Functional Interpretations

Lecture 1: The parametrised interpretation

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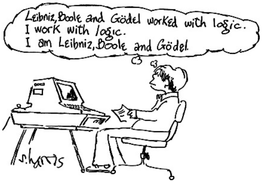

Munich, 27 March 2006

Plan of lectures

Lecture 1:

- Parametrised interpretation
- Intuitionistic logic

Lecture 2:

- Classical and linear logic
- Arithmetic

Lecture 3:

- Analysis
- Applications

Aim of lectures

- Broad overview of functional interpretations
- Variants

  Dialectica, m.r., Diller-Nahm, b.f.i., m.f.i.
- Different contexts

  logic, arithmetic, analysis
- Details of (some) recent developments

Outline

1. Introduction
   - Historic digression
   - Preliminaries
2. Intuitionistic logic
   - Parametrised interpretation
   - Instantiations

Outline

BHK interpretation

L.E.J. Brouwer

A. Heyting

A. N. Kolmogorov

Define "A is constructively true" or "p is a constructive proof of A.

- a pair of constructions $\langle p_0, p_1 \rangle$ is a proof of $A \wedge B$ if p_0 is proof of A and p_1 is a proof of B.
- a construction p is a proof of $A \to B$ if whenever a is a construction for A then $p(a)$ is a construction for B.

Curry-Howard isomorphism

H. Curry

W. A. Howard

Isomorphism between formulas and types $[[\cdot]] : \mathbf{Form} \to \mathbf{Type}$

$$
\begin{array}{lll}
[[P]] & :\equiv & \tau_P \\
[[A \wedge B]] & :\equiv & [[A]] \times [[B]] \\
[[A \to B]] & :\equiv & [[A]] \to [[B]]
\end{array}
$$

So that proofs of A correspond (one-to-one) to λ-terms of type $[[A]]$

Intuitionistic Type Theory

Martin Löf

Extend isomorphism to predicate logic, using *dependent types*

$$[[\forall x^X A(x)]] \quad :\equiv \quad \Pi_{x:X}[[A(x)]]$$
$$[[\exists x^X A(x)]] \quad :\equiv \quad \Sigma_{x:X}[[A(x)]]$$

Gödel's Dialectica interpretation

- Developed by Gödel since the 1930s
- Finally published in 1958
- **Aim**:

  solution to Hilbert's consistence program

  reduce consistency of arithmetic to the consistency of a "finitary calculus" T

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- **Interpretation**:

  associate formulas A with new formulas $(A)^D$

  If HA $\vdash A$ then T $\vdash (A)^D$

  If Con(T) then Con(HA) (since $(\perp)^D \equiv \perp$)

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  If HA $\vdash A$ then T $\vdash (A)^D$

  If Con(T) then Con(HA) (since $(\bot)^D \equiv \bot$)
- **Functional interpretations**:

  Variations of Dialectica interpretation

K. Gödel

Functional interpretations

- Associate formulas A to specifications $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$
- **Intuitively**:
  - A is interpreted by $\exists \boldsymbol{x} \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$
  - A associated with the "type" $\{\boldsymbol{x} : \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}\}$
- Proof of A provides a witness $\boldsymbol{t}$ for interpretation $\forall \boldsymbol{y} |A|^{\boldsymbol{t}}_{\boldsymbol{y}}$

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	Formula A	Proof π
Curry-Howard	$[[A]]$	$t_\pi : [[A]]$
Functional Interpretations	$\{\boldsymbol{x} : \forall \boldsymbol{y} \vert A\vert_{\boldsymbol{y}}^{\boldsymbol{x}}\}$	$\forall \boldsymbol{y} \vert A\vert_{\boldsymbol{y}}^{\boldsymbol{t}}$

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Functional Interpretations	$\{\boldsymbol{x} : \forall \boldsymbol{y} \|A\|^{\boldsymbol{x}}_{\boldsymbol{y}}\}$	$\forall \boldsymbol{y} \|A\|^{\boldsymbol{t}}_{\boldsymbol{y}}$

- **Modular**: analysis of sub-proofs reused in analysis of main proof
- **Flexible** definition of negation, using "counter-examples" $\boldsymbol{y}$

From unwinding to proof mining

1951 Kreisel launches his “unwinding program”, describes the notion of an “interpretation” and defines his no-counterexample interpretation

1952 Kreisel attempts an application to Littlewood's theorem

1958 Gödel's publishes Dialectica interpretation presenting relative consistency proof of *classical arithmetic*

1958 Kreisel revisits his program, asking for shift from “Hilbert's consistency program” to concrete mathematical applications

1958 Kreisel analyses Artin's proof of Hilbert's 17th problem

1959 Kreisel defines a variant of Dialectica called “modified realizability”

1962 Spector extends the Dialectica interpretation to *classical analysis*

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1992 Kohlenbach develops a “monotone” version of the Dialectica interpretation and applies to uniqueness proofs

2001- New case studies in approximation theory and fixed-point theory

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Finite types

We will normally consider multi-sorted first-order theories.

