Unifying Functional Interpretations

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Outline

Functional Interpretation

- A formula mapping

$$A \quad \mapsto \quad |A|^x_y$$

  - x marks the witness required by A (i.e. $\forall y |A|^t_y$)
  - y marks the refutation of a given witness for A.

- A proof mapping

$$\vdash A \quad \mapsto \quad \vdash B,$$

  for some B such that $B \vdash \exists x \forall y |A|^x_y$.

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$$\vdash A \quad \mapsto \quad \vdash B,$$

for some B such that $B \vdash \exists x \forall y |A|^x_y$.

E.g. $B \equiv \forall y |A|^t_y$, for some term t.

History

1958. Gödel's Dialectica interpretation

- *Relative consistency of* PA

1959. Kreisel's modified realizability

- *Independence results, unwinding proofs*

1974. Diller-Nahm variant of Dialectica interpretation

- *Solve contraction problem*

1978. Stein's family of functional interpretations

- *Relate modified realizability and Diller-Nahm's*

1992. Kohlenbach's "monotone" interpretations

- *Proof mining*

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Goal

Relation between Dialectica interpretation
and modified realizability.

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and modified realizability.

Common framework for all functional interpretations
via a **parametrised functional interpretation**.

Logical Framework

$A \wedge B$	conjunction
$A \vee B$	disjunction
$A \triangledown B$	classical disjunction
$\neg A$	negation
$\exists x\, A$	existential quantifier
$\forall x\, A$	universal quantifier

Logical Framework

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$\exists x\, A$	existential quantifier
$\forall x\, A$	universal quantifier

$$A \rightarrow B \;:\equiv\; \neg A \triangledown B$$

$$A \vdash A \quad \text{(Axiom)}$$

$$\frac{\Gamma_0 \vdash A, \Delta_0 \quad \Gamma_1 \vdash B, \Delta_1}{\Gamma_0, \Gamma_1 \vdash A \wedge B, \Delta_0, \Delta_1} \wedge\text{I} \qquad \frac{\Gamma \vdash A \wedge B, \Delta}{\Gamma \vdash A, \Delta} \wedge\text{E}$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} \vee\text{I} \qquad \frac{\Gamma_0, A \vdash \Delta \quad \Gamma_1, B \vdash \Delta}{\Gamma_0, \Gamma_1, A \vee B \vdash \Delta} \vee\text{E}$$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \triangledown B, \Delta} \triangledown\text{I} \qquad \frac{\Gamma \vdash A \triangledown B, \Delta}{\Gamma \vdash A, B, \Delta} \triangledown\text{E}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \neg\text{I} \qquad \frac{\Gamma \vdash \neg A, \Delta}{\Gamma, A \vdash \Delta} \neg\text{E}$$

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$$\frac{\Gamma \vdash A(z), \Delta}{\Gamma \vdash \forall z A(z), \Delta} \ \forall\mathrm{I} \qquad \frac{\Gamma \vdash \forall z A(z), \Delta}{\Gamma \vdash A(s), \Delta} \ \forall\mathrm{E}$$

$$\frac{\Gamma \vdash A(s), \Delta}{\Gamma \vdash \exists z A(z), \Delta} \ \exists\mathrm{I} \qquad \frac{\Gamma, A(z) \vdash \Delta}{\Gamma, \exists z A(z) \vdash \Delta} \ \exists\mathrm{E}$$

$$\frac{\Gamma_0 \vdash A \quad \Gamma_1, A \vdash \Delta}{\Gamma_0, \Gamma_1 \vdash \Delta} \ \text{(cut)} \qquad \frac{\Gamma \vdash \Delta}{\Gamma' \vdash \Delta'} \ \text{(per)}$$

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} \ \text{(con)} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, \Gamma_0 \vdash \Delta, \Delta_0} \ \text{(wkn)}$$

$$\frac{\Gamma \vdash A(z), \Delta}{\Gamma \vdash \forall z A(z), \Delta} \forall\text{I} \qquad \frac{\Gamma \vdash \forall z A(z), \Delta}{\Gamma \vdash A(s), \Delta} \forall\text{E}$$

