)$	$\dfrac{\vdash \Gamma}{\vdash \Gamma, ?A}\,(\text{wkn})$	$\dfrac{\vdash \Gamma}{\vdash \pi\{\Gamma\}}\,(\text{per})$
$\dfrac{\vdash \Gamma, A_i}{\vdash \Gamma, A_0 \vee A_1}\,(\vee_r)$	$\dfrac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\,(\text{con})$	$\vdash \alpha^{\perp}, \alpha \quad (\text{id})$
$\dfrac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\,(\otimes)$	$\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,(!)$	$\dfrac{\vdash \Gamma, A \quad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\,(\text{cut})$
$\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,(\parr)$	$\dfrac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\,(?)$	

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$\dfrac{\vdash \Gamma, A_i}{\vdash \Gamma, A_0 \vee A_1}\ (\vee_r)$	$\dfrac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}$ (con)	$\vdash \alpha^{\perp}, \alpha$ (id)
$\dfrac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ (\otimes)$	$\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ (!)$	$\dfrac{\vdash \Gamma, A \quad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}$ (cut)
$\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \invamp B}\ (\invamp)$	$\dfrac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ (?)$	

Shirahata's interpretation

- Symmetric logic should lead to a symmetric interpretation
- **Intuition**: A interpreted as a two-player game $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$
  $\exists$-player chooses $\boldsymbol{x}$ and $\forall$-player chooses $\boldsymbol{y}$ simultaneously

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\begin{aligned}
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\qquad
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\end{array}
$$

$$
\begin{array}{lcl}
|A \otimes B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{f},\boldsymbol{g}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{v}} \wedge |B|^{\boldsymbol{v}}_{\boldsymbol{g}\boldsymbol{x}} \\
|A \parr B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{y},\boldsymbol{w}} & :\equiv & |A|^{\boldsymbol{g}\boldsymbol{w}}_{\boldsymbol{y}} \vee |B|^{\boldsymbol{f}\boldsymbol{y}}_{\boldsymbol{w}}
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\end{array}
$$

- $|A^{\perp}|^{\boldsymbol{y}}_{\boldsymbol{x}} \equiv \neg |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$
- $|A \multimap B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} \equiv |A|^{\boldsymbol{x}}_{\boldsymbol{g}\boldsymbol{w}} \to |B|^{\boldsymbol{f}\boldsymbol{x}}_{\boldsymbol{w}}$

Soundness

Theorem (Shirahata)

If $\vdash_{\mathsf{LL}} A_0, \ldots, A_n$ *then there are sequences of terms* $\boldsymbol{t}_0, \ldots, \boldsymbol{t}_n$ *(*$\boldsymbol{y}_i \notin \mathsf{FV}(t_i)$*) such that* $\vdash_{\mathsf{CL}^\omega} |A_0|^{\boldsymbol{t}_0}_{\boldsymbol{y}_0}, \ldots, |A_n|^{\boldsymbol{t}_n}_{\boldsymbol{y}_n}$.

Proof.

□

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Proof.

Consider "promotion". Assume we have

$$\vdash_{\mathsf{CL}^\omega} |?\Gamma|^{\boldsymbol{s}[\boldsymbol{y}]}_{\boldsymbol{w}}, |A|^{\boldsymbol{t}[\boldsymbol{w}]}_{\boldsymbol{y}}$$

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Unwinding the definition of $?\Gamma$ we have

$$\vdash_{\mathsf{CL}^\omega} \exists \boldsymbol{v} \lhd \boldsymbol{s}[\boldsymbol{y}](\boldsymbol{w}) \, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}}, |A|^{\boldsymbol{t}[\boldsymbol{w}]}_{\boldsymbol{y}}$$