Sorts will be taken from the set "$\mathbf{Type}$" of types:

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  Finite sequences

Quantifications over all finite types: $\forall x^{\rho \to \tau} \exists y^{\sigma} \ldots$

Majorizability

Partial order between terms of type ρ

$$
\begin{aligned}
n \leq_o^* m &\quad :\equiv \quad n \leq m \\
f \leq_{\rho\to\tau}^* g &\quad :\equiv \quad \forall x^\rho \forall y \leq_\rho^* x (fy \leq_\tau^* gx \wedge gy \leq_\tau^* gx)
\end{aligned}
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Define new type structure $\mathcal{M}$ as

$$
\begin{aligned}
\mathcal{M}_o \quad &:\equiv \quad \mathbb{N} \\
\mathcal{M}_{\rho\to\tau} \quad &:\equiv \quad \{f \in \mathcal{M}_\tau^{\mathcal{M}_\rho} \; : \; \exists g \in \mathcal{M}_\tau^{\mathcal{M}_\rho} (f \leq^* g)\}
\end{aligned}
$$

Majorizability: properties

Monotonicity	$f \leq^* g \wedge x \leq^* y \to fx \leq^* gy$
Self-majorizability	$f \leq^* g \to g \leq^* g$
Joins (a, b monotone)	$(a \leq^* \max\{a, b\}) \wedge (b \leq^* \max\{a, b\})$
Model of T	closed terms t of T have majorant t^*
Weak continuity	$\forall Y^{\mathbb{N}^\omega \to \mathbb{N}}, f \exists n \forall g (\forall i \leq n (fi = gi) \to Y(g) \leq n)$

- g is called *monotone* if $g \leq^* g$
- $\max\{a, b\}$ defined as $\begin{cases} \max_o\{n, m\} & := & \max_{\mathbb{N}}\{n, m\} \\ \max_{\rho \to \tau}\{f, g\}(x) & := & \max_\tau\{fx, gx\} \end{cases}$

Outline

$\wedge \rightarrow \forall$ fragment of IL

$$A \vdash A \quad \text{(id)} \qquad\qquad \bot \vdash A \quad \text{(efq)}$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ \wedge\mathsf{E}_l \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ \wedge\mathsf{E}_r \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\ \wedge\mathsf{I}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}\ \rightarrow\mathsf{I} \qquad \frac{\Gamma \vdash A \rightarrow B}{\Gamma, A \vdash B}\ \rightarrow\mathsf{E}$$

$$\frac{\Gamma \vdash A(x)}{\Gamma \vdash \forall x A(x)}\ \forall\mathsf{I} \qquad \frac{\Gamma \vdash \forall x A(x)}{\Gamma \vdash A(s)}\ \forall\mathsf{E}$$

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}\ \text{(wkn)} \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\ \text{(con)} \qquad \frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\ \text{(cut)}$$

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Basic interpretation

Associate formulas A to "types" $|A|^{\boldsymbol{x}}$

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\begin{aligned}
|A \wedge B|^{\boldsymbol{x},\boldsymbol{v}} &:\equiv |A|^{\boldsymbol{x}} \wedge |B|^{\boldsymbol{v}} \\
|A \to B|^{\boldsymbol{f}} &:\equiv \forall \boldsymbol{x}(|A|^{\boldsymbol{x}} \to |B|^{\boldsymbol{f}\boldsymbol{x}}) \\
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Commutative Monoids

For each type ρ let $([\rho], *)$ be a commutative monoid

- $a * b : [\rho]$ given that $a : [\rho]$ and $b : [\rho]$

with $\eta(\cdot) : \rho \to [\rho]$ and $\mu(\cdot)(\cdot) : (\rho \to [\sigma]) \to ([\rho] \to [\sigma])$

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Define a partial order $\sqsubseteq$ on $[\rho]$ as $a \sqsubseteq b$ if $\exists a'. a * a' = b$
Functionals $\eta(\cdot)$ and $\mu(\cdot)(\cdot)$ should satisfy

$$\begin{array}{lcl} \mu(f)(a) * \mu(f)(b) & \sqsubseteq & \mu(f)(a * b) \\ f(x) & \sqsubseteq & \mu(f)(\eta(x)) \end{array}$$