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Logical Framework

- Language of finite types $\mathcal{T}$
  - $\mathbb{N} \in \mathcal{T}$
  - $\rho, \sigma \in \mathcal{T} \quad \Rightarrow \quad \rho \to \sigma \in \mathcal{T}$
- Variable and quantifiers for each finite type $\rho \in \mathcal{T}$
- Combinatorial completeness
- Equality for $\mathbb{N}$

Heyting Arithmetic HA^ω

- Universal axioms for 0 and S
- Gödel's primitive recursion
- Induction rule

$$\frac{\vdash A(0) \qquad A(n) \vdash A(n+1)}{\vdash A(n)} \text{ (IND)}$$

Outline

1. Functional interpretations
2. **Parametrised Formula Translation**
3. Parametrised Proof Translation

The Formula Translation

$|A_{\mathrm{at}}| :\equiv A_{\mathrm{at}}$, when A_{at} is atomic.

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$|A_{\mathrm{at}}| :\equiv A_{\mathrm{at}}$, when A_{at} is atomic.

Assume we have already defined $|A|^x_y$ and $|B|^v_w$, we define

$$|A \wedge B|^{x,v}_{y,w} \quad :\equiv \quad |A|^x_y \wedge |B|^v_w$$

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Notation: $|A|^x_y \vee_n |B|^v_w :\equiv (n = 0 \wedge |A|^x_y) \vee (n \neq 0 \wedge |B|^v_w)$

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|\neg A| & :\equiv & ?
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Interpretation of negation

Assume A has interpretation $|A|^x_y$

- Gödel's Dialectica interpretation

  Functionals producing counter-examples for A, i.e.

  $|\neg A|^f_x :\equiv \neg |A|^x_{fx}$

- Modified Realizability

  $\neg A$ does not ask for witnesses, i.e.

  $|\neg A|_x :\equiv \neg \forall y |A|^x_y$

Interpretation of negation

Assume A has interpretation $|A|_y^x$

- Gödel's Dialectica interpretation

  Functionals producing counter-examples for A, i.e.

  $|\neg A|_x^f :\equiv \neg |A|_{fx}^x \equiv \neg \forall y (y = fx \rightarrow |A|_y^x)$

- Modified Realizability

  $\neg A$ does not ask for witnesses, i.e.

  $|\neg A|_x :\equiv \neg \forall y |A|_y^x \equiv \neg \forall y (\text{true} \rightarrow |A|_y^x)$

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- In General

  Functionals producing "bound" on counter-examples

  $|\neg A|^f_x :\equiv \neg \forall y \sqsubset fx |A|^x_y \equiv \neg \forall y (y \sqsubset fx \to |A|^x_y)$

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\end{aligned}
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$$
\begin{aligned}
|y < x| &\equiv (y < x) \\
|\forall y(y < x)|_y &\equiv (y < x) \\
|\exists x \forall y(y < x)|^x_y &\equiv (y < x) \\
|\neg \exists x \forall y(y < x)|^f_x &\equiv \neg \forall y \sqsubset fx (y < x)
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|\neg \exists x \forall y (y < x)|^f_x & \equiv & \neg \forall y \sqsubset fx (y < x) \quad \left\{ \begin{array}{r} \neg(fx < x) \\ \neg \forall y (y < x) \end{array} \right.
\end{array}
$$

Choice for abbreviation	Provable in HA^ω
Modified realizability $\forall x \sqsubset^{\mathrm{r}} a A(x) :\equiv \forall x A(x)$	$\exists x \forall y \lvert A \rvert^x_y \leftrightarrow \exists x (x \text{ mr } A)$
Dialectica interpretation $\forall x \sqsubset^{\mathrm{g}} a A(x) :\equiv A(a)$	$\exists x \forall y \lvert A \rvert^x_y \leftrightarrow \exists x \forall y A_D(x, y)$
Diller-Nahm variant $\forall x \sqsubset^{\wedge} a A(x) :\equiv \forall x \in a A(x)$	$\exists x \forall y \lvert A \rvert^x_y \leftrightarrow \exists x \forall y A_{\wedge}(x, y)$
Stein's family of interp. $\forall x^\tau \sqsubset^n a A(x) :\equiv \begin{cases} \forall x \in a A(x) \\ \forall x A(x) \end{cases}$	$\exists x \forall y \lvert A \rvert^x_y \leftrightarrow \exists x \forall y A_n(x, y)$