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Let $\tilde{s}[\boldsymbol{f}](\boldsymbol{w}) := \bigcup_{\boldsymbol{v} \in \boldsymbol{f}(\boldsymbol{t}[\boldsymbol{w}])}((\lambda \boldsymbol{y}.\boldsymbol{s}[\boldsymbol{y}](\boldsymbol{w}))\boldsymbol{v})$, then

$$\vdash_{\mathsf{CL}^\omega} \exists \boldsymbol{v} \lhd \tilde{s}[\boldsymbol{f}](\boldsymbol{w})\, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}}, \forall \boldsymbol{y} \lhd \boldsymbol{f}(\boldsymbol{t}[\boldsymbol{w}])\, |A|^{\boldsymbol{t}[\boldsymbol{w}]}_{\boldsymbol{y}}$$

i.e. $\vdash_{\mathsf{CL}^\omega} |?\Gamma|^{\tilde{s}[\boldsymbol{f}]}_{\boldsymbol{w}}, |!A|^{\boldsymbol{t}[\boldsymbol{w}]}_{\boldsymbol{f}}$. □

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Soundness Extension I

- We have seen how to interpret IL into IL^ω
- Interpretation extends easily to $\mathrm{IL}^\omega \mapsto \mathrm{IL}^\omega$
- ... with a neutral treatment of equality:
  - $x =_\rho y$ for each finite type ρ
  - reflexivity
  - transitivity
  - $x =_\rho y \rightarrow z(x) =_\tau z(y)$
  - $x =_{\rho\rightarrow\tau} y \rightarrow xz =_\tau yz$

Extensional equality

- We would like to have an extensional treatment of equality, i.e. $\forall z^{\rho}(xz =_{\tau} yz) \to x =_{\rho\to\tau} y$

Theorem (Howard)

Any witness for the Dialectica interpretation of

$$\mathsf{EXT} \;:\; \forall Y, f, g(\forall n(fn =_o gn) \to Y(f) =_o Y(g))$$

is not majorizable.

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  $\forall z^\rho (xz =_\tau yz) \to x =_{\rho \to \tau} y$

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is not majorizable.

Proof.

The Dialectica interpretation of EXT asks for a functional Φ satisfying

$$\forall Y, f, g ((f(\Phi Y f g) =_o g(\Phi Y f g)) \to Y(f) =_o Y(g))$$

Let $\Phi \leq^* \Phi^*$ and $Y, f, g \leq^* 1$ so that $\Phi Y f g \leq \Phi^* 111 = k$, for some k. Let f, g coincide up to k and differ at $k+1$, and $Y(f) = f(k+1)$. □

Soundness Extension I

Theorem (Soundness - Extension I)

Let the monoidal embedding be fixed. If

$$\Gamma \vdash_{\mathsf{IL}^\omega} A$$

then there are sequences of terms $\boldsymbol{t}, \boldsymbol{s}$ *such that*

$$\forall \boldsymbol{w} \lhd \boldsymbol{s}\boldsymbol{v}\boldsymbol{y}\, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}} \vdash_{\mathsf{IL}^\omega} |A|^{\boldsymbol{t}[\boldsymbol{v}]}_{\boldsymbol{y}}$$

Interpretable principles

Definition (Interpretable principles)

If P is such that $\mathsf{IL} \nvdash P$ but $\mathsf{IL}^\omega \vdash \exists \boldsymbol{x} \forall \boldsymbol{y} |P|^{\boldsymbol{x}}_{\boldsymbol{x}}$ then P is called a $\lhd$-interpretable principle (for short, $\mathsf{I}_\lhd$).

One such example (for all instantiations) is the axiom of choice

$$\mathsf{AC} \quad : \quad \forall x^\rho \exists y^\tau A(x, y) \to \exists f^{\rho \to \tau} \forall x A(x, fx)$$

given $|A(x, y)|^{\boldsymbol{v}}_{\boldsymbol{w}}$ the interpretation of premise is

$$|A(x, y)|^{\boldsymbol{v}, fx}_{\boldsymbol{w}, x}$$

which is the same as $|A(x, fx)|^{\boldsymbol{v}, f}_{\boldsymbol{w}, x}$ (the interpretation of conclusion).