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Finally, define $x^\rho \lhd a^{[\rho]}$ as $\eta(x) \sqsubseteq a$. We have:

(A1) $(x \lhd a) \vee (x \lhd b) \to (x \lhd a * b)$
(A2) $x \lhd \eta(x)$
(A3) $(x \lhd a) \wedge (y \lhd fx) \to (y \lhd \mu(f)(a))$

Commutative Monoids: Examples

Axioms:

$$\begin{array}{lcl} \mu(f)(a) * \mu(f)(b) & \sqsubseteq & \mu(f)(a * b) \\ f(x) & \sqsubseteq & \mu(f)(\eta(x)) \end{array}$$

Instances:

$[\rho]$	$a * b$	$\eta(x)$	$\mu(f)(a)$	$x \lhd a$

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ρ	?	x	$f(a)$	$x = a$

Basic parametrised interpretation

$$
\begin{array}{lcl}
|A \wedge B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \wedge |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
|A \to B|^{\boldsymbol{f}}_{\boldsymbol{x},\boldsymbol{w}} & :\equiv & \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to |B|^{\boldsymbol{f}\boldsymbol{x}}_{\boldsymbol{w}} \\
|\forall z A(z)|^{\boldsymbol{f}}_{z,\boldsymbol{y}} & :\equiv & |A(z)|^{\boldsymbol{f}z}_{\boldsymbol{y}}
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Let the monoidal embedding be fixed, so that $\lhd$ is defined.

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Definition

The parametrised interpretation is defined as:

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\end{array}
$$

Paremetrised soundness

Theorem (Soundness)

Let the monoidal embedding be fixed. If

$$\Gamma \vdash_{\mathsf{IL}} A$$

then there are sequences of terms $\boldsymbol{t}, \boldsymbol{s}$ *such that*

$$\forall \boldsymbol{w} \lhd \boldsymbol{s}\boldsymbol{v}\boldsymbol{y} \, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}} \vdash_{\mathsf{IL}^\omega} |A|^{\boldsymbol{t}[\boldsymbol{v}]}_{\boldsymbol{y}}$$

Proof.

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Proof. Contraction

$$\cfrac{\cfrac{\forall \boldsymbol{y} \lhd \boldsymbol{r}_0\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}, \forall \boldsymbol{y} \lhd \boldsymbol{r}_1\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash |B|^{\boldsymbol{t}}_{\boldsymbol{w}}}{\forall \boldsymbol{y} \lhd \boldsymbol{r}_0 * \boldsymbol{r}_1\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}, \forall \boldsymbol{y} \lhd \boldsymbol{r}_0 * \boldsymbol{r}_1\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash |B|^{\boldsymbol{t}}_{\boldsymbol{w}}}\;(\text{A1})}{\forall \boldsymbol{y} \lhd \boldsymbol{r}_0 * \boldsymbol{r}_1\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash |B|^{\boldsymbol{t}}_{\boldsymbol{w}}}\;(\text{con})$$

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Proof. Axiom

$$\forall \boldsymbol{y}' \lhd \eta(\boldsymbol{y}) \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}'} \vdash |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \qquad \text{(A2)}$$

Paremetrised soundness

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Proof. Cut

$$\dfrac{\dfrac{\dfrac{\forall \boldsymbol{y} \lhd \boldsymbol{r}[\boldsymbol{z}]\, |\Gamma|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash |A|^{\boldsymbol{s}}_{\boldsymbol{z}}}{\forall \boldsymbol{z} \lhd \boldsymbol{q}'\, \forall \boldsymbol{y} \lhd \boldsymbol{r}[\boldsymbol{z}]\, |\Gamma|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash \forall \boldsymbol{z} \lhd \boldsymbol{q}'\, |A|^{\boldsymbol{s}}_{\boldsymbol{z}}} \quad \dfrac{\forall \boldsymbol{z} \lhd \boldsymbol{q}\, |A|^{\boldsymbol{v}}_{\boldsymbol{z}} \vdash |B|^{\boldsymbol{t}}_{\boldsymbol{w}}}{\forall \boldsymbol{z} \lhd \boldsymbol{q}'\, |A|^{\boldsymbol{s}}_{\boldsymbol{z}} \vdash |B|^{\boldsymbol{t}'}_{\boldsymbol{w}}}}{\forall \boldsymbol{z} \lhd \boldsymbol{q}'\, \forall \boldsymbol{y} \lhd \boldsymbol{r}[\boldsymbol{z}]\, |\Gamma|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash |B|^{\boldsymbol{t}'}_{\boldsymbol{w}}}}{\forall \boldsymbol{y} \lhd \mu(\lambda \boldsymbol{z}.\boldsymbol{r}[\boldsymbol{z}])(\boldsymbol{q}')\, |\Gamma|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash |B|^{\boldsymbol{t}'}_{\boldsymbol{w}}}\ (\mathrm{A3})$$