Conditions on $\forall x \sqsubset aA(x)$

For all formulas A there are $\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3$ such that

(A1) $\forall y' \sqsubset \mathsf{a}_1\, y\, |A|^x_{y'} \vdash |A|^x_y$

(A2) $\forall y' \sqsubset \mathsf{a}_2\, y_0\, y_1\, |A|^x_{y'} \vdash \forall y' \sqsubset y_0\, |A|^x_{y'} \wedge \forall y' \sqsubset y_1\, |A|^x_{y'}$

(A3) $\forall y' \sqsubset \mathsf{a}_3\, h\, z\, |A|^x_{y'} \vdash \forall y \sqsubset z \forall y' \sqsubset hy\, |A|^x_{y'}$

(Standard) Proof Translation

Theorem (Over HA^ω)

If conditions (A1), (A2), (A3) hold and

$$\vdash A$$

then there is a sequence of terms t such that

$$\vdash \forall y |A|^t_y$$

(Standard) Proof Translation

Theorem (Over HA^ω)

If conditions (A1), (A2), (A3) hold and

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Theorem (Over HA^ω)

If conditions (A1), (A2), (A3) hold and

$$\Gamma \vdash A$$

then there are sequences of terms t and q such that

$$\forall w \sqsubset q[v, y] |\Gamma|^v_w \vdash |A|^{t[v]}_y$$

Condition (A1) - Logical axioms

- Any counter-example has a bound
  $\forall y' \sqsubset \mathsf{a}_1\, y\ |A|^x_{y'} \vdash |A|^x_y$

$$A \vdash A \quad \text{(axiom)}$$

Condition (A1) - Logical axioms

- Any counter-example has a bound
  $\forall y' \sqsubset a_1 y \, |A|^x_{y'} \vdash |A|^x_y$

$$A \vdash A \quad \text{(axiom)}$$

$$\Downarrow$$

$$\forall y' \sqsubset a_1 y \, |A|^x_{y'} \vdash |A|^x_y$$

Condition (A2) - Contraction

- Joining two sets of counter-examples into one

$\forall y' \sqsubset \mathrm{a}_2\, y_0\, y_1\, |A|^x_{y'} \vdash \forall y' \sqsubset y_0\, |A|^x_{y'} \wedge \forall y' \sqsubset y_1\, |A|^x_{y'}$

$$\frac{A, A \vdash B}{A \vdash B}\ (\text{con})$$

Condition (A2) - Contraction

- Joining two sets of counter-examples into one

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$$\frac{A, A \vdash B}{A \vdash B}\ (\text{con})$$

$$\Downarrow$$

$$\frac{\forall y' \sqsubset q_0 |A|^{x_0}_{y'}, \forall y' \sqsubset q_1 |A|^{x_1}_{y'} \vdash |B|^t_w}{\forall y' \sqsubset \mathsf{a}_2\, q_0\, q_1 |A|^x_{y'} \vdash |B|^t_w}\ (\text{A2})$$

Condition (A3) - Cut

- Bounded family of sets into a single set

  $\forall y' \sqsubset \mathsf{a}_3\, h\, z\, |A|^x_{y'} \vdash \forall y \sqsubset z \forall y' \sqsubset hy\, |A|^x_{y'}$

$$\frac{\Gamma \vdash A \quad A \vdash B}{\Gamma \vdash B}\,(\mathrm{cut})$$

Condition (A3) - Cut

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$$\frac{\Gamma \vdash A \quad A \vdash B}{\Gamma \vdash B}\,(\mathrm{cut})$$

