Computational Interpretations of the Contraction Axiom

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Outline

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Purpose of the talk

Role of contraction in the computational analysis of proofs

$$A \rightarrow A \wedge A$$

equivalently

$$\frac{\Gamma, A, \ldots, A \vdash B}{\Gamma, A \vdash B}$$

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- with classical logic (Herbrand's theorem)

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- with classical logic (Herbrand's theorem)
- without classical logic (Functional interpretations)

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On Herbrand theorem

- **Contraction**

$$\frac{\Gamma, A, \ldots, A \vdash B}{\Gamma, A \vdash B}$$

- **Stability**

$$\frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}$$

On Herbrand theorem

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 & \Leftrightarrow & \vdash A(t_0) \vee \ldots \vee A(t_n)
\end{array}$$

Girard on contraction

When talking about the fact that LL has the existence property, despite identifying A with $(A^{\perp})^{\perp}$, he writes:

"*This exceptional behaviour of 'nill' (the linear negation) comes from the fact that $A^{\perp}$ negates a single action of type A, whereas usual negation only negates some (unspecified) iteration of A, what usually leads to a Herbrand disjunction of unspecified length*"

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Linear logic does not deal with contractions, simply avoids it!

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Theorem (A)

$\exists n(P(n) \rightarrow P(n+1))$

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Proof.

Assume $\forall n(P(n) \wedge \neg P(n+1))$, then we have both

- $P(0) \wedge \neg P(1)$
- $P(1) \wedge \neg P(2)$

which implies $\perp$. Hence $\exists n(P(n) \rightarrow P(n+1))$. $\square$

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Theorem (A^+)

$(P(0) \rightarrow P(1)) \vee (P(1) \rightarrow P(2))$

Intuitionistic version

Theorem (B)

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Theorem (B^+)

$\neg ((P(0) \wedge \neg P(1)) \wedge (P(1) \wedge \neg P(2)))$

Proof analysis

- Classically

$$\exists x A(x) \quad \Longrightarrow \quad A(t_0) \vee \ldots \vee A(t_n)$$

- Intuitionistically

$$\neg \forall x A(x) \quad \Longrightarrow \quad \neg (A(t_0) \wedge \ldots \wedge A(t_n))$$

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- Intuitionistically

$$\neg \forall x A(x) \quad \Longrightarrow \quad \neg(A(t_0) \wedge \ldots \wedge A(t_n))$$

$$\forall x A(x) \to B \quad \Longrightarrow \quad (A(t_0) \wedge \ldots \wedge A(t_n)) \to B$$

Outline

Direct interpretation

$|A|^{\vec{x}} \equiv \vec{x}$ is a **witness** for A, or $\vec{x}$ realizes A

$$
\begin{aligned}
|A \wedge B|^{\vec{x},\vec{y}} &:\equiv |A|^{\vec{x}} \wedge |B|^{\vec{y}} \\
|A \vee B|^{\vec{x},\vec{y},b} &:\equiv |A|^{\vec{x}} \vee_b |B|^{\vec{y}} \\
|A \rightarrow B|^{\vec{f}} &:\equiv \forall \vec{x}(|A|^{\vec{x}} \rightarrow |B|^{\vec{f}\vec{x}}) \\
|\forall z A(z)|^{\vec{f}} &:\equiv \forall z(|A(z)|^{\vec{f}z}) \\
|\exists z A(z)|^{\vec{x},z} &:\equiv |A(z)|^{\vec{x}}
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$A \vee_b B \equiv$ if b then A else B

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$A \vee_b B \equiv$ if b then A else B

$|A \to A \wedge A|^{\lambda x.\langle x,x\rangle}$, i.e. $\lambda x.\langle x, x\rangle$ witnesses $A \to A \wedge A$

Non-linear negation

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  some set of consequences of A is inconsistent

- Made precise via the notion of **counter-example**:

  Witness for $A \rightarrow \perp$ consists of a (unspecified but finite) collection of **counter-examples** for A