Instantiations

	$x \lhd a$	$\forall \boldsymbol{y} \lhd \boldsymbol{gxw}\, \lvert A \rvert^{\boldsymbol{x}}_{\boldsymbol{y}} \to \lvert B \rvert^{\boldsymbol{fx}}_{\boldsymbol{w}}$
Modified realizability	true	$\forall \boldsymbol{y}\, \lvert A \rvert^{\boldsymbol{x}}_{\boldsymbol{y}} \to \lvert B \rvert^{\boldsymbol{fx}}_{\boldsymbol{w}}$
Diller-Nahm	$x \in a$	$\forall \boldsymbol{y} \in \boldsymbol{gxw}\, \lvert A \rvert^{\boldsymbol{x}}_{\boldsymbol{y}} \to \lvert B \rvert^{\boldsymbol{fx}}_{\boldsymbol{w}}$
Bounded f.i.	$x \leq^* a$	$\forall \boldsymbol{y} \leq^* \boldsymbol{gxw}\, \lvert A \rvert^{\boldsymbol{x}}_{\boldsymbol{y}} \to \lvert B \rvert^{\boldsymbol{fx}}_{\boldsymbol{w}}$
Dialectica	$x = a$	$\lvert A \rvert^{\boldsymbol{x}}_{\boldsymbol{gxw}} \to \lvert B \rvert^{\boldsymbol{fx}}_{\boldsymbol{w}}$

Instantiations

	$x \lhd a$	$\forall \boldsymbol{y} \lhd \boldsymbol{g x w}\, \lvert A \rvert^{\boldsymbol{x}}_{\boldsymbol{y}} \to \lvert B \rvert^{\boldsymbol{f x}}_{\boldsymbol{w}}$
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- Modified realizability the most natural instantiation
- Diller-Nahm requires multi-sets

Application: Intuitionistic Herbrand theorem

Theorem (Herbrand, intuitionistic)

If

$$\mathsf{IL} \vdash \neg \forall x A_{\mathsf{qf}}(x)$$

then, for some sequence of terms $t_0, \ldots, t_n$ *we have*

$$\mathsf{IL} \vdash \neg (A_{\mathsf{qf}}(t_0) \wedge \ldots \wedge A_{\mathsf{qf}}(t_n))$$

Proof.

1. $\mathsf{IL} \vdash \neg \forall x A_{\mathsf{qf}}(x)$ (assumption)
2. $\mathsf{IL}^\omega \vdash \neg \forall x \in t \, A_{\mathsf{qf}}(x)$ (by f.i.)
3. $\mathsf{IL}^\omega \vdash \neg \forall x \in t_0 * \ldots * t_n \, A_{\mathsf{qf}}(x)$ (by normalisation)
4. $\mathsf{IL}^\omega \vdash \neg (A_{\mathsf{qf}}(t_0) \wedge \ldots \wedge A_{\mathsf{qf}}(t_n))$
5. $\mathsf{IL} \vdash \neg (A_{\mathsf{qf}}(t_0) \wedge \ldots \wedge A_{\mathsf{qf}}(t_n))$ (by conservation)

□

Instantiations: Dialectica

- Take $x \lhd a$ as $x = a$, so that

$$\forall \boldsymbol{y} \lhd \boldsymbol{gxw}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to |B|^{\boldsymbol{fx}}_{\boldsymbol{w}}$$

simplifies to

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  ... unless the type ρ comes equipped with the specification $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$

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- Problem: Not really a monoid...

... unless the type ρ comes equipped with the specification $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$

- Then monoid can be defined as:

$[\rho]_{\lvert A\rvert^{\boldsymbol{x}}_{\boldsymbol{y}}}$	$a * b$	$\eta(x)$	$\mu(f)(a)$	$x \lhd a$
ρ	$\text{if}(\lvert A\rvert^{\boldsymbol{x}}_{b}, a, b)$	x	$f(a)$	$x = a$

Instantiations: Bounded f.i.