$$\Downarrow$$

$$\dfrac{\dfrac{\dfrac{\forall w \sqsubset q[y] |\Gamma|^v_w \vdash |A|^t_y}{\forall y \sqsubset r \forall w \sqsubset q[y] |\Gamma|^v_w \vdash \forall y \sqsubset r |A|^t_y} \quad \dfrac{\forall y \sqsubset r |A|^x_y \vdash |B|^s_z}{\forall y \sqsubset r |A|^t_y \vdash |B|^s_z}}{\forall y \sqsubset r \forall w \sqsubset q[y] |\Gamma|^v_w \vdash |B|^s_z}\,(\mathrm{cut})}{\forall y \sqsubset \mathrm{a}_3\, q\, r \vdash |B|^s_z}\,(\mathrm{A3})$$

Outline

Parametrised Proof Translation

$$\vdash A \quad \Rightarrow \quad \vdash B$$

for some B such that $B \vdash \exists x \forall y |A|^x_y$

Parametrised Proof Translation

$$\vdash A \quad \Rightarrow \quad \vdash B$$

for some B such that $B \vdash \exists x \forall y |A|_y^x$

- $B \equiv \forall y |A|_y^t$, for some term t
- $B \equiv \exists x \leq^* t \forall y |A|_y^t$ ($\leq^*$ is Howard/Bezem majorizability)
- $B \equiv \exists x \forall y |A|_y^x$

Parametrised Proof Translation

$$\vdash A \quad \Rightarrow \quad \vdash B$$

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- $B \equiv \forall y |A|_y^t$, for some term t
- $B \equiv \exists x \leq^* t \forall y |A|_y^t$ ($\leq^*$ is Howard/Bezem majorizability)
- $B \equiv \exists x \forall y |A|_y^x$

- $B \equiv \exists x \prec t \forall y |A|_y^x$

Conditions on $\forall x \sqsubset aA(x)$ and $\exists x \prec aA(x)$

For all formulas A there are terms $\mathsf{a}_1^*, \mathsf{a}_2^*, \mathsf{a}_3^*$ such that

(A1)* $\vdash \exists \nu \prec \mathsf{a}_1^* \forall x, y (|\neg A|_x^{\nu y} \triangledown |A|_y^x)$

(A2)* $\vdash \exists \chi \prec \mathsf{a}_2^* \forall x, y_0, y_1 (|\neg A|_x^{\chi y_0 y_1} \triangledown \bigwedge_{i=0}^{1} \forall y' \sqsubset y_i |A|_{y'}^x)$

(A3)* $\vdash \exists \xi \prec \mathsf{a}_3^* \forall x, h, z (|\neg A|_x^{\xi hz} \triangledown \forall y \sqsubset z \forall y' \sqsubset hy |A|_{y'}^x)$

Condition on $\exists x \prec a A(x)$

(E) For each formula A, closed term s and term $t[z]$, if

$$\vdash \exists z \prec s \forall a, y |A|_y^{t[z]a}$$

then there exists a closed term t^* such that

$$\vdash \exists F \prec t^* \forall a, y |A|_y^{Fa}.$$

We call t^* a $\prec$-*majorizing term* for t.

Parametrised Proof Translation

Theorem (Over HA^ω)

If conditions (A)* and (E) hold, and

$$\Gamma \vdash A,$$

then there are sequences of closed terms t, r such that

$$\vdash \exists f \prec t \exists g \prec r \forall v, y |\neg \Gamma \triangledown A|^{g,f}_{v,y}.$$

Summary

- $$\forall x \sqsubset a A(x) :\equiv \begin{cases} A(a) \\ \forall x \in a\, A(x) \\ \forall x \in a\, A(x) \text{ or } \forall x\, A(x) \quad (\text{type of } x) \\ \forall x\, A(x) \end{cases}$$

- $$\exists x \prec a A(x) :\equiv \begin{cases} A(a) \\ \exists x \leq^* a\, A(x) \\ \exists x\, A(x) \end{cases}$$

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   - different principles interpreted
   - concrete case studies

2. Extend to classical context

3. Interpretations not covered
   - Bounded functional interpretation [Ferreira, O. '04]
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