Generalised interpretation

- $\forall \vec{y} |A|^{\vec{x}}_{\vec{y}} \equiv \vec{x}$ is a **witness** for A
- $\neg |A|^{\vec{x}}_{\vec{y}} \equiv \vec{y}$ is a **counter-example** to the witness $\vec{x}$

$$|A \to B|^{\vec{f}} \quad :\equiv \quad \forall \vec{x}(|A|^{\vec{x}} \to |B|^{\vec{f}\vec{x}})$$

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$$|A \to B|^{\vec{f}, \vec{g}}_{\vec{x}, \vec{w}} \quad :\equiv \quad \forall \vec{y} \in \vec{g}\vec{x}\vec{w} |A|^{\vec{x}}_{\vec{y}} \to |B|^{\vec{f}\vec{x}}_{\vec{w}}$$

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- (Diller-Nahm'74)

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- (Dialectica'58)

$$|A \to B|^{\vec{f},\vec{g}}_{\vec{x},\vec{w}} \quad :\equiv \quad |A|^{\vec{x}}_{\vec{g}\vec{x}\vec{w}} \to |B|^{\vec{f}\vec{x}}_{\vec{w}}$$

If $|A|$ decidable then $\forall \vec{y} \in \vec{g}\vec{x}\vec{w} |A|^{\vec{x}}_{\vec{y}}$ equivalent to $|A|^{\vec{x}}_{\tilde{\vec{g}}\vec{x}\vec{w}}$

Generalised interpretation

- (Diller-Nahm'74)

$$\forall \vec{y} \in \{\vec{y}_0, \vec{y}_1\} |A|^{\vec{x}}_{\vec{y}} \to |A|^{\vec{x}}_{\vec{y}_0} \wedge |A|^{\vec{x}}_{\vec{y}_1}$$

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- (Dialectica'58)

$$|A|^{\vec{x}}_{\mathsf{C}_{|A|}(\vec{y}_0, \vec{y}_1)} \to |A|^{\vec{x}}_{\vec{y}_0} \wedge |A|^{\vec{x}}_{\vec{y}_1}$$

where

$$\mathsf{C}_{|A|}(\vec{y}_0, \vec{y}_1) :\equiv \begin{cases} \vec{y}_0 & \text{if} \quad \neg |A|^{\vec{x}}_{\vec{y}_0} \\ \vec{y}_1 & \text{otherwise} \end{cases}$$

Generalised interpretation and MP

- Markov Principle

$$\neg \forall x P(x) \rightarrow \exists x \neg P(x)$$

Interpreted by Dialectica (under decidability condition)

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- Markov Principle

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Interpreted by Dialectica (under decidability condition)

- a weaker (but more natural) MP

$$\neg \forall x P(x) \rightarrow \exists s \neg \forall x \in s\, P(x)$$

is interpreted by Diller-Nahm

Other functional interpretations

1978. Stein's family of functional interpretations

- *Relate modified realizability and Diller-Nahm's*

1992. Monotone functional interpretation (Kohlenbach)

- *Proof mining*

2005. Bounded functional interpretation (Ferrreira & O.)

- *Conservation results in feasible analysis*

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Parametrised functional interpretation

- $|A \to B|^{\vec{f},\vec{g}}_{\vec{x},\vec{w}} :\equiv \forall \vec{y} \sqsubset \vec{g}\vec{x}\vec{w} |A|^{\vec{x}}_{\vec{y}} \to |B|^{\vec{f}\vec{x}}_{\vec{w}}$

Outline

Functional interpretations and proof mining

- Semi-classical proofs
  - Markov principle
  - Independence of premise
- Analytical proofs
  - (intuitionistic) axiom of choice
- $(\forall \rightarrow \forall)$-theorem
  - Unique approximation
    $\wedge_{i=0}^{1} ||x_i - c|| = \text{dist} \rightarrow x_0 = x_1$
  - Convex
    $||1/2(x_0 + x_1)|| = 1 \rightarrow x_0 = x_1$
  - Function
    $x_0 = x_1 \rightarrow f(x_0) = f(x_1)$

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- Compare functional interpretations

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- Applications to mathematics and CS
  - Arithmetic?
  - Linearity, complexity?