Soundness Extension II

Theorem (Soundness - Extension II)

Let the monoidal embedding be fixed. If

$$\Gamma \vdash_{\mathsf{IL}^\omega + \mathsf{I}_\lhd} A$$

then there are sequences of terms $\boldsymbol{t}, \boldsymbol{s}$ *such that*

$$\forall \boldsymbol{w} \lhd \boldsymbol{svy} \, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}} \vdash_{\mathsf{IL}^\omega} |A|^{\boldsymbol{t}[\boldsymbol{v}]}_{\boldsymbol{y}}$$

Image and kernel of interpretation

Definition ($\forall\lhd$-bounded formulas)

The $\forall\lhd$-bounded formulas (we use $A_{\rm b}$ and $B_{\rm b}$) are those built out of

- prime formulas
- conjunction $(A_{\rm b} \wedge B_{\rm b})$
- implication $(A_{\rm b} \to B_{\rm b})$ and
- bounded quantification $(\forall x \lhd t A_{\rm b})$

- The verifying system only needs $\lhd$-bounded formulas
- Most interpretations are idempotent, so that $||A|| \leftrightarrow |A|$
- $\lhd$-bounded formulas also form kernel $\mathsf{K}_{\lhd}$ of interpretation
- Formulas in the kernel can be trivially added to soundness

Image of Functional Interpretations

	$x \lhd a$	$\lhd$-bounded formulas
Modified realizability	true	$\exists$-free formulas
Diller-Nahm	$x \in a$	$\forall \in$-bounded formulas
Bounded f.i.	$x \leq^* a$	$\forall \leq^*$-bounded formulas
Dialectica	$x = a$	quantifier-free formulas

Soundness Extension III

Theorem (Soundness - Extension III)

Let the monoidal embedding be fixed. If

$$\Gamma \vdash_{\mathsf{IL}^\omega + \mathsf{I}_\lhd + \mathsf{K}_\lhd} A$$

then there are sequences of terms $\boldsymbol{t}, \boldsymbol{s}$ *such that*

$$\forall \boldsymbol{w} \lhd \boldsymbol{s}\boldsymbol{v}\boldsymbol{y}\, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}} \vdash_{\mathsf{IL}^\omega + \mathsf{K}_\lhd} |A|^{\boldsymbol{t}[\boldsymbol{v}]}_{\boldsymbol{y}}$$

Interpreting induction

We consider the induction rule and recursor

$$\frac{A(0) \quad A(k) \to A(k')}{A(n)}\ \mathsf{IND} \qquad \begin{array}{lcl} \mathsf{Rec}(g, f, 0) & = & g \\ \mathsf{Rec}(g, f, n') & = & f(n, \mathsf{Rec}(g, f, n)) \end{array}$$

That can be interpreted as follows:

$$\dfrac{\dfrac{\dfrac{\vdash |A(0)|^{\boldsymbol{s}}_{\boldsymbol{y}}}{\vdash \forall \boldsymbol{y} |A(0)|^{\boldsymbol{s}}_{\boldsymbol{y}}}}{\vdash \forall \boldsymbol{y} |A(0)|^{\mathsf{Rec}(\boldsymbol{s},\boldsymbol{t},0)}_{\boldsymbol{y}}} \quad \dfrac{\dfrac{\dfrac{\forall \boldsymbol{y}' \lhd \boldsymbol{q}[\boldsymbol{x}, \boldsymbol{y}]\, |A(k)|^{\boldsymbol{x}}_{\boldsymbol{y}'} \vdash |A(k')|^{\boldsymbol{tx}}_{\boldsymbol{y}}}{\forall \boldsymbol{y} |A(k)|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash \forall \boldsymbol{y} |A(k')|^{\boldsymbol{tx}}_{\boldsymbol{y}}}}{\forall \boldsymbol{y} |A(k)|^{\mathsf{Rec}(\boldsymbol{s},\boldsymbol{t},k)}_{\boldsymbol{y}} \vdash \forall \boldsymbol{y} |A(k')|^{\boldsymbol{t}\,\mathsf{Rec}(\boldsymbol{s},\boldsymbol{t},k)}_{\boldsymbol{y}}}}{\forall \boldsymbol{y} |A(k)|^{\mathsf{Rec}(\boldsymbol{s},\boldsymbol{t},k)}_{\boldsymbol{y}} \vdash \forall \boldsymbol{y} |A(k')|^{\mathsf{Rec}(\boldsymbol{s},\boldsymbol{t},k')}_{\boldsymbol{y}}}}{\vdash \forall \boldsymbol{y} |A(n)|^{\mathsf{Rec}(\boldsymbol{s},\boldsymbol{t},n)}_{\boldsymbol{y}}}\ \mathsf{IND}$$