- Take $x \lhd a$ as $x \leq^* a$, so that

$$\forall \boldsymbol{y} \lhd \boldsymbol{gxw}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \rightarrow |B|^{\boldsymbol{fx}}_{\boldsymbol{w}}$$

simplifies to

$$\forall \boldsymbol{y} \leq^* \boldsymbol{gxw}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \rightarrow |B|^{\boldsymbol{fx}}_{\boldsymbol{w}}$$

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  ... unless x is monotone, in which case $\eta(x) = x$

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- Problem: mapping $\eta(x) = x^*$ not effective

  ... unless x is monotone, in which case $\eta(x) = x$

  Achieve this through relativization

$$\forall x A(x) \quad \Rightarrow \quad \forall x \in \mathcal{M}\, A(x) \quad \Leftrightarrow \quad \forall a \forall x \leq^* a\, A(x)$$

$$\exists x A(x) \quad \Rightarrow \quad \exists x \in \mathcal{M}\, A(x) \quad \Leftrightarrow \quad \exists a \exists x \leq^* a\, A(x)$$

Instantiations: Bounded f.i.

Definition

Interpretation as before, except:

$$
\begin{aligned}
|\forall z \leq^* a\, A(z)|^{\boldsymbol{b}}_{\boldsymbol{c}} \quad &:\equiv \quad \forall z \leq^* a\, |A(z)|^{\boldsymbol{b}}_{\boldsymbol{c}} \\
|\forall z A(z)|^{\boldsymbol{f}}_{\boldsymbol{b},a} \quad &:\equiv \quad \forall z \leq^* a\, |A(z)|^{\boldsymbol{f}a}_{\boldsymbol{b}}
\end{aligned}
$$

Adding $\vee\exists$

New rules:

$$\frac{\Gamma, A_0 \vee A_1 \vdash B}{\Gamma, A_i \vdash B}\ \vee\mathsf{I} \qquad \frac{\Gamma, A_0 \vdash B \quad \Gamma, A_1 \vdash B}{\Gamma, A_0 \vee A_1 \vdash B}\ \vee\mathsf{E}$$

$$\frac{\Gamma, \exists x A(x) \vdash B}{\Gamma, A(s) \vdash B}\ \exists\mathsf{I} \qquad \frac{\Gamma, A(x) \vdash B}{\Gamma, \exists x A(x) \vdash B}\ \exists\mathsf{E}$$

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Extended interpretation:

$$|A \vee B|^{\boldsymbol{x},\boldsymbol{v},b}_{\boldsymbol{y},\boldsymbol{w}} \quad :\equiv \quad \mathsf{if}(b, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}, |B|^{\boldsymbol{v}}_{\boldsymbol{w}})$$

$$|\exists z A(z)|^{\boldsymbol{x},z}_{\boldsymbol{y}} \quad :\equiv \quad |A(z)|^{\boldsymbol{x}}_{\boldsymbol{y}}$$

Adding $\vee\exists$: bounded f.i.

Bounded functional interpretation only searches for bounds...

Adding ∨∃: bounded f.i.

Bounded functional interpretation only searches for bounds...

Therefore:

$$
\begin{aligned}
|\exists z \leq^* a\, A(z)|^{\boldsymbol{b}}_{\boldsymbol{c}} \quad &:\equiv \quad \exists z \leq^* a\, \forall \boldsymbol{c}' \leq^* \boldsymbol{c}\, |A(z)|^{\boldsymbol{b}}_{\boldsymbol{c}'} \\
|\exists z A(z)|^{\boldsymbol{b},a}_{\boldsymbol{c}} \quad &:\equiv \quad \exists z \leq^* a\, \forall \boldsymbol{c}' \leq^* \boldsymbol{c}\, |A(z)|^{\boldsymbol{b}}_{\boldsymbol{c}'} \\
|A \vee B|^{\boldsymbol{b},\boldsymbol{d}}_{\boldsymbol{c},\boldsymbol{e}} \quad &:\equiv \quad \forall \boldsymbol{c}' \leq^* \boldsymbol{c}\, |A|^{\boldsymbol{b}}_{\boldsymbol{c}'} \vee \forall \boldsymbol{e}' \leq^* \boldsymbol{e}\, |B|^{\boldsymbol{d}}_{\boldsymbol{e}'}
\end{aligned}
$$

Quiz

Consider the following game with 3 people.

1. Each person i builds a function g_i which given her number $x_i > 0$ should give the (predicted) sum of all numbers $x_1 + x_2 + x_3$.
   E.g. $g_2(x_2) := 7x_2^2 + 111$
2. Person $i \in \{1, 2, 3\}$ is then assigned the number $x_i := g_i(i)$
3. It should be the case that $g_i(x_i) = x_1 + x_2 + x_3$

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3. It should be the case that $g_i(x_i) = x_1 + x_2 + x_3$

How should the participants proceed in choosing g_i?