Feasible arithmetic

- Functional interpretations can also be used for very weak subsystem

- Consider induction restricted to Σ^b_1-formulas

$$A(0) \wedge \forall n(A(n/2) \to A(n)) \to \forall n A(n)$$

where $A(n)$ is of the form $\exists k \leq s\,(t[k,n] = 0)$.

- **Restricted induction** can be interpreted via **restricted recursion**

$$R(n) = \begin{cases} a & n = 0 \\ b(n) & b(n) < c(n, R(n/2)) \\ c(n, R(n/2)) & \text{otherwise} \end{cases}$$

Application: Parikh's theorem

Theorem (Parikh)

Let $A_{\mathrm{b}}(x,y)$ *be a bounded formula. If*

$$\mathsf{CPV}^{\omega} \vdash \forall x^{o} \exists y^{o} A_{\mathrm{b}}(x,y)$$

then there exists a term t *such that*

$$\mathsf{CPV}^{\omega} \vdash \forall x \exists y \leq t[x] A_{\mathrm{b}}(x,y).$$

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Proof.

The bounded functional interpretation of $\forall x \exists y A_{\mathrm{b}}(x, y)$ is

$$\forall a \forall x \leq^{*} a \exists b \exists y \leq^{*} b A_{\mathrm{b}}(x, y)$$

$$\forall a \exists b \forall x \leq^{*} a \exists y \leq^{*} b A_{\mathrm{b}}(x, y)$$

$$\exists f \forall a \forall x \leq^{*} a \exists y \leq^{*} f a A_{\mathrm{b}}(x, y)$$

Proof gives monotone witness t such that
$\forall a \forall x \leq^{*} a \exists y \leq^{*} t[a] A_{\mathrm{b}}(x, y)$ □

Parikh's theorem: extension

Let Σ^0_1-UB be the following *uniform boundedness principle*:

$$\forall f \leq_1 h \exists e^1 A_0(f, e) \to \exists g \forall f \leq_1 h \exists e \leq_1 g A_0(f, e)$$

Theorem

Let $A_{\rm b}(x, y)$ *a bounded formula. If*

$$\text{CPV}^\omega + \Sigma^0_1\text{-UB} \vdash \forall x^o \exists y^\rho A_{\rm b}(x, y)$$

then there exists a term t *such that*

$$\text{CPV}^\omega \vdash \forall x^o \exists y \leq^*_\rho t[x] A_{\rm b}(x, y).$$

Quiz

Consider the following game with 3 people.

1. Each person i builds a function g_i which given her number $x_i > 0$ should give the (predicted) sum of all numbers $x_1 + x_2 + x_3$.
   E.g. $g_2(x_2) := 7x_2^2 + 111$
2. Person $i \in \{1, 2, 3\}$ is then assigned the number $x_i := g_i(i)$
3. It should be the case that $g_i(x_i) = x_1 + x_2 + x_3$

How should the participants proceed in choosing g_i?

Restriction: Functions g_i must be linear, i.e. $g_i(x) = a_i x + b_